Vehicle Dynamics

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Lecture - 18
Lateral Dynamics-Bicycle Model

In the last class we were looking at governing the equation for dynamics and we had in fact derived a very general expression for body where the coordinates are centered at its say CG location for example and that we said this is the body-centered coordinates. We looked at the inertial frame of reference and we also found out that there are some additional terms that are required especially when the body-centered coordinates.

In other words, which is attached to the body of say for example vehicle and if it now starts moving, the vehicles start moving as well its rotating and so on. The body-centered coordinates also rotates along with it and hence certain changes are needed okay some additional terms are required in order that we take into account these motion of the body and this is what we said.

And we derived a very general three dimensional expression for this body-centered coordinate. You would have studied this in dynamics anyway what I am going to do is that some of the things which you would have studied anyway we will go back and just refresh it a bit so that you will know that I am going to apply what you have studied before. I am going to do that for example you would done a course in control systems. I am going to do that again here in today's lecture.

So ultimately as usually the case in engineering you start with a grand equation and then slowly say that this I will neglect that I will neglect this I will neglect that I will neglect so that you know it ultimately becomes so to say handle-able. So that you do not lose the Physics at the same time the equations are simple. This is the usual fact of life in engineering. So we will write that way for what is called as the bicycle model.

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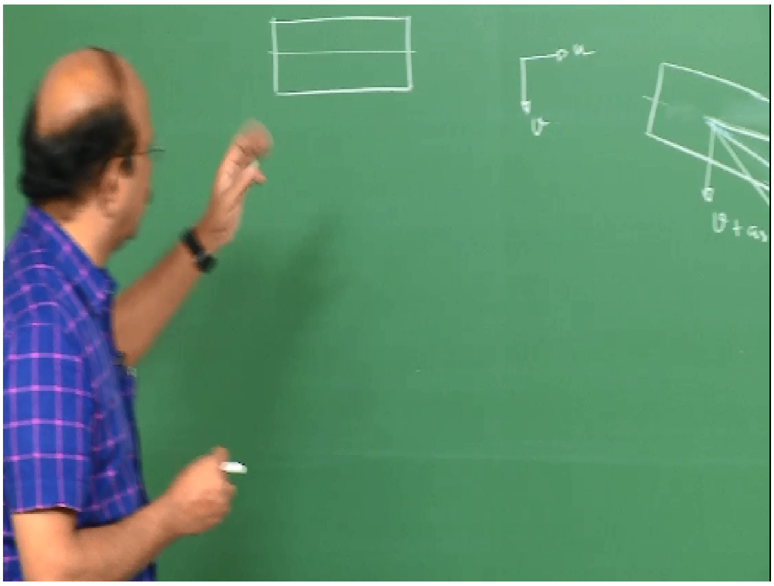

So that is the bicycle model which we had just started. So what I told please remember what I said in the last class that I am going to collapse the 2 wheels into 1 both in the front and the rear. Hence it looks like a bicycle so it is called a bicycle model nothing to do with actual bicycle, stability of bicycle and so on. It is just called a bicycle model because it has 2 wheels collapse the 2 wheels into 1 wheel.

The result of collapsing this is to reduce the degrees of freedom.

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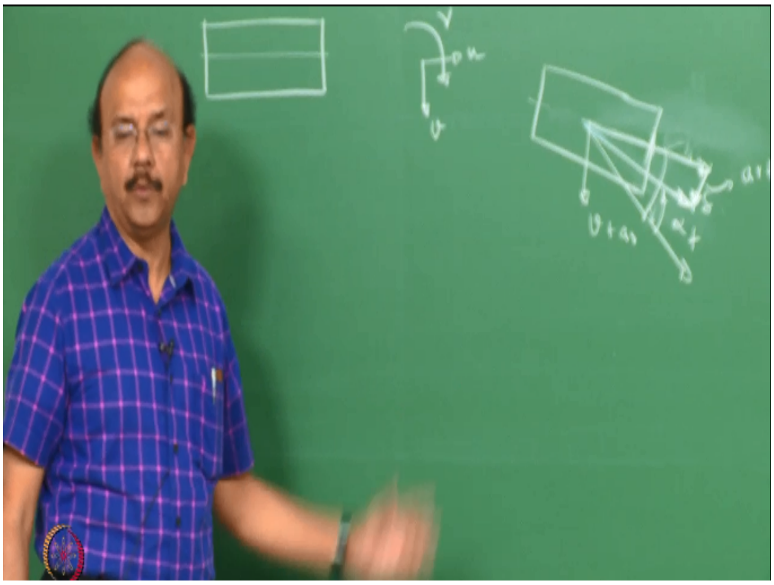

So I am going to look at this with or in other words the equations which I am going to write down with consists of v and r where r is the yaw rate. So we already said that R is going to be the yaw rate. We are going to look at a system where U is a constant. In other words, steady state U=constant U we are not going to vary U. I am going to write down all the equations in terms of v and r. In other words, my differential equation which I am going to reduce that huge guy 6 equations into simple 2 equations.

So I am going to write down that equation into Izr dot what is Izr dot? It is= tau. It is the yaw which is perpendicular to the plane of the board. So for that I need to put down the forces. So that is the force F alpha R. Why alpha R because as I told you before alpha R is the slip angle and f alpha r is the result of this alpha r. In the same fashion we will put down a force that is acting in the front and we will call this as f alpha f.

This angles are very small so I will make lot of again assumptions there. The whole idea most of the time is to bring down the equation to a linear range. The whole question in engineering is whether assumption is value or you are really operating the linear range and so on. Sometimes you would not be and that is where your judgment matters. Let us finish this expressions and then we will come back to we will take the questions.

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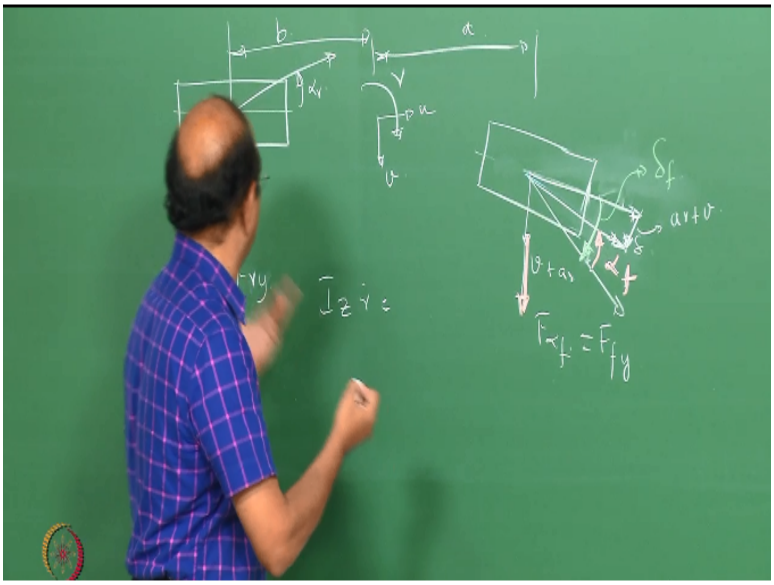

Okay now let us look at the front. So this is the alpha R this is the f alpha r. Let us look at the front. The front you have I hope this is not a very confusing picture maybe I will use a chock piece. So I have first look at that. So that is what is the steering angle I have given. So that green one is the steering angle. This is not I am not steering the rear wheel it is possible to do that. There are lot of interesting things that happen when we can steer the rear wheel as well.

For example if you have huge vehicles for examples vehicles which are used by space agencies, you know, where they want to transport the rockets which are to be launched. So you would see that there are 32 axles, 64 axles huge ones where you cannot obviously drive them with only 1 driven axle or steered axle only. So it has to be more than one, you know, the problem becomes very interesting. Well let us not get into all that.

We are only looking at now the front axle steered vehicle. So that is the alpha r and the delta F rather or delta and that since I have developed a force note that force requirement that I develop a slip angle which I call as alpha F. So that is the slip angle. So with the result that I kept the force which is F alpha f. So let us we will keep that as alpha F and alpha r if you want front Y and that will be easier this is F front in the Y direction and F rear in the Y direction.

Either way you can call this as alpha r or does not matter as long as you understand what we are talking about. No this is the turning please note that please understand the coordinate system we are going to take a turn. This is some sort of a plan view. Okay you are looking at X and Y and that is X, Y and the Z is into the ground that is a coordinate. So you are looking at the plan view of the vehicle. So obviously Fy is the centripetal force.

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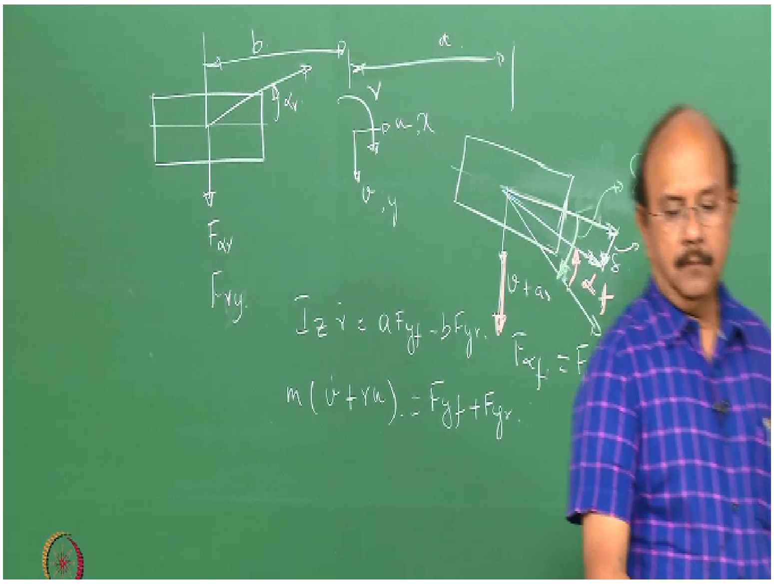

So this is=moment. So what is that A*Fy front or Fyf let us call that. It is easy to call that as Fyf because many books the problem why I am writing like this is because we follow textbooks then you would have given different terminologies you would understand that clearly. Fyf-b*Fyr. Okay Y in the rear direction. So what is the second expression. It is go back and look at what we had done m v dot+ru.

So it is not mv dot because of the body-centered coordinate mv dot+ru and that is=Fyf+fyr. Okay so these are the two expressions we have. Now I am going to use again the dynamics along with geometry in order to write down deltas and so on.

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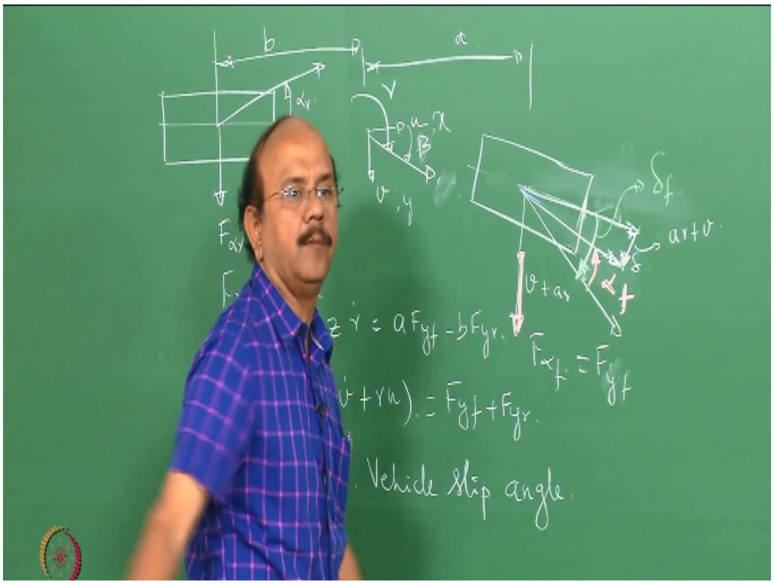

Now let me define a new term I hope you will not confuse with that which I would call as beta. Obviously there is a V there is a U and hence vehicles actually moves along this angle beta that is the angle beta and beta is called as the vehicle slip angle. We will call beta as the vehicle slip angle.

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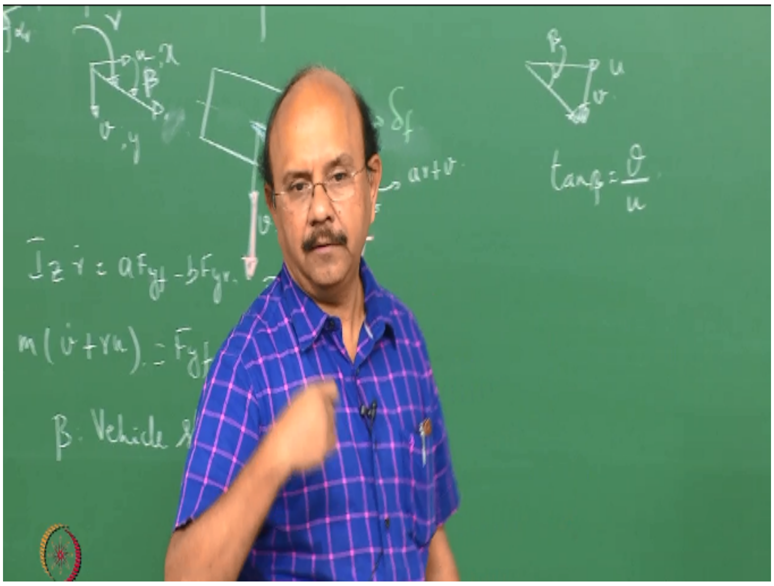

Obviously from this diagram okay so you have U you have that angle beta and so that is V. So tan beta=v/u. Now let us look at alpha r. Now I have a r. R is now what is r we said that it is the yaw rate. So if I have to write down the velocity at the rear. This is the velocity V and

U, r at the CG location. Hence if I have to write down the velocity at the rear what would be the velocity there would be look at this there would be a velocity which is due to r which is b*r that will be I will just put that here and then transfer it there.

So that will be in this direction b*r. There will be a velocity which is v in this direction that will be a V in this direction. So b*r is due to r and V is the velocity so when you transfer the velocity it becomes v and br.

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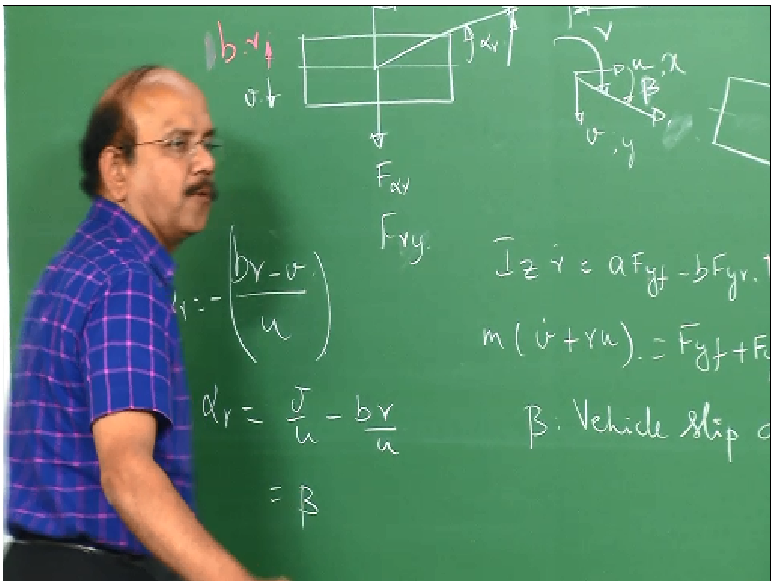

Since it is in the opposite direction I can write that as so in this direction I can write that as br-v. Okay br-v and tan alpha r because that is in the opposite direction to our coordinate system that becomes divided by U. So that is very simple look at that clear. Yeah of course longitudinal velocity U is constant. We are not touching U so that is the same because the vehicle is moving in the same direction with a constant velocity U.

Please note this in other words when I have to look at U the variation of U I am looking at longitudinal dynamics. The first equation in what we did in the last class we had what is that we had done we have looked at the first equation at the last class. So we had looked at the longitudinal dynamics. Now I am going to remove the longitudinal dynamics. So I am saying now that I am going to only look at the lateral dynamics.

So that is what we are going to do now. **"Professor - student conversation starts"** Sir here ply steer conicity. Yeah right we will come to that. We have not looked at tire yet I am going to have a very simple tire model. Ply steer conicity all those things will come. Everything we will have to then put the model just wait for some time. Let us finish this just hold your questions let us go forward many of your questions will be answered as we go forward. **"Professor - student conversation ends"**.

I am going to make an assumption that tan alpha r is approximately=alpha r. So alpha r=v/u-b*r/u. I am just going to so what is v/u we had already seen that v/u is beta. Tan beta which is again approximately=beta so that is =beta. So let us assume that this vehicle is taking a turn in the turning radius=R. So R is the let say that R is the turning the radius. I will finish this here and write it down there.

So now how do I write the second expression. R is like omega U is the tangential velocity so that is the R omega R R*R=U. So R/U so that is=-b/R.

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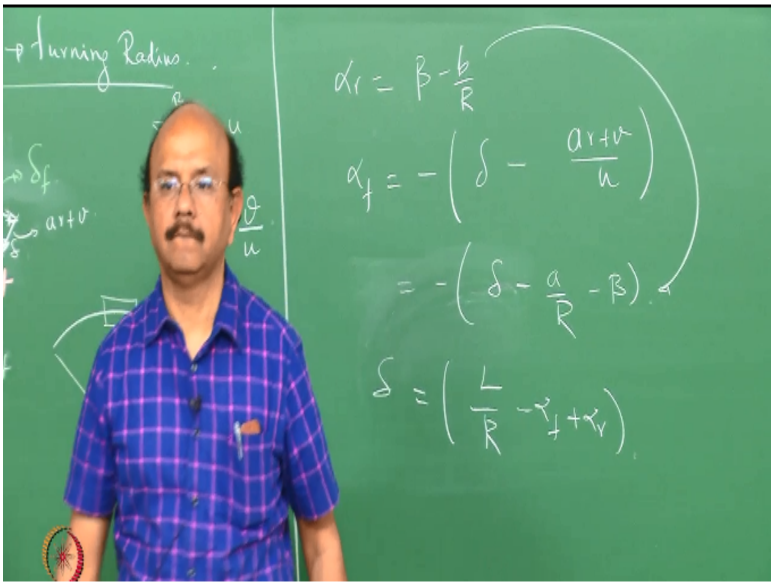

So the first expression of interest to us I will write down here I will do all these things. So alpha r=beta-b/R that is my first expression. Let us now look at alpha f is there any questions right similar way I will look at alpha f. Please understand that just wait for sometimes. Let us finish this expression okay. We will take the questions towards the end just hold your questions we will do that later.

So just say that this is an r. R is positive is right hand side so inside so you will see that this is the tangential velocity here due to R is actually in this direction. So b*r is in that direction that is all we had said. I mean closely watch that is why I had put 1 br like that and v like this clear. Now I have to write alpha f. So what is alpha f look at that this whole thing is delta okay and that is alpha f. So alpha f is=delta I am just giving a small gap there.

So here that is Br this is now Ar right A*R that is in the same direction as that of v. So you will get delta-that is the ar+v/u. Now note that alpha f is in a direction which is in a negative direction of z so I am going to put a –here. So I have to be careful because note that a negative direction alpha f produces a positive Fy. So that is why I put a negative so that I am clear so I do not have confusion.

So in many books you would see that it is alpha f and when I develop Ffy or Fy it is just multiplied by C alpha multiplied by alpha. There is a small confusion there is nothing wrong with this, but you have to be very careful on the way you are writing it. So I am going to write that you will see that I will put two – so that I will take care of that clearly the signs are clear.

Now simplify that let us say that I am going to simplify that expression -delta-ar/u. So that is=-a/R –beta. So that delta now can be written in terms of alpha bring that delta the other side. And then rewrite that expression and substitute my this expression for beta, substitute that into this expression and you rewrite that expression. So in other words I can say that when I take it delta the other side do you have any questions.

Note the difference between capital R and small r. **"Professor - student conversation starts"** I know that but what can I signify in this model like capital R. Capital R is the radius of this turning, you know the turning radius. Actually let me give a bigger picture you are taking a turn. **"Professor - student conversation ends"**. So your vehicle is here and that radius of your turn=capital R.

So rearranging the terms this becomes L/R because A+B=L-alpha f+alpha r clear okay.

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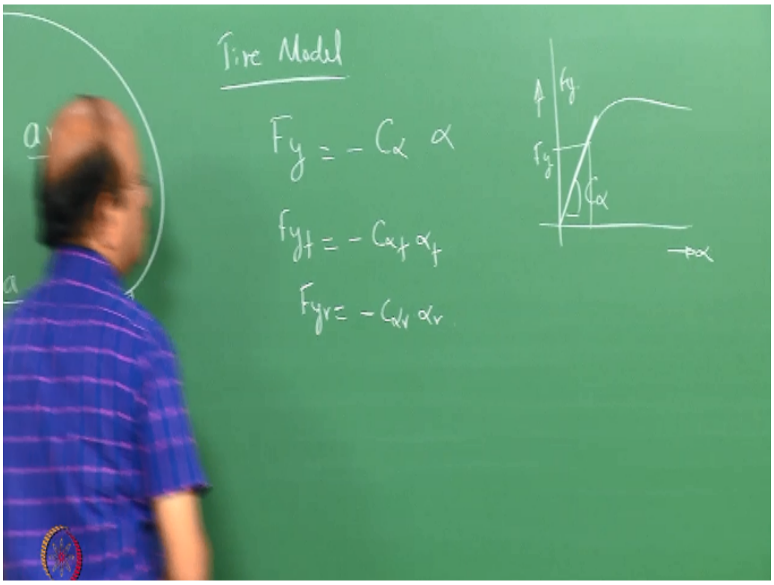

Now we will come to the tire model. There are number of as I told you there are number of tire models available. We will go only for a simple model because I want to capture the physics of maneuvering so that I will write down Fy=-C alpha*alpha. So why I am writing this as – because I know that a negative alpha produces negative alpha produces a positive Fy okay and negative alpha produces a positive Fy.

So in other words when I put alpha to be negative actually becomes Fy becomes positive so that is why we put a -. There are books which treat this to be negative which is not correct okay C alpha stiffness is not correct. So this is a simple model. What is this model? This is a linear model. So in other words we saw already that there is a region that suppose I plot alpha versus Fy. We saw that the curve look something like this.

And I am going to take a linear model where that is the slope is what I call a C alpha. So C alpha *alpha gives me Fy. So this is a simple linear model. From here when I put Fy front I want I put C alpha f *alpha f and Fyr I put –C alpha r*alpha r.

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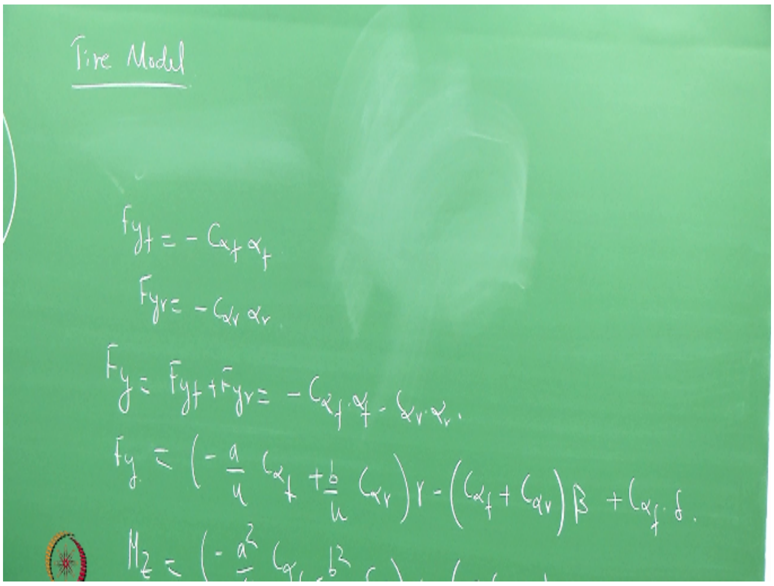

So the Fy now the total Fy note that this is a general expression. I am just going to remove it so that you do not get confused. So now Fy=Fyf+Fyr and that is=-C alpha f* alpha f –c alpha r *alpha r. I will give you a small job write down alpha f and alpha r. We had already derived it if I think it is there in that expression. So just put that so –is that so just substitute it –of that. So I am going to write down completely the final expression.

You can look at the final expression that is=-a/u c alpha f+b/u C alpha r*r-C alpha f+C alpha r*beta +C alpha f*delta. So that is going to be my Fy. The second is –A*C alpha f*alpha f okay +b*c alpha r *alpha r substitute that so that Mz =-A square/ U C alpha f –B square/u C alpha r. So substituting that you get these 2 expressions. Now what is the next step? I have Fy and Mz go and substitute it back into my expression or my governing equation.

So substitute that into governing equation so I get these are the 2 equations. Now what I have essentially done is to write down the right hand side. So write down the right hand side I have written down the right hand side and so I have now okay substitute that back and write down the complete equation the left hand side and the right hand side. Now what the 2 variables or two degrees of freedom here R not U, R and V. Which one b/R right.

No, no I am put R outside. I am sorry. I have put R outside so that U here so U is fine R is outside here. R is outside and that is fine. Now I am going to rewrite this expression substitute that and rewrite this expression in a very familiar form. So those 2 equations you know. So those are the 2 expressions I am doing. So that is very simple. The only thing is we have to be careful in that algebra.

So the first expression now becomes I do not want to rewrite it. Iz r dot=Mz that is the one and that is the second expression. Now I am going to write this down.

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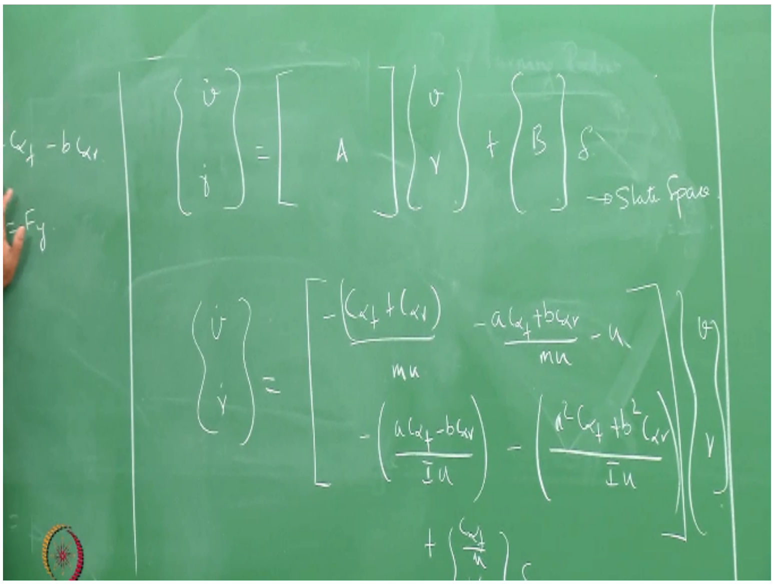

I leave a couple of steps to you to do it this is nothing in a form which is familiar to you. I am going to write this as V what are the things that are available there or my expression. So V dot okay that is the second expression I have and R dot those are the 2 things. I am going to write down that as A readjust those terms vr+b*delta. Essentially what is that I have done I have just put that rearrange that terms. Clear any questions nothing, no big calculations.

If I now put that form you know when I rearrange it I am going to write that final form v dot r dot=-C alpha f+c alpha r/mu. It is good to have that expression this we had Iz r dot=ac alpha f-bc alpha r this one we write and m*v dot+ru=Fy. So r dot and v dot you know that is what we are writing there-ac alpha f+bc alpha r/mu-u that is the second term and –ac alpha f-bc alpha r/ Iu-A square C alpha f+b square C alpha r/Iu that would be my A and vr+c alpha f/m ac alpha f/I delta.

Okay any questions on this? I will skip 2, 3 simple lines which I think you can fill up. Now this is the familiar form to you. This is a very familiar form to you. What is this form what is this called as state space form. So this is nothing, but a state space form. What are the state variables here V and R. The whole idea of writing this in a state space form is to look at the stability of the system that is the first idea.

Later that we will use this in order to look at the response of the vehicle to dynamics inputs which is the steering input. So the first job here is to look at stability of the system. Now I will take question any questions. The only thing I have to be careful is C alpha F C alpha R - +whatever you want to use it is fine as long as ultimately Fy is in the positive direction. So that you have to be that the only thing you have to be careful whatever you do ultimately you will get that expression only. So –to-+and all that we would get only that expression.

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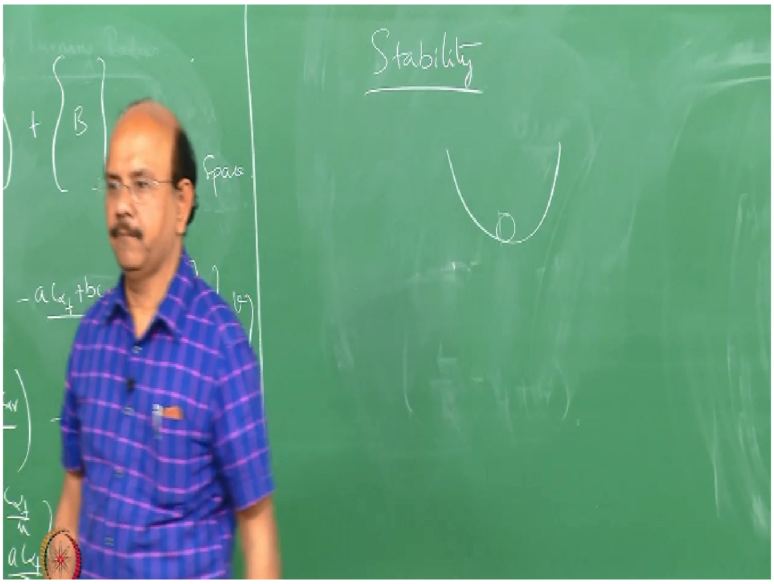

So now we are going to look at the stability of the system. **"Professor - student conversation starts"** U is also input right. Yes, U is a particular velocity it does not change, but it is an input. It does not change we are not going to calculate it right. Please understand that this is a lateral dynamics. **"Professor - student conversation ends"**. So what do you mean by stability of the system? How do we determine the stability of the system?

The immediate thing that comes to your mind when we talk about the stability of the system is an Eigenvalues problem. So you had studied that in your control system course and that there is no input the right hand is=0 and then U the left hand side which is the governing equation. Right hand side which is the F is put to 0 you go back and solve for the Eigenvalues and so on and so forth this is what you have been doing.

Here also you have to understand this carefully. Now we are going to do exactly the same thing. We are going to put the inputs to the system to be=0. In other words, I am going to put delta to be equal to 0. This is the input to the system =0. Now there is the confusion. If I put delta=0 which means that actually I go straight. Now what is that I am doing when I go straight and I am putting delta=0 and we are talking about stability in the lateral dynamics then what does it mean that is the question.

So for that you have to understand what we mean by stability. This is the usual thing because we are going to apply the concept of stability. So we need to understand what is stability. Even if you are going straight we say that the system is stable in simple terms when there is a perturbation to the system. The perturbation can be due to some suddenly wind that is blowing or due to porthole you take a small turn and so on.

You know a perturbation to the system you give the system returns back to its equilibrium position. What is in equilibrium position? The system continuous to be in that position unless it is forced to move out due to an input. So a small perturbation which you give it gets back to its equilibrium position so that is why you put a very high very high school example of putting a ball in a well and when you disturb it. It comes back to this position because you are actually perturbing the ball about an equilibrium position and then it gets back to that.

So that is the philosophy here. In other words, you cannot make a statement which I have seen many people making. When I do not take a turn in a vehicle why am I worried about stability of the system. No it is not true even if you go straight a small perturbation is enough for a system to become unstable. So keep this in mind because this is going to be very useful for results later.

So we will get into this step what is meant by stability, how do we calculate stability and so on maybe you have done this before, but we will do that again quickly.

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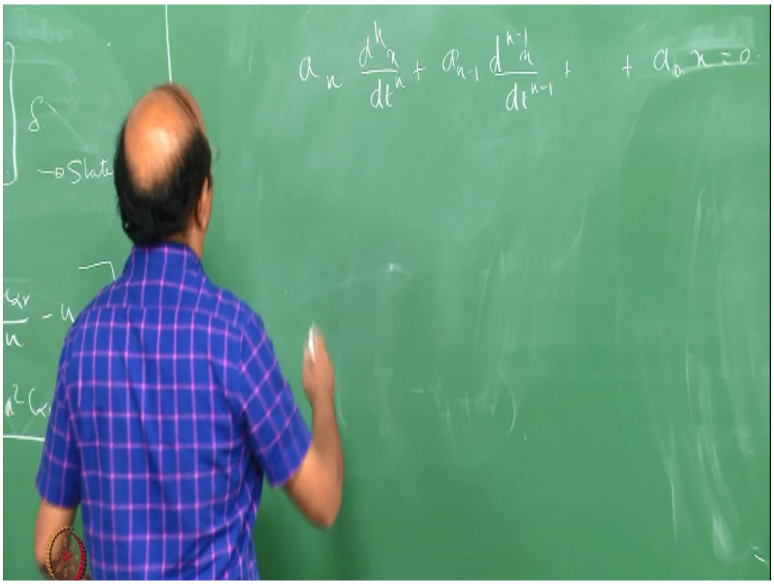

Now generally system is expressed in terms of a differential equation which we will say that a dt power N we will put that as dt power n+an-1 dn-1 x dt n-1. This is the nth order system so the last term should be available +An *x=0. Note that An cannot be 0 if you are looking at an nth order system. This is A0.

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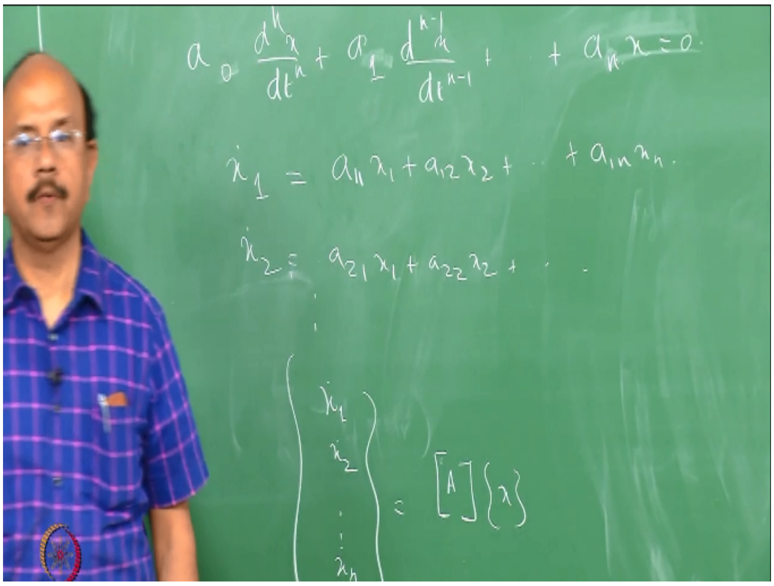

Many times you write it the other way you can write this to be 0,1 and also as the another. Either way you can do that text books follow different procedures. You can also write down this is a differential equation form. The most immediate application if you look at this is n in dynamics is your vibration equation you can write it as Mx double dot+Cx dot+Kx. You can also write this down in the state space form which we had written down which I can write down as x dot 1 in terms of what are called the state variables.

X dot 1=a 11 x1+a12x2+so on+a1n xn and then x dot 2=a21 x1+ and a22 x2 and then so that x dot 1, x dot 2 x dot n =ax is another way of writing it down. There are 2 ways in which you can write down it does not matter in what way you write down.

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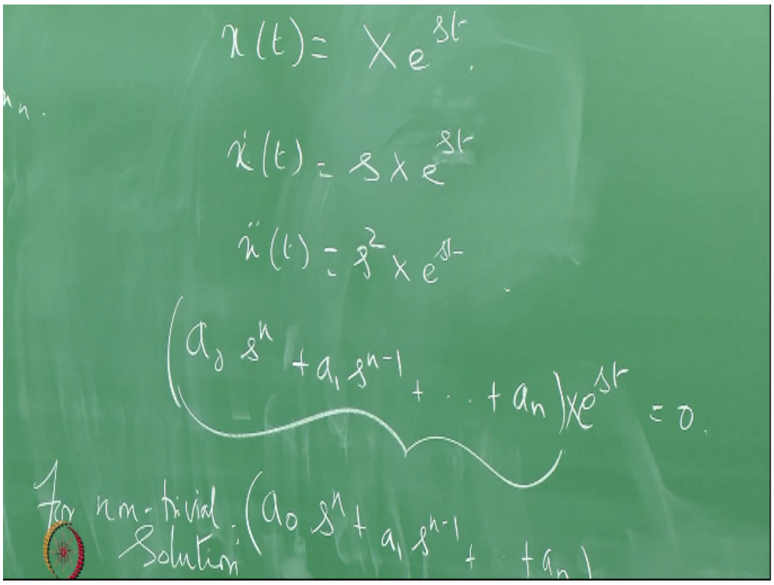

If I want to look at the solution of this equation. The solution of the equation can be written as x of t=X e power st. So the solution of equation can be written as x of t=X e power st. This form of equation is valued only if we have certain characteristics for s. I cannot put any s of course s has you know it can be real, imaginary, so on and so forth we will come to that in a minute.

So this s okay has a characteristics which you had studied in your earlier courses and we will see what this characteristics of S has to be or what should this s be in order that this becomes a solution to the system. Since x=this thing x dot t=s x e power st and x double dot t=s square x e power st and so on. Substituting that into the governing equation we will do that for this case in a minute.

So substituting that in that expression. What is that you get? A0 s power n+a1 s power n-1+ so on +an X e power st=0. The next step is obvious we have done this what is that for the non trivial solution to exist okay this has to be=0. What is a trivial solution x=0 because once this is not equal to 0 then x has to be=0 and that is a trivial solution. So the trivial solution or non trivial solution to exist=+A1 s power n-1 + an=0 for non trivial solution.

So there are s is nothing but the routes of this equation there are N routes and so the solution of the equation can be expressed as the sum of these routes and so on. Now that is known. Now let us now look at the states space form. Please understand that this is one way. There is no unique way of writing there is one way of writing the state space. It is not necessary that you will have or you need to follow only a set called x dot 1, x dot 2, x dot and so on for state space.

You can have another form as the state variables and express this equation in another form. It does not matter you will ultimately see that the stability of the system obviously will not be affected by the way you write down the state space equation. So let us now look at how to write this is clear it exactly now extended here.

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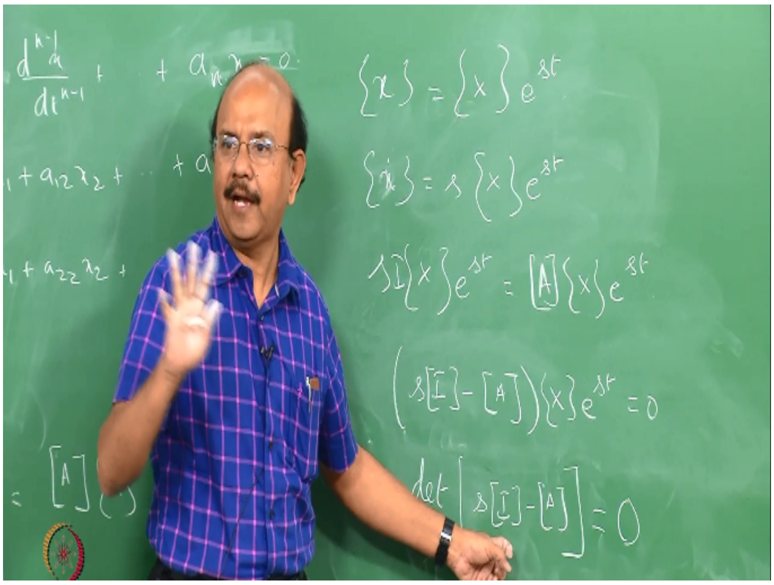

So how do I write this instead of x I will write that as x e st. So that x dot now becomes s x e st. Substitute that into this expression okay x dot=A x. S x e st okay=A x e st put an I here so that which is the identity matrix and so I will write this down as s I-A*X e st=0. Okay this is just a revision I know you guys know this once we finish it we will go and apply it to this expression that is what we are going to do. So for non trivial solution for this expression to exist what is it. What is determinant of this?

That is a characteristics what is called as characteristics equation and so the same condition for non trivial solution to exist. So the determinant of s of I-A should be=0. Okay that is simple or else if it happens to be non singular then obviously you will get end up with only the trivial solution. Hence this has to be=0 and that throws up values of S. No s1, s 2 you know that is what we are going to get now s1 and s2.

We are looking still at stability. We will come back to other quantities a bit later. We are still we have not solved for S. We have not found out the Eigenvalues yet. We are still looking at the equation for stability, what is the condition for stability. I have not solved it yet for s1, s2 and so on for this that is what I am going to do now. First let me look at the stability of this equation.

Yes, that is what I am saying S. I have not yet solved it. I am just writing down. I said that there is S1 and S2 that is all. What are they I have not yet determined them. **"Professor - student conversation starts"** According to this my S1 is capital S1 *e power st and my S2 is capital S2* again E power st. Yeah so we will S1 and S2 there will be two routes, you know, just wait for a minute.

Let us finish this. **Professor - student conversation ends"**. We will look at that that is what we did here. I wrote this if it is a second order system it will be A square that is what I said that there are N routes just wait for some time. Let us first finish it and then we will go to the stability. So now that is the determinant of that system should be=0. Okay now what I am going to do now is to express I know what is A that is A.

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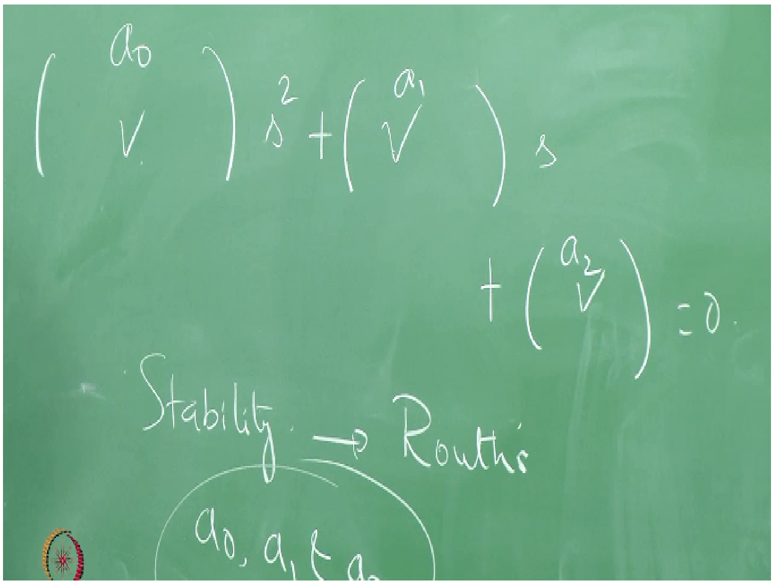

What I am going to do to substitute that A here and expand it and write down some quantity of s square+ some quantity as s + some quantity=0. What are those things we will do that in the next class just quickly so that you do not have a confusion as to what I am doing? So I am going to write this down like this then look at stability of the system that is what first I am doing. So look at stability of the system.

The stability of the system for which an expression governing the characteristics equation is like this is given by a very simple criteria Routh's criteria which states that what is here should be positive. If I call this as A0 this as A1 and this as A2 the necessary and sufficient condition is that A0, A1 and A2 should be positive. So that is the first thing I am going to do. We are going to do problems later. I have to look at the transient analysis and so on then we will see.

We will rework this out and see how am I going to write down expression of E power At matrix called E power At and so on. We will do that later. So my first step is to find out A0, A1 and A2 and see under what condition A0, A1 and A2 are going to be positive. So the question which we are going to answer in the next class is it possible for a system to be unstable for a vehicle to be unstable.

If so under what condition the vehicle is going to be unstable. We will finish this class by saying that note what I mean by unstable. If you are going straight, it is possible that if the system is unstable or vehicle is unstable a disturbance that comes out or comes about in your system may not die down or may not bring you back to your equilibrium position. So that is the unstable case. Our job is to substitute A here write down that expression. We will do that in the next class.