Graph Theory

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Module No. # 01
Lecture No. # 08
Gallai - Millgram theorem, Dilworth's theorem

Welcome to the eighth lecture of Graph theory.

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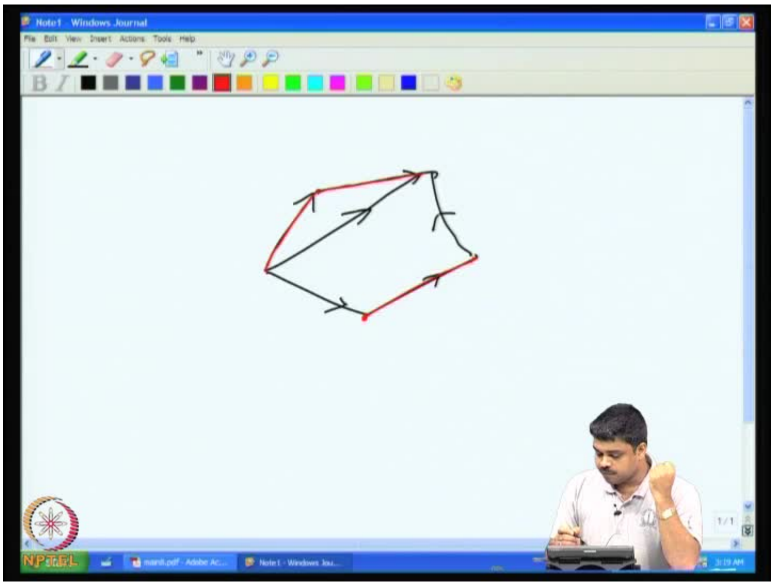

In the last class, you are we were discussing the problem of covering a directed graph with a collection of directed paths. So, here, this time we have, we are given a directed graph; that means, So there are edges and each edge has a direction.

Now, So these are the directed graphs. Now, we want to find out a collection of paths such that every vertex belongs to exactly one path; means this path should be disjoint and also it should cover all the vertices here. For instance, for this graph, I can select this path and say this path - these two directed path will cover all the vertices of this graph. So, this kind of paths is called a path cover.

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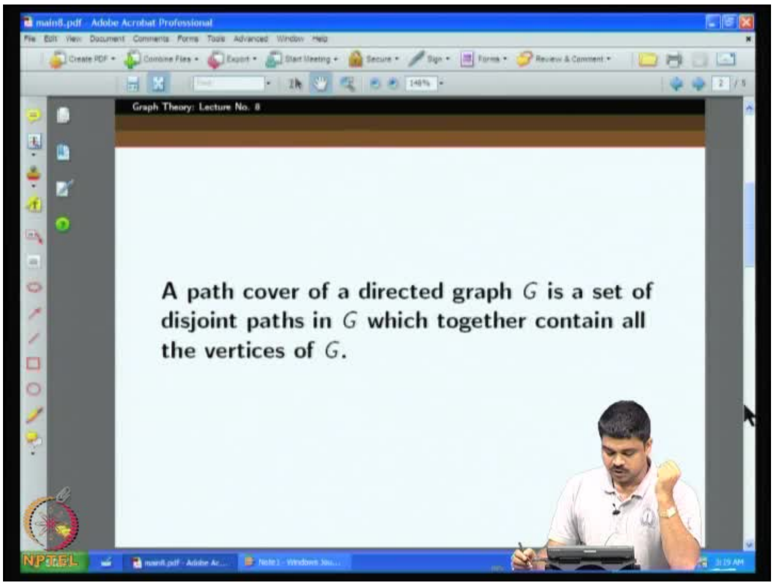

For instance, we can look here, a path cover of a directed graph G is a set of disjoint paths in G, which together contain all the vertices of G. Now, we are going to prove the Gallai Millgram theorem. And before that, let us take the theorem like this. So, suppose we are interested to find the minimum number of paths that can cover a directed graph in this way, that means we want a path cover of minimum cardinality.

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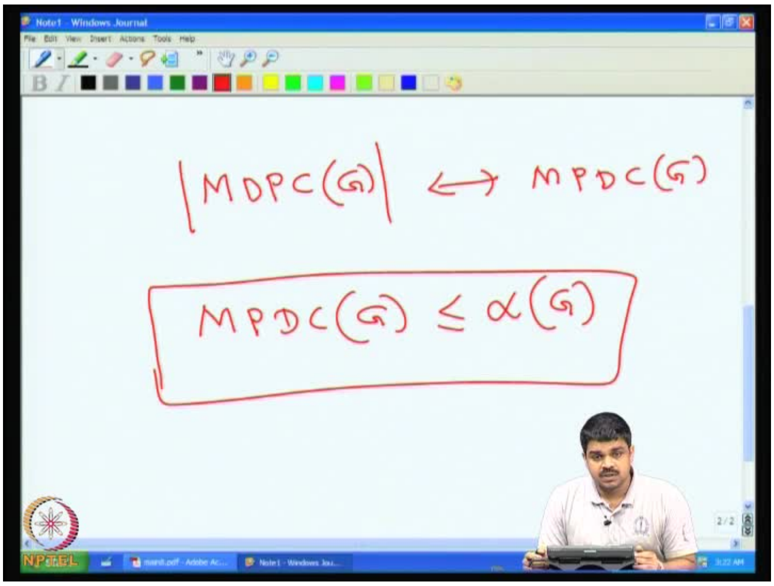

We are interested in the cardinality of the minimum path cover. We will call this parameter say minimum path directed path cover of G and we are interested in the cardinality of this. So, we may may not always write like this. We may use the notation and say say MPDC of G to represent the path cover as well as its cardinality which achieves the minimum. And now, to find out the minimum, a path cover… So, this is the question. Now, this parameter always is upper bound by MPDC of G is at most alpha of G is what we are going to prove. This is called Gallai Millgram theorem. So, slightly stronger straight statement will prove here alpha of G is the independent set.

Remember, the independent set of a directed graph is the same as the independent set of the underlined undirected graph. So, there is no difference between the direction. It will not make any difference in the independent sets, but a path cover can be different in a directed graph and an undirected graph. For instance, you are given a directed graph and you find a path cover of that. We have to cover the vertices of the graph with directed paths. The path should be disjoint and each vertex should be present in exactly one path. So, I should be present in some of these paths. That means all the vertices should be covered, but now, if you remove the directions on the edges and look at the underlying undirected graph and look for the minimum path cover, you might be able to cover with smaller number of paths. For instance, you can easily see this example.

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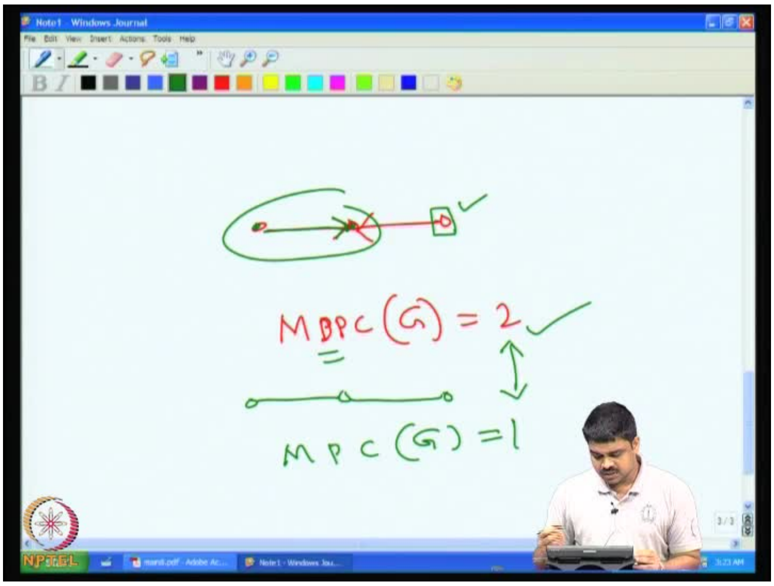

See let us take this simple example. So, here, suppose this is my directed graph. If I am asking for a minimum cardinality path cover here, I can clearly say that minimum path cover, minimum directed path cover of this graph is equal to 2 because… so once you take, see you may take this one; we cannot go further; only in any path, any directed path you can get only two vertices because their arrows does not allow us to go further here.

Now, you will have to take another one singleton path. So, this and this will be the two, but on the other hand, the underlying undirected graph will look like this because if you remove the directions on the edges to look like this, what will be minimum path cover of G? That means this D denoted the directed graph. So, this is only one. So, it can be different in both cases. Of course, this is smaller because any directed path cover when you remove the edges will give you a path cover in the corresponding undirected graph, but the other way is not true. For instance, a path cover in the underlying undirected graph may not convert to a directed path cover.

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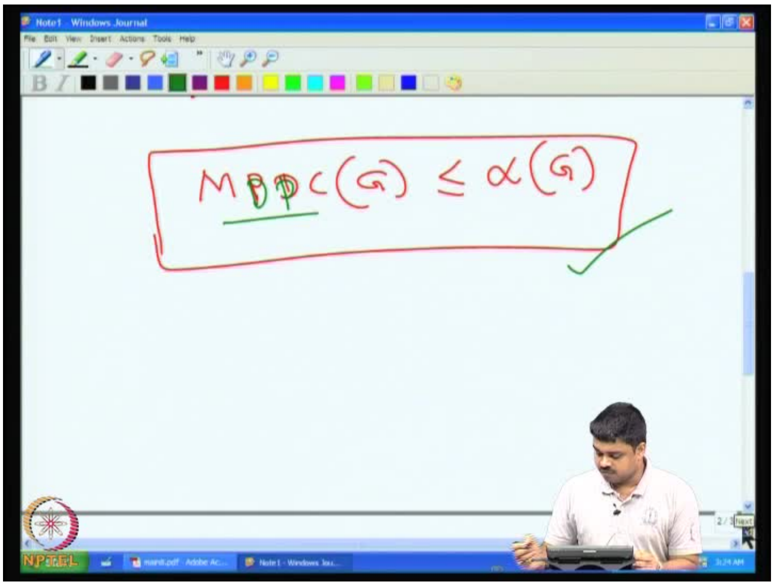

Yes. So, what we wanted to prove? As we As I wrote here, this is what I want. Given a directed graph, the cardinality of the minimum directed sorry MDPC directed path cover is less than equal to alpha of G is what I wanted to show. Suppose I look at the question, so just to understand this question, we may look at the undirected version of this thing.

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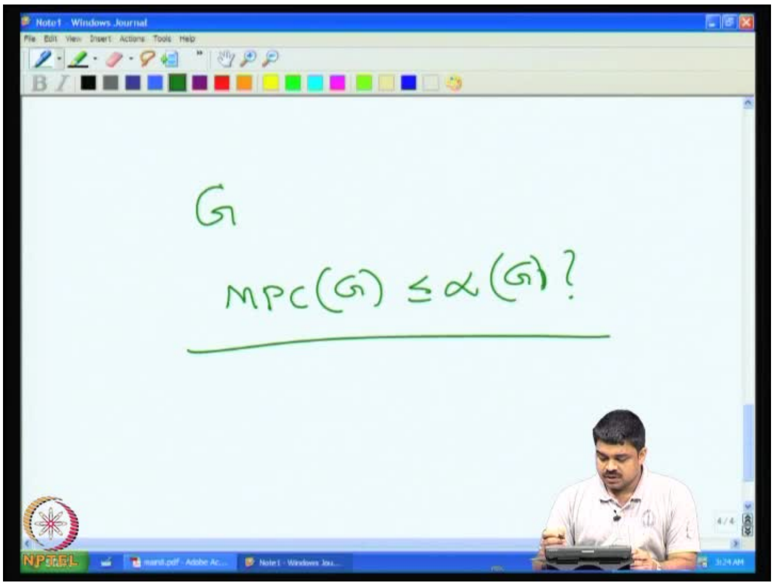

So, given a direct undirected graph G, I am interested to prove, say, M is this true minimum path cover of G is always less than equal to alpha of G. This is what the question is in undirected version.

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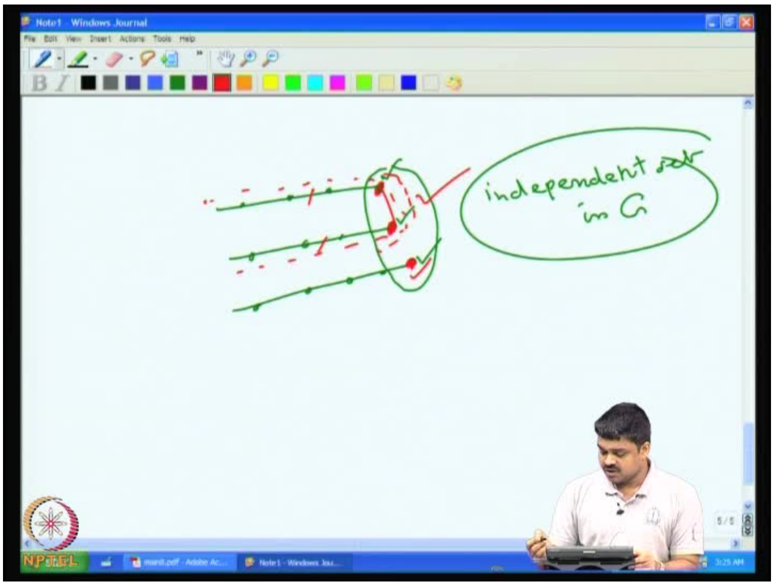

So, as I mentioned in the last class, this is the very easy question because you can always take a path cover; say, you take a minimum path cover. Suppose, this is a minimum path cover of the given graph, so, these are disjoint paths and then you pick up the n vertices of this path cover. They should always form a independent set in G. Why? Because otherwise, suppose if there is an edge here, suppose you had an edge here, now you can of course, this is, I could have replaced these two paths with this one path; is not it? Therefore, we could have got a smaller path cover and which is not possible because we assume that we have already taken a minimum path cover. So, you can see that, in this undirected case not only that we have got an independent set of cardinality at least the cardinality of the minimum path cover, that independent set was such that, the n vertices, this independent set actually is the n vertices of the path set; means I selected one n vertex from each path; that is what the independent set was.

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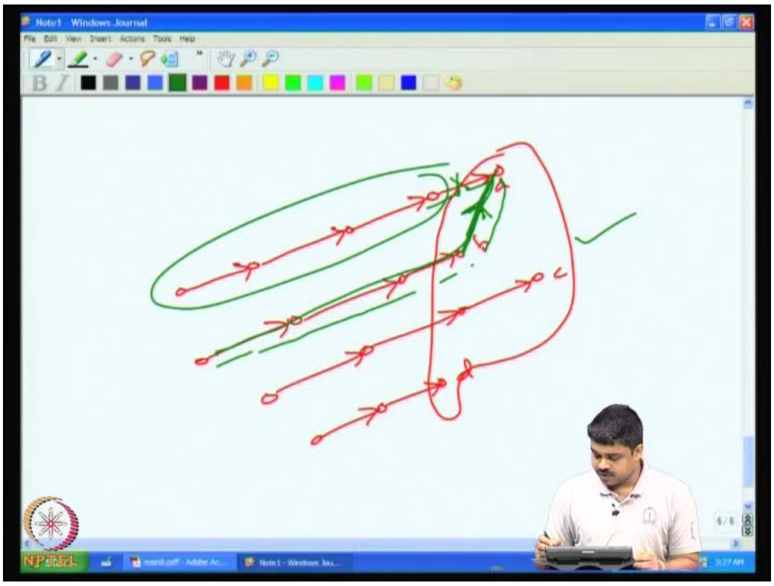

So, will the same trick work in the directed case also? But it does not work. Why? Why does not it work? Suppose this is the path cover of the directed case; so, this may be the directions. Now, let us try to pick up the n vertices like say this is a b c d. These four n vertices, let me consider. Should they form a always form an independent set? Not necessarily we can have some edge like this because the direction may be like this (Refer Slide Time: 08:48). In that case, we may try to follow this, try to construct a bigger path, but it will go here and then, it will not able to move further because the arrows are in the wrong direction. You cannot take, unlike the earlier case, we cannot continue like this.

So, we still will have two paths of case. We will cut here and then you will get little slightly longer path here, but slightly shorter path here. That is all, but the cardinality will remain same; there would not be any contradiction. So, there is no guarantee that you would not have edges of this sort. Therefore, that strategy is not true here. So, of case, one should ask whether this theorem itself is a very reasonable one because in the undirected case it is true, but given an undirected graph, you can make several directed graphs from that.

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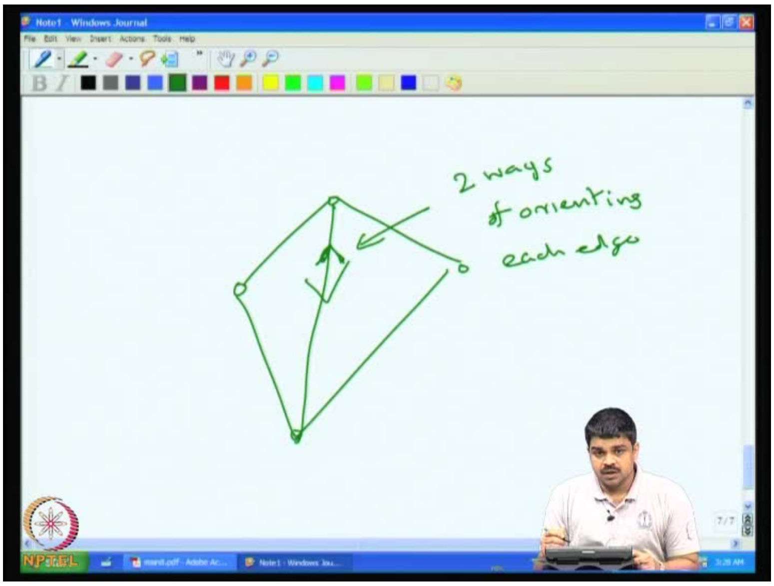

For instance, every edge can be… so given an undirected graph, so this is for see here if i for instance if I take this undirected graph of case there are several possible ways of orienting the edges. So, this can be oriented this way or the other way; this way.

So therefore, there are two ways of orienting each edge. So, there are two, two to the power number of edges, ways of orienting it. So, there are several directed graph which can be formed an undirected… some of them may be same, but as far as we are concerned, we can get several directed graph from one undirected graph.

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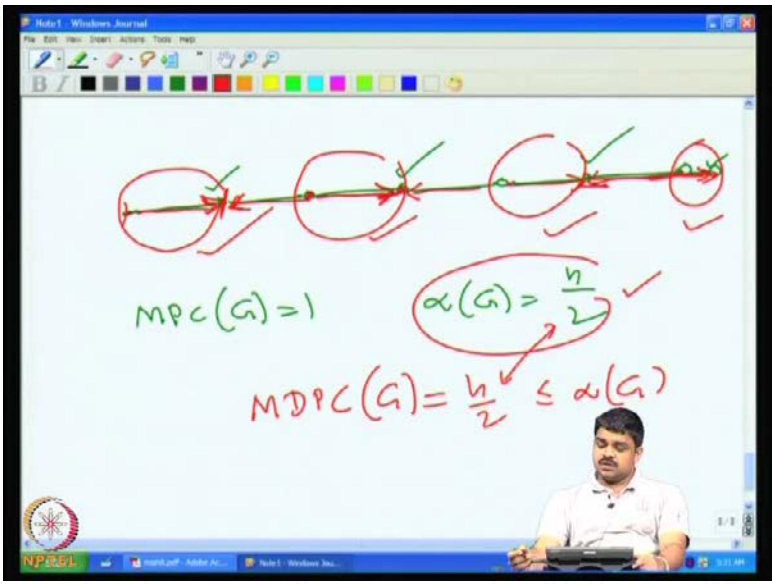

So, what if even one of them violate the theorem? If you understand that alpha independent set, the biggest independent set will not change by giving some direction to the edges; that will remain same while the path cover cardinality mean drastically increase. So, let us take one example. For instance, this is the very easy example that comes to our mind. So, this is a path; so, here, one, two, three, four, five, six, seven. Let us take eight vertices. So, we know that, in the undirected case, minimum path cover cardinality is equal to one here because there is just one path; that is enough. And of case alpha of G is equal to 4. So, let us say n by 2; if n is the number of vertices, so here this vertex, this vertex, this vertex, like that we can pick up.

Now, suppose I direct the edges this way, first I directed it this way; that means this is like this. Then, I directed this edge in the opposite direction and then I directed this way and then I direct this one, this way; so, this one, this way; this one, this way. So, on the wrong direction, in the opposite directions, I from each vertex, each vertex will see two one arrow, two arrows going out or two arrows coming in. Here, it is coming in two arrows (Refer Slide Time: 11:50 to 12:15).

So, it means that if I try to take a path, I start from anywhere. I will have to stop; if I start from here, I will have to stop with the next vertex. I will not able to go further; if I start from here, I will have to stop here. If I start from here, I will have to stop here or here (Refer Slide Time: 12:20 to 12:35). So, there is no way of getting a path containing more than two vertices in it. That is a way we have. This way, we have to increase the cardinality of the minimum path cover so much because now a path can only cover two vertices at most; n by 2 as is the MDPC; so, MDPC of G has become n by 2. So, this is one path; this is another path; this is another path; this is another path like that. This is only way we can we can cover it, but alpha remains mean same. So, it has touched, it has reached here, but still it is less than equal to alpha of G, interestingly, but look at the way we have oriented the edges that cardinality of the minimum directed path cover has gone up to n by 2 and it touched the alpha. So, this is only simple example. It is just possible that I can have some complicated examples, such that it may cross alpha. So, may be the theorem itself may not be true; that comes to our mind.

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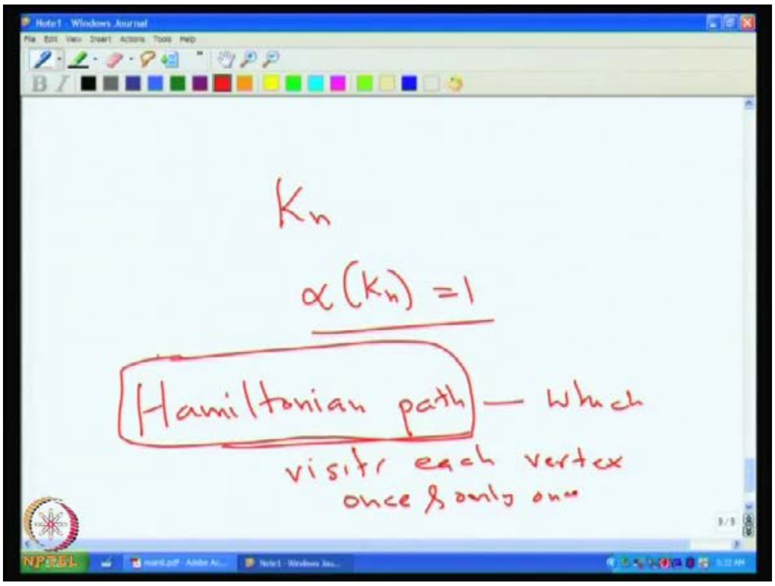

So, then thinking like that, another example which will come easily to our mind is the complete graph k n. So, what is so interesting about k n? The k n, the independent set is the maximum independent set cardinality is only 1. You cannot have an independent set of cardinality 2 or more. So, alpha of k n is equal to 1. Why? Because, now, any, so of case in k n, there is a Hamiltonian path. What is a Hamiltonian path? That means a Hamiltonian path is a path which touches which touches for which visits each vertex once and only once. So, this is one term which we should be familiar. Hamiltonian path means it will go from vertex to… path will move from vertex to vertex, and every vertex will come exactly once; every vertex will come in only once. The Hamiltonian cycle means, so it is a cycle. In the same way, every vertex will be present in the cycle; exactly a cycle; so, a cycle which goes through all the vertices, but exactly once. This will be Hamiltonian cycle.

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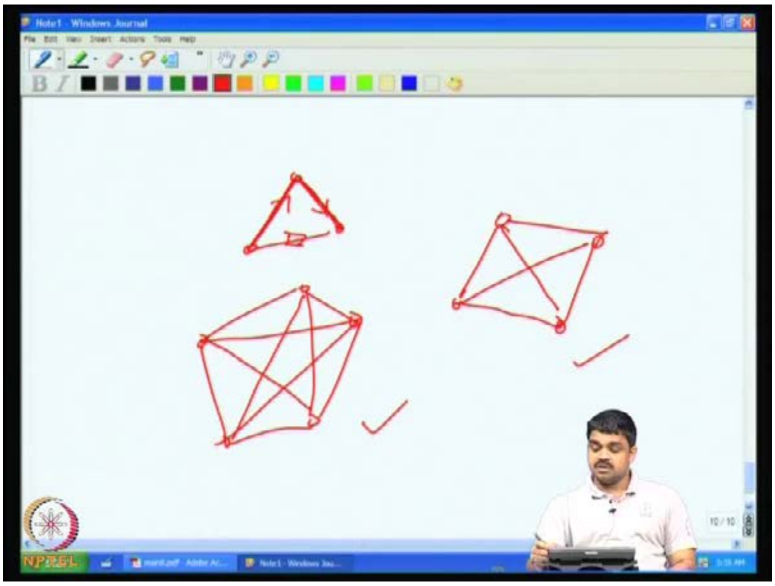

Now, we know that, in a complete graph, any sequence of vertices if I take one, two, three, four, up to n and if you travel the edges, you will get a Hamiltonian path. Therefore, that is not big deal in a undirected k n, but what if I try to orient the edges of k n, in such a way that… see my tension is to somehow beat this; that means to show that I may get an orientation, such that the minimum path cover cardinality increases to more than 1, 2 at least or in other words I will somehow try to orient the edges in such a way that there are no there is no Hamiltonian directed path after orienting the edges. Is it possible?

For instance, you may want to try orienting a triangle, but unfortunately, whatever way your triangle you will orient your triangle, you will end up with a Hamiltonian; this way or this way, whatever Hamiltonian path here, but then triangle is quite small. You can take a k 4 and try. It will be interesting to see whether k 4 or k 5; interesting to see whether you can give direction to the edges of these graphs, in such a way that there is no Hamiltonian directed path in these things. So, that is, after some attempts you may realize that. See you can always get whatever way you orient the edges, you end up getting one Hamiltonian path; sorry directed path.

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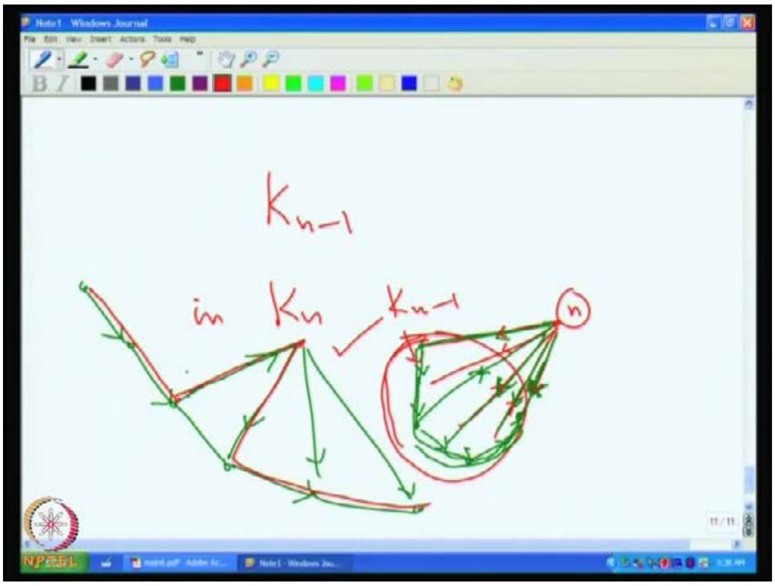

So, for instance, I can give a quick proof for that. For instance, so let us let us try with the 2. It is very easy to prove k equal to n equal to 2; k 2 has of case irrespective of the orientation. You will have a Hamiltonian directed path. Similarly, k n, k 1 k 3 if you take, it is very easy to show. So, let us say up to k n minus 1. We have shown that, irrespective of the way we orient the edges of the graph, then we can end up with a directed Hamiltonian path; we can end up with a directed Hamiltonian path.

So, it is a… Now, how do I show that in k n also in k n also, it is possible. So, we will we will take the last vertex; say, this is the last vertex. You see it is connected to all the vertices in this k n minus 1; this is the k n minus 1 and this is the last n th vertex. So, n th vertex. Now, see all these are irrespective… some then you are giving some direction to this. Some of them may be directed like this; some of them may be directed like this, whatever it is; so, there will be some directions. What you do is here you will try to find because by induction, you know that there is a directed path here; may be some directed path is there here; like, the directions are like this (Refer Slide Time: 18:50).

Now, the idea is… So, you will first look here - the last vertex of this directed Hamiltonian path in this thing and see how this edge is oriented. If it is oriented backward, that means then we are done because if it is if it is like this, then we are done because we can just attach that vertex to it; you get a Hamiltonian. So, it should be oriented like this (Refer Slide Time: 19:19).

Now, we know that. So, if all the edges are oriented like this, so if all the this in the n th this also will be oriented like this. So, then, I would start like this and complete the directed path. Therefore, if I keep going like this just checking the orientation from this vertex to the edges, going back in this path like this, you will see the orientation changing once. So, at some point you will see that the path is like this (Refer Slide Time: 20:11to 20:40). So, the path has orientation like this at some points; say, this point you will see that. Here, this is oriented like this. From here onwards, all the orientation are like this. Now, what will I do? I will rather, I will trace path like there I will go, come up to here and then go here; then go here and then follow this. So, then I can include it here. So, this is one way of seeing it. I you can try to get other proves also.

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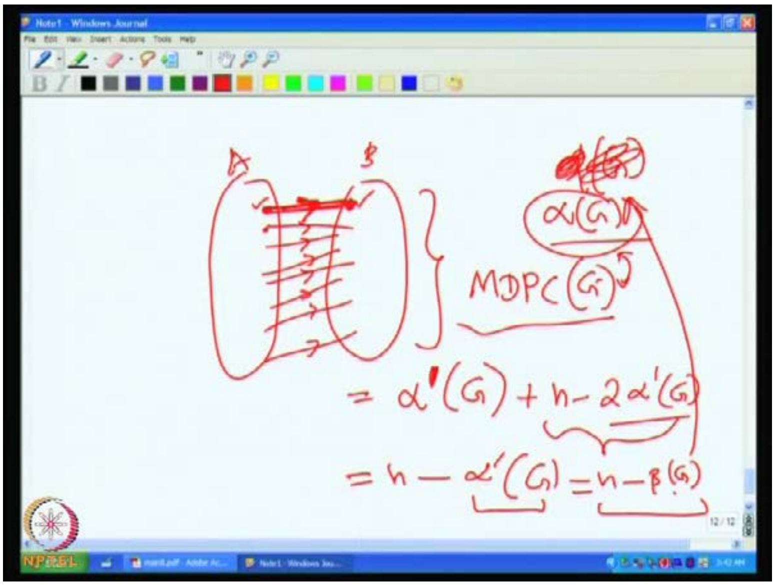

So, therefore, there is a Hamiltonian path in k n minus 1 always. Therefore, though initially it looks a little puzzling, it is always there. Therefore, in complete graph also the theorem is true. Another thing we may want to check is the bipartite graph. So, what will happen in a bipartite graph? So, if you are given a bipartite graph, these are the edges here; this is undirected bipartite graph.

Now, if I want to make the cardinality of the minimum path cover as higher as possible, the best strategy to orient the edges would be to orient all the edges from one side to the other. Why? Because you know in the bipartite graph, if I trace a path, I will first go to the other side; then I will have to come back. That is only way I can trace a path. Now, you see that every edge is oriented from the A side to the B side. So, once you reach the B side, there is no coming back to A side. So, every directed path have to be of cardinality; I mean have to contain at most two vertices; one or two vertices that is it.

So, now, if you ask for this bipartite graph, what is the cardinality of the minimum directed path cover, we will immediately be able to say that, that will be equal to the see the number of two length; that means length two; that means the number of length – two; means number of vertices in the directed path is two paths plus for the remaining vertices you have just take singleton path, but how many lengths, how many paths can be there with exactly two vertices in it?

So, see there are edges and they are independent edges directed of case from A side to the B side. But clearly without some, with some thought we can realize that it is essentially alpha dash of G; that means the biggest matching in G is to be considered; the biggest matching in G is to be considered; that is alpha dash of G. That will give you the maximum number of paths with the two vertices in it. That is only where we can take it. And for the remaining vertices, how many remaining vertices are there? Totally n are there. n minus 2 times alpha dash of G because each edges covered two of them. So, two times alpha dash already covered. The remaining uncovered things are this much. So, we have to add that.

So, this is the total cardinality of the minimum directed path cover of this graph if you orient a bipartite graph from A side to the B side. Every edge from this will be n minus alpha dash of G. And by Konig's theorem, this alpha dash of G is essentially beta of G and what is that? So, this one is essentially our alpha of G; is not it? This one is alpha of G. n minus beta of G we have already seen that alpha of G plus beta of G equal to n; then n minus beta of G is equal to alpha of G. And so, it so happens that in any bipartite graph this becomes this MPDC happens to be equal to not just less than equal to it is equal to this thing. Somehow, it is reaching the bound if it is true; this theorem Gallai Millgram theorem that, that minimum directed path cover cardinality in a directed graph is, at most its alpha is true; then in all bipartite graph that tight is their equality; is not it?

So, its reaching theorem. Therefore, therefore, this is an interesting theorem. There are several cases where it is really touching the ceiling. So, let us try to prove it now. So, as I pointed out, we have seen that in the directed case we can get a collection of independent sets. So, independent sets in G such that it has a representative. This independent set has a representative, set representative vertex in each of the paths and the minimum path cardinality, path cover Ii am going to show that same thing is true in the directed case also; not only that we have an independent set of cardinality, at least the minimum cardinality, path cover cardinality of a path cover, but also that we will get one vertex of the independent set sitting situated placed on the paths; each path on the path cover.

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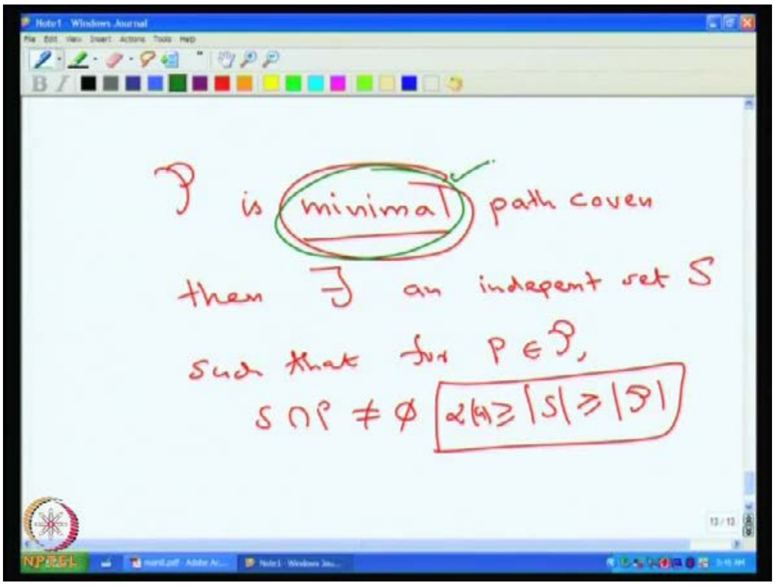

Earlier in the directed case, we could take it as the terminal vertices, the n vertices of each path and path cover. But in the undirected directed case, we may not be able to able to do that. So, I will take some internal vertices sometimes. So, let us see how we can do this thing, but we will be showing a little extra here; not just that if this p is a minimal path cover, this is what I am saying. Instead of minimum path cover, I will say, if it is a minimal path cover, then there exists an independent set s such that for each p, in p s intersection p not equal to 5. So, we have some representatives from each path of the path cover in p. Therefore, the cardinality of s will be definitely greater than or equal to cardinality of my path cover and definitely this is less than equal to alpha of G. This is what I am going to show, but then what is this minimal path cover?

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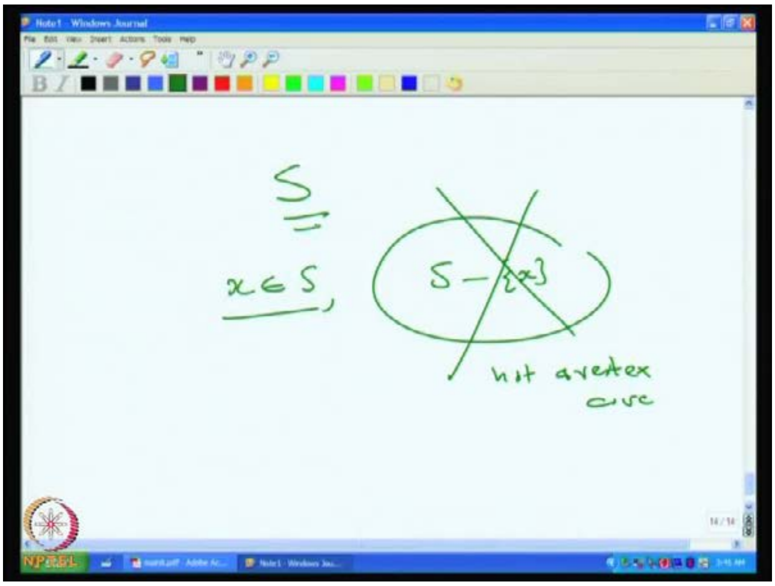

So, we have seen other instances where we use this word minimal. Minimal and maximal, for instance we talked of minimal vertex cover. So, what was the minimal vertex cover? So, it is a vertex cover s such that if you drop any vertex on s, so taken x element of s and consider s minus x, this would not be a vertex cover; not a vertex cover see irrespective of which x you take. So, that is the minimality of s; that is a minimal vertex cover. Similarly, we talked of maximal independent set in the last class.

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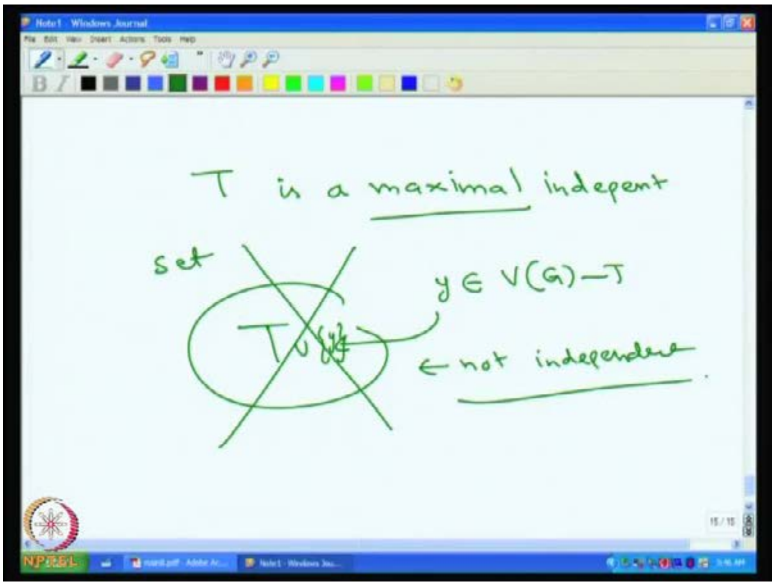

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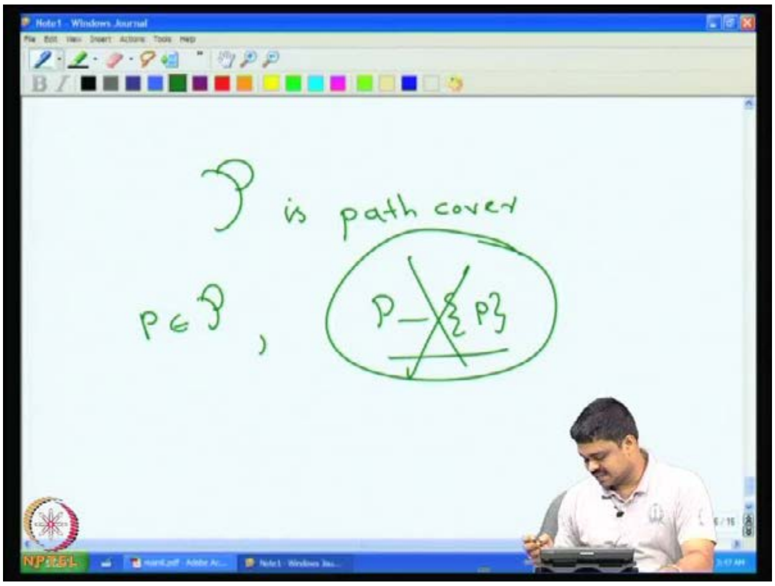

Suppose, some T is a maximal is not maximal this is maximum maximal independent set, what was the property, then you take any y element of V of G minus T outside vertex and then trying, adding it to u then T union y is not an independent set. This was the maximal independent set. So, similarly, we are familiar with this word. So, in the case of path cover, what should I mean by that? So, in the case of path cover, if I define it like this, it should be like p is a path cover and I will say, I will have to say that it is a minimal path cover.

If I remove any p, element of p is removed, then p minus this p is not a path cover is what I should say. But then, that is trivial because any path cover, if you remove one path from that, it cannot cover all the vertices because there were some vertices covered by that removed path and now the remaining thing is not a path cover. So, in that sense, every path cover will be a minimal path cover. So, that definition does not make sense. So, the minimum path cover is defined in more different way.

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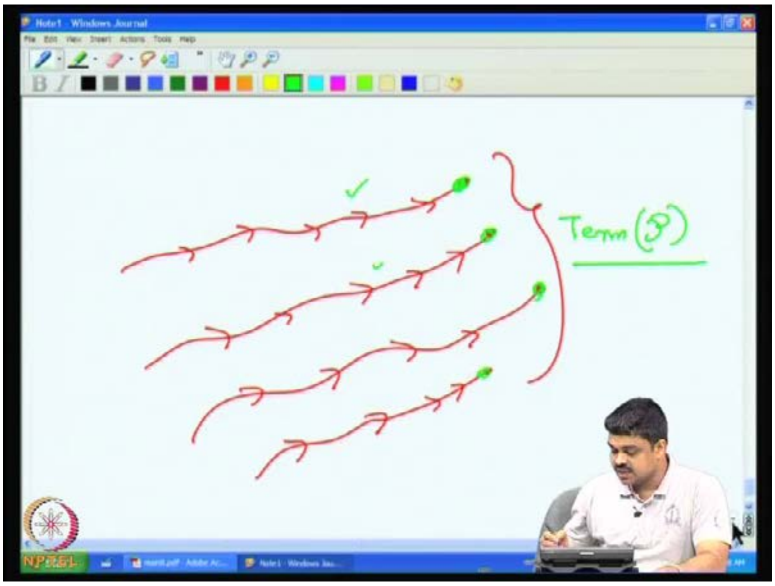

So, we rather look at suppose this is a path cover, I will just draw it like this. So, it can be in the graph. So, in however way it comes, we will just pick up the paths and then this and this, remember vertex disjoint things. So, they do not share any vertices. Now, if what is important for us is this last vertices; this is the last vertices; this vertices which are marked here. Now, these vertices are called the terminal vertices of the path cover p; terminal vertices. This is the terminal vertices of the path cover.

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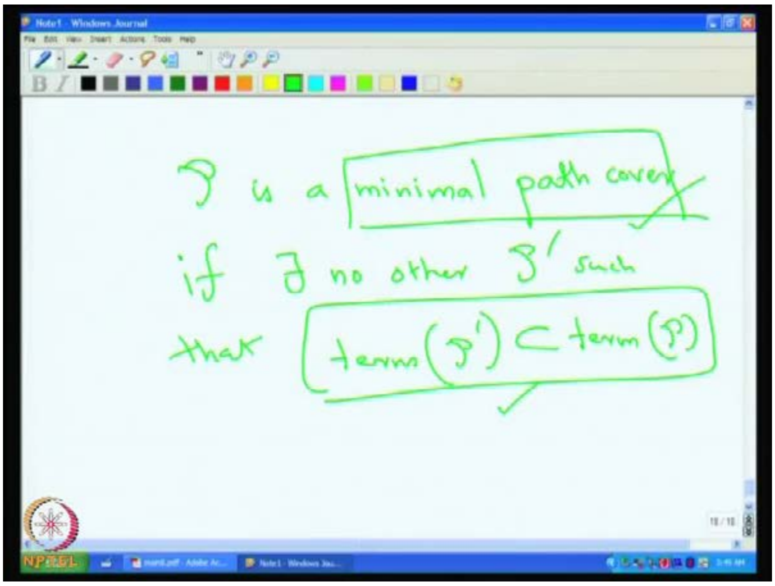

Now, you see that cardinality of the terminal vertex will be equal to the cardinality of path, path cover because each path has one terminal vertex; that is it. Therefore, the path cover cardinality is essentially the terminal vertex; cardinality of the terminal vertex. Now, p is called is a minimal path cover if there exist no other path cover p dash, such that terminal vertex of p dash is a proper subset of the terminal vertex of p. This is the condition; this is the main condition; that subset condition with respect to the terminal vertices; that means, I will say that p is a minimal path cover. If you cannot find any other path cover in G, such that the set of terminal vertices of the second is a proper subset of the original; proper for the for it to be a proper subset it has to be smaller and the the path cover itself has to be smaller in cardinality.

So, of case, the minimal path cover need not be a minimum path cover, but of case it is very clear that if it is indeed a minimum path cover, it is a smallest cardinality set. If it is a sorry if it is a minimal, it is a minimal path cover; any, if it is a minimum path cover, it has to be a minimal path cover. This is what we can clearly say.

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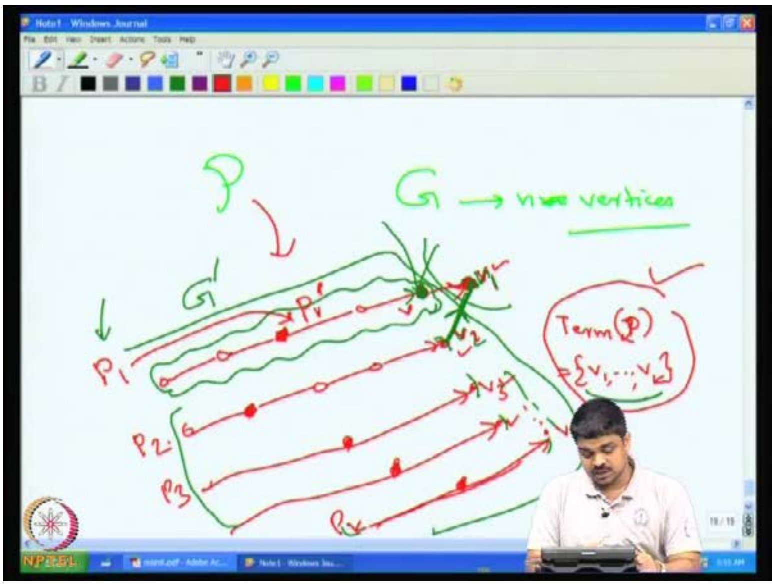

Now, let us look at a minimal path cover. So, let p be a minimal path cover. I will claim that I can get a independent set, in such a way that every path in this path cover has a representative from the independent set. So, I will do it by induction for small graph. It is very easy to verify; for instance, one node graph, two node graph, I leave it to you. To verify this, the basis for the case of n equal to 1, n equal to 2, n being the number of vertices.

Now, let us say, for n minus 1 vertex graph, it is true or more or below. Now, we consider a graph G with n vertices, n vertices. Now, this n vertex graph, let us consider a minimal path cover of this n; any minimal path cover. Let it be p; this is the p. So, let us I am drawing this like this. So, it can be anything. So, I can just draw like this without marking the vertices. Something is marked. Now, let us say, I am interested in the…this is p 1, this is p 2, this is p 3 like that; this is p k; k number of paths are there. k paths are there in this path cover suppose and I am interested in the terminal vertices v 1, v 2, v 3 of case v k. So, of case, this is not an independent set. If it is an independent set, I am done because this is terminal vertex; this term of p namely the v 1 to v k is not an independent set in G. If it is, then I am done because I have already got independent set with representative from each path in the path cover.

Now, let us say, there is one edge here in this thing. So, without also, it will in general, let us say the edge is like this; this is the edge; possible quite possible, any direction. Let us say with, we can draw this like this. So, or in other words, we draw this edge first and then name it as y 1 and v 2 and the remaining v - v 3 to v k. Now, I decide to remove this vertex on that graph. This v 1 will be cut away from the graph and consider I will consider the graph, remaining graph. This graph, let this graph, new graph this smaller graph be called G dash. In G dash, if I take this one, this one, this one, this one and this one, definitely it is a path cover of G dash. So, one question we can ask is - is it possible that this entirely this disappeared in G dash or not? This portion, for instance what if p 1 was a single node path? Initially, can p 1 be a single node path in the original path cover? So, that means its portion was not there.

So, that will never happen because in that case, we would have extended this path p 2 to include v 1. And say, now, the terminal set of this p 2 to p k, if you had looked it will become v 1 v 2 is gone. Now, v 1 v 3 up to v k; that will be a subset of the original terminal set. So, how can the original one can be minimal.

So, therefore, that will never happen. So, there will be something here. Let us call this vertex, this new terminal vertex – v; this v. So, this is maybe you can I can use this v. Now, suppose this p 1, still I can say that this new path cover is p 1 dash because this is coming from p 1. So, essentially, after cutting this p 1, I got p 1 dash. This p 1 dash p 2 p 3, p k; possibly that is needed path cover of G, but that need not be a minimal path cover.

Suppose, it is a minimal path cover, suppose it is a minimal path cover of G, if it is a minimal path cover of G, then clearly, I will get by induction assumption, I will get an independent set in G dash; minimal path cover of G dash. I mean, so I will get a independent set in G dash, such that some vertex of the independent set is present in each of the path; may be something like this. I will get some something like this; I will able to get some independent set. So, this red mark vertex will be an independent set. So, if it happens, then this p 1 dash p 2, p 3 up to p k is indeed a minimal path cover of G dash, but that need not happen. Why it need not be a minimal path cover? It may be a path cover, but need not be a minimal path cover and what will we do then? What will we do?

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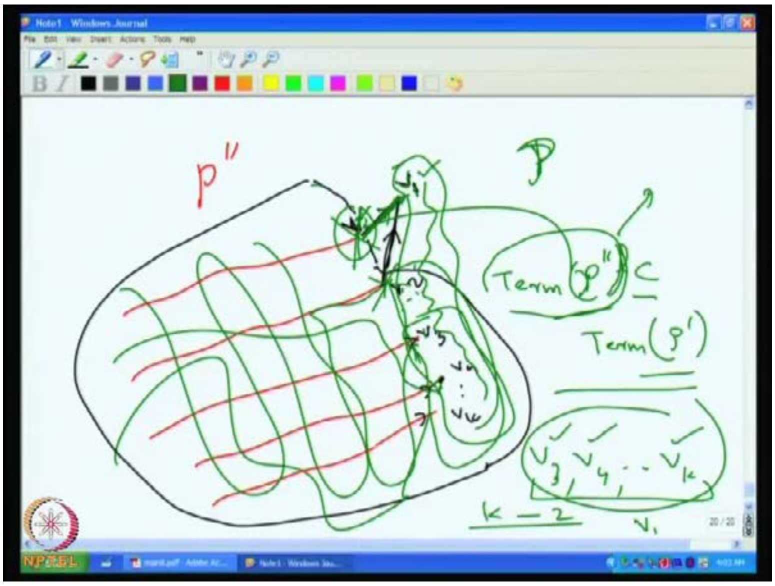

So, the idea is to somehow prove that this is indeed a minimal path cover. It is not possible that it is not a minimal path cover. So, opposite is not a minimal path cover; then there is some other path cover. So, there is some other path cover p double dash and this p double dash, let me draw this original path cover; this original picture I will draw here with color. So, remember, this was the vertex v, we cut here; there was this edge; this edge was there because originally this was this was a this was the p 1 path. This was the p 1 path and then this was the this was the edge we assume to be there from.

So, v 1, this was v 2 and this was v. For that matter, this v 3 v 4 up to v k. So, this connections are… So, this is ending; these are the terminal vertices. So, those paths and our vertex, the graph G dash was like this; remember this. This was the graph G dash. v dash was like this. Now, our this is v, sorry this is v; this is v 1.

Now, we are saying that this is not a minimal path cover. Why is it not a minimal path cover because there is another path cover. So, let us draw it like this. So, there is some other path cover; something like this, such that it somehow ends in this terminal vertices Only. Somehow it will always end in these terminal vertices only. So, I am just drawing in some different way. So, this will take the same vertices, but in some different way they will, but then, they are terminal vertex of this new path cover; terminal vertex of this new path cover p double dash has to be a subset or proper subset of the terminal vertex of p dash. This is the problem.

Now, we will our analysis will be for this green path cover; that means p double dash. Its terminal vertex set can contain this vertex v. It should be from So, it should be from here. So, suppose it contains v, this one contains v, then what will I do? I would, I would simply add. I have this edge; add v 1 to that; I will extend that path; include v 1 also and that p dash can be extended to become a path cover of original graph G. And now, see the p dash terminal vertices are this and probably this one and this one is now gone and this is added. So, all other terminal vertices are from this collection only and then the essentially the terminal vertex, this extended path cover will be from this and naturally there was because it is a proper set originally, something should be missing. Also, the original path cover p was not a minimal path cover for G; that is why the original graph G, its path cover itself was not minimal.

In that case, can I repeat if our p double dash the supposed the path cover whose terminal set is a proper set of the proper subset of the terminal set of p dash, if the terminal set of p double dash contains its vertex v. In that case, I can extend the path of p double dash which ends in v; it include v one also because there is directed edge from v 2 v 1 and now we get a path cover for the original graph G, which is essentially that p double dash with the simple extension of one path and naturally the number of paths here strictly less than the number of paths in the original one. As we can see, this is a proper subset here. So, naturally and also the terminal set vertices of that path cover is a proper subset or the terminal vertex by the original one, namely v 1 v 2 v 3 v 4 because the only one which is outside was v, but that now, instead of v, we have a v 1 as a terminal vertex. So, naturally, the minimality of the original path cover p is violated. So, it so happens that, we cannot be there in the terminal set of p double dash. Now, we ask – then, if v is not there, can v 2 be there? Same argument, if v 2 is there, that particular path which ends in v 2 can be extended to include v 1 also, and then we will get a path cover of the original graph G.

Now, you can see the terminal set has to be again has subset of the original v 1 v 2 v 3 v 4; that means, the terminal set of p. And how can the original path cover p be a minimal path cover? The same contradiction comes. So, v 2 also cannot be a vertex in the terminal set of p dash. Now, this and this v and v 2 are not a part of p double dash. Now, essentially, the terminal set of p dash can only take from v 3 v 4 up to v k which means the cardinality of p double dash is only k minus 2 or less because only you see only at most k minus 2 terminal vertices there. So, it cannot be the cardinality of the path cover itself cannot be more than k minus 2.

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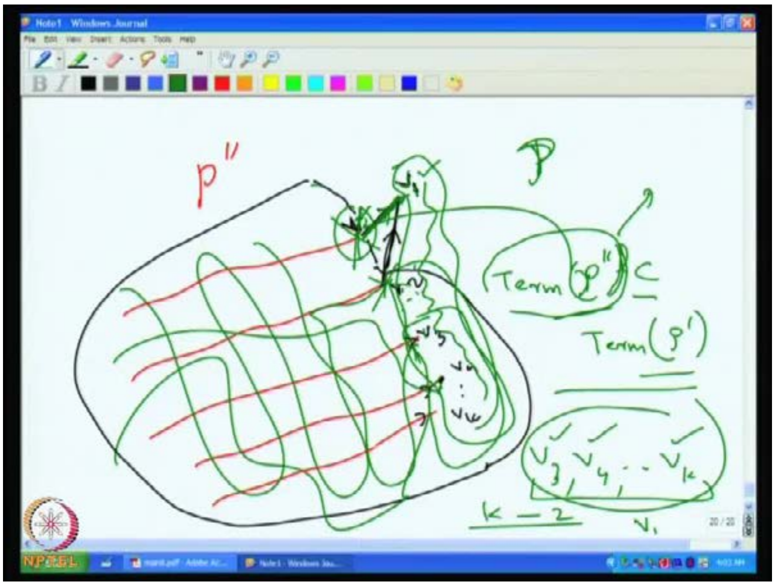

So, now, what I would do is I could simply add v 1 as a simpleton path along with this path cover p double dash and that will make a path cover for the original graph G. Now, we get a path cover, new path cover for the original graph where the terminal vertices are this, up to this plus v 1. This is definitely a subset of the original terminal set v 1 v 2 v 3 v 4. v 2 is missing there and v something also is missing. So, it is indeed a subset. So, we have a smaller path cover than p with the terminal set of it in a proper subset of the original one which is the contradiction. So, because the original path cover p was supposed to be a minimal path cover. So, we got a contradiction. So, it is not possible that our p dash is not a minimal path cover. It has to be a minimal path cover though. So, if it is a minimal path cover, by induction hypothesis we know that there is a independent set in G dash itself with one representative vertex in each of the path of the path cover.

So, we got the necessary independent set and that same independent set will work for the G. Therefore, we complete the proof. So, this is why the Gallai Millgram theorem is true. Essentially, we can say that any if you take the cardinality of the minimum directed path cover in a directed graph, that has to be less than equal to the alpha of G; independent number of G.

So, we conclude the Gallai Millgram theorem with that. Now, we will just explain, quickly explain an application of Gallai Millgram theorem to the partially ordered sets. So, you have learnt what relations are and you know what equivalents relations are the so called symmetric transitive and reflexive relations. Similarly, there is this partial orders which are reflexive and transitive and anti-symmetric. That means, if a and b are related b a are not related. So, if a is related to b, b will not be a related to a, unless a equal to b.

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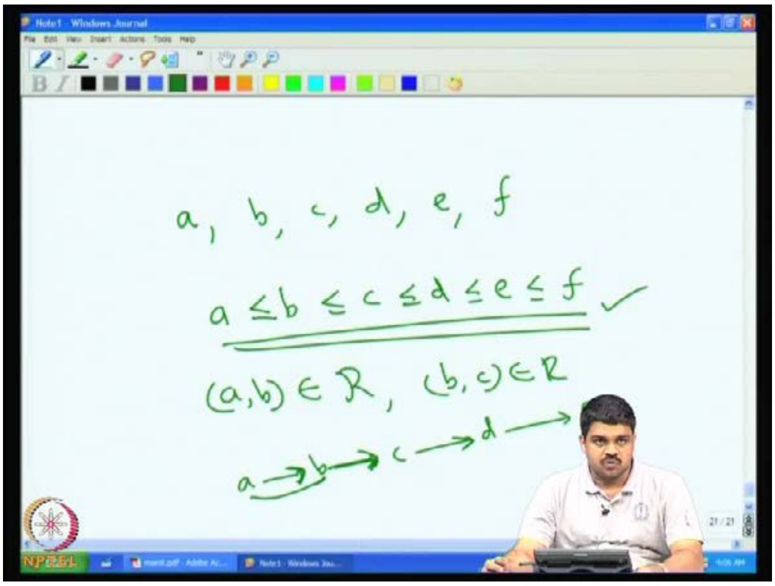

So, this partial orders are… So, an interesting topic and see suppose look at this question. Suppose, we are interested in covering all the vertices of a partial ordering some way; so, let us say, we consider a sequence of members of the partial order a, b, c, d, e, f like that, such that a is related to b and b is related to c, c is related to d d is related to e, e is related to f like that or may be a b element in the relation and then b c element of the relation like that, a sequence.

So, in a directed graph notation, we will have a to b b to c edges because a b is represent the a b pair is represented by an edge here; the directed edge here. So, b c pair is represented by a directed from b to c; c d pair is represented from a directed edge from c to d and so on. So, this kind of a sequence is called a chain. So, essentially, it is a total order and partial subset, the partial order where the induced relation gives, say, a total order. Total order means every two elements are comparable; comparable or they are related either one of… So, if a and b are in the set, then either a less than equal to b or b less than equal to a; so, a is related to b or b is related to not both. So, should this such kind of subset is called a chain; a total order.

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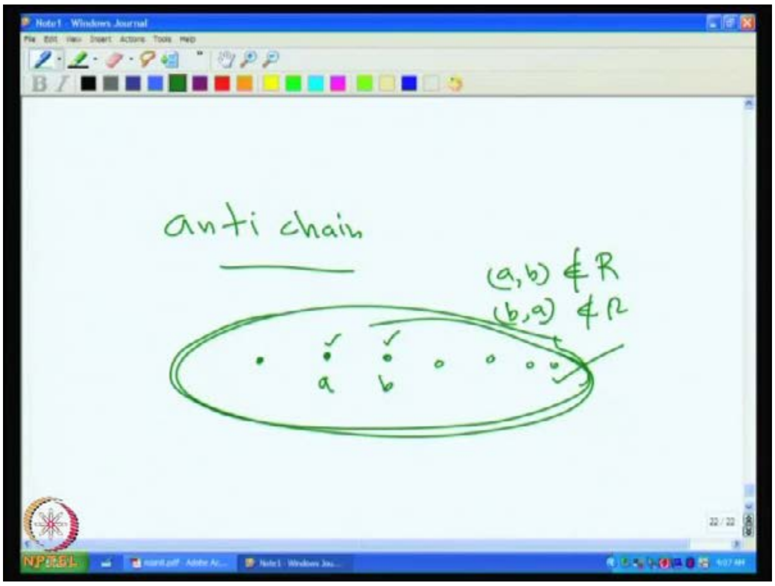

Now, similarly, that is something called an anti-chain. So, anti-chain is a subset of the partial order where there is node 1, node 2 are related with each other. If I take a and b, neither a related to b in r b related to a neither r. So, get the maximum biggest such collection that is maximum anti chaining; its equivalent independent set in our this thing. So, for instance, one way of easily visualizing this is to draw the partial order.

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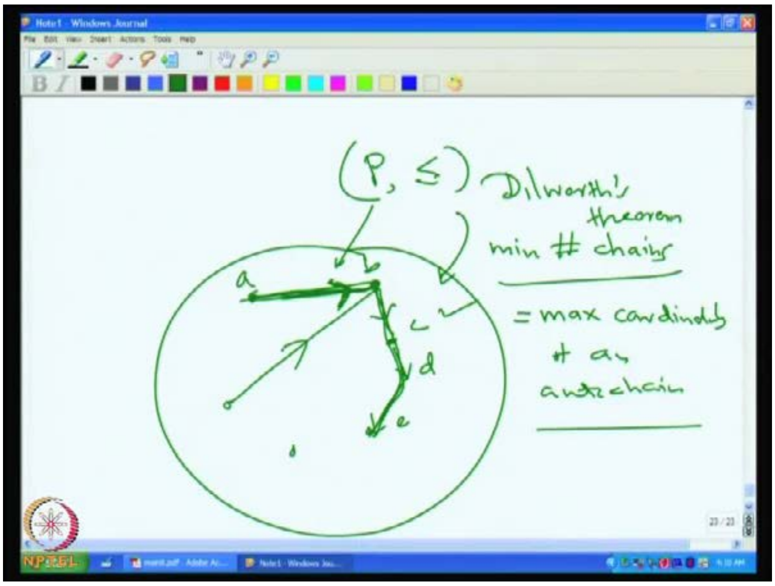

See, partial order has a base element set; set of elements say p and then there is a relation which is usually put like this. So, the things, the elements of p can be seen as the vertices. So, vertices and then the relation if at all toward the elements are related, I can put an edge between them and the way. Depending on the pair, I can put on the edge. There you will never see the edge. There will only be a conflict; only one direction you will have because of the anti-symmetric. If a is related to b, I will put this thing; we would not be (()).

If a b is a pair in the relation then b a would not be a pair in the relation. Therefore, we can draw a directed graph such that each edge gets exactly, between any pair, there is only one direction. So, there is one direction. So, we will get a directed graph from the partial order that way. So, we may if anybody is not understanding what this partial order thing is, I will advise you to just refresh your twelfth standard material on partial order. So, it is not very difficult to pick up that. We do not need that much either.

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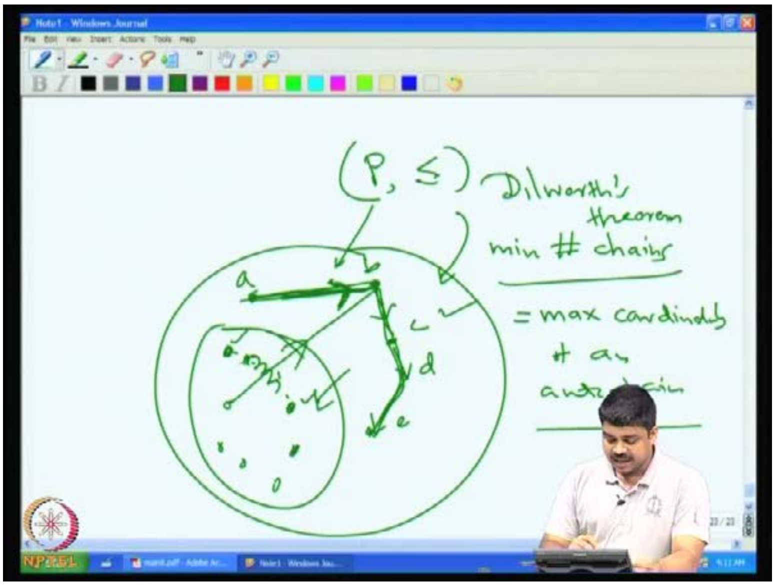

So, what we are interested in now is this question; I want to cover the elements of the partial order with the minimum number of chains. How will a chain look like here? So, the chain will look like, you know, a; say to a is related to b, b is related to c, c is related to d, d is related to e, like that; I will get a directed path starting from a to b to c to d to e to like that; so, I want to cover the elements of the partial order with the minimum number of chains. Then, there is a famous theorem that states this minimum number of chains to cover a partial order actually equal to the maximum cardinality of an anti chain in the partial order; cardinality of an anti-chain in the partial order; this is called Dilworth's theorem.

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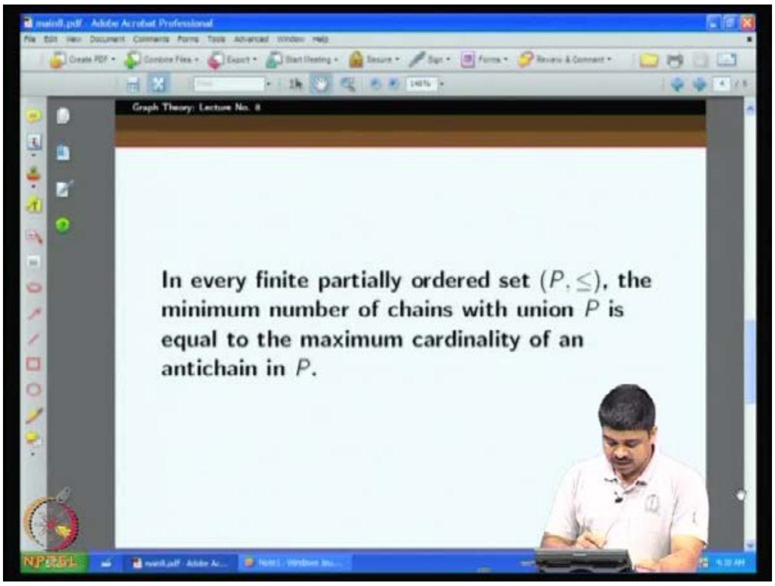

Dilworth's theorem - may be you can look at this. See every finite partially ordered set, the minimum number of chains with union p that means the union of the members of the chain elements of this chain is equal to p that is equal to the maximum cardinality of an anti-chain in p, the biggest.

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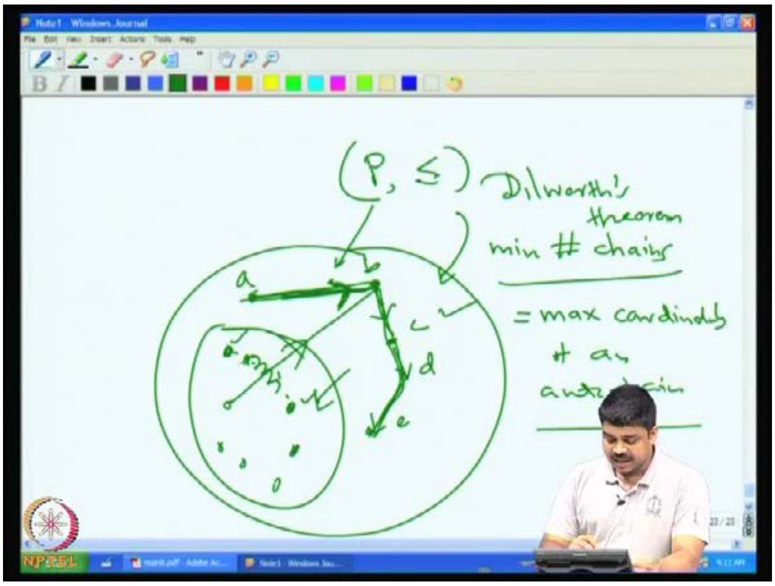

And that is, we can easily see the maximum anti-chain or the any anti-chain for that matter is going to be an independent set because for any pair, we are not supposed to have either of these directions; that means should not be any edge in that, any edge. So, that means anti-chain correspond to an independent set in the corresponding directed graph. So, what we see is because no two vertex in an anti-chain can be in the same chain. So, we definitely need at least as many chains as the number of element in the anti-chain.

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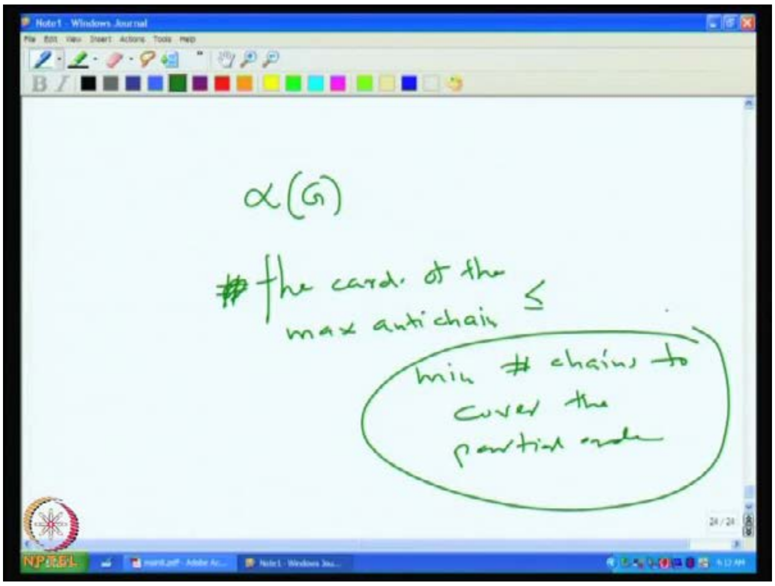

So, this is equivalent to saying that see the number of chains number of those are the cardinality of the anti-chain, biggest anti-chain. The cardinality of the maximum anti chain should be a lower bound for the minimum number of chains required to cover the partial order; cover the elements of curve p, but the other way, that means actually this minimum number of chains to cover the partial order is the minimum directed path cover in the corresponding directed graph; is not it? Because if you take a chain, it is a directed path. You are covering all the things and they are disjoint. Therefore, it is indeed a minimum directed path cover.

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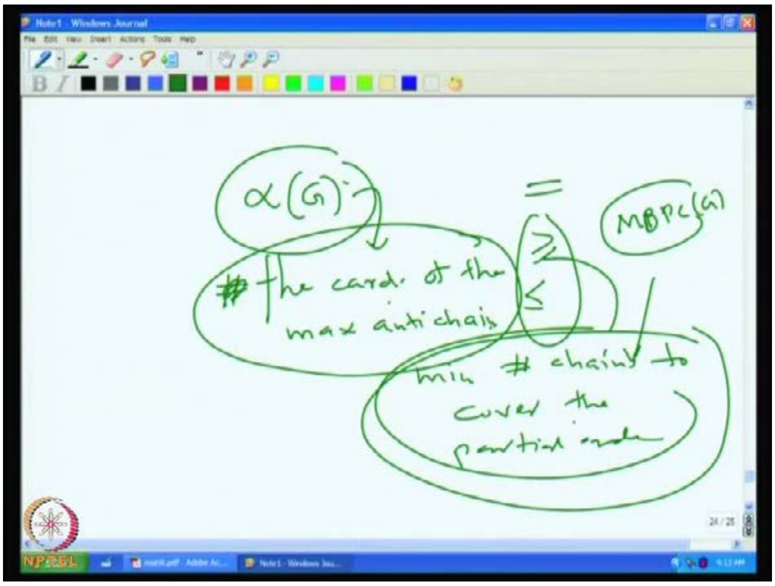

So, we know by Gallai Millgram theorem that the number of directed minimum directed path cover cardinality is less than equal to the alpha. So, essentially this will be… So, I can say less than equal to; therefore, this will become equality. The cardinality of the maximum anti-chain is alpha remember right in with respect to the graph and this will become MDPC of G. You know that this has to be… this has to be greater than this (Refer Slide time: 55:18); then, this is less than this and this follows immediately from the definition because any anti-chain, two vertices and the anti-chain two elements in anti-chain cannot be the part of the same chain.

So, therefore, we can understand that both is in equality. This is less than equal to this and this is less than equal to this. From that, we can infer that both are same; the Dilworth's theorem says that the number of minimum number of chains required to cover the partial order is equal to the maximum cardinality of anti-chain in the partial order. So, it is now, that is Dilworth's theorem.

So, in this class what we have done is the proof of Gallai Millgram theorem, essentially it says the cardinality of the minimum directed path cover is less than equal to the cardinality of the biggest independent set in it. And then, we used this to show Dilworth's theorem that this will immediately imply one side of the Dilworth's theorem the other side is trivial. Therefore, there it becomes equality because the partial order will give us a kind of graph, directed graph. Corresponding to the elements of the partial order, we have the vertices and corresponding to the directed edges, we have… so essentially corresponding to the pairs in the relation, we have the directed edges. This graph, in this graph the chains correspond to directed paths. So, minimum number of chains to cover is essentially the minimum directed path cover and anti-chains maximum anti-chains is equal to alpha. So, that becomes equal in this particular type of directed graph. So, we will in the next class, we will mention a little bit about the kind of undirected graph that we will obtain, if you remove the direction from this kind of graphs. So, they are called comparability graphs; that I will briefly mention in the next lecture.

Thank you.