Vehicle Dynamics
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Lecture - 20
Understeer Gradient and State Space Approach

In the last class we defined a very important parameter which we called as understeer gradient. We looked at it from a state space form, the equations were written in a state space form and we found out that this understeer gradient plays a very important role in the stability of the vehicle. In fact, if you go and look at that equation again, you see that the BC alpha or – CSE alpha F is the key term.

And the whole concept of understeer gradient is based on that particular expression, right? So that is what is going to make it positive negative and so on, right? Then we moved over, we have not yet given a very physical meaning to it, we have not given yet. We said that that is definitely a parameter which has a problem or which is going to create a problem on the stability of the vehicle, right?

We also made it very clear that it is not that when it is negative, the vehicle immediately becomes unstable. We also said that there is a speed, a critical speed at which this vehicle becomes unstable. That is what we said. In other words, we had 2 terms and we have to make sure or we have to get that sum of those 2 terms to be negative, in order that it should be unstable.

This is where we left and then we started with another small digression on steady state cornering. Why are we doing this, in order to understand what physically this understeer gradient is. Okay, we will go back to the state space. We are going to spend some more time in state space. I know you have done this in controls. But we will go and refresh your memory on the state space as well. Okay, before that let us look at this.

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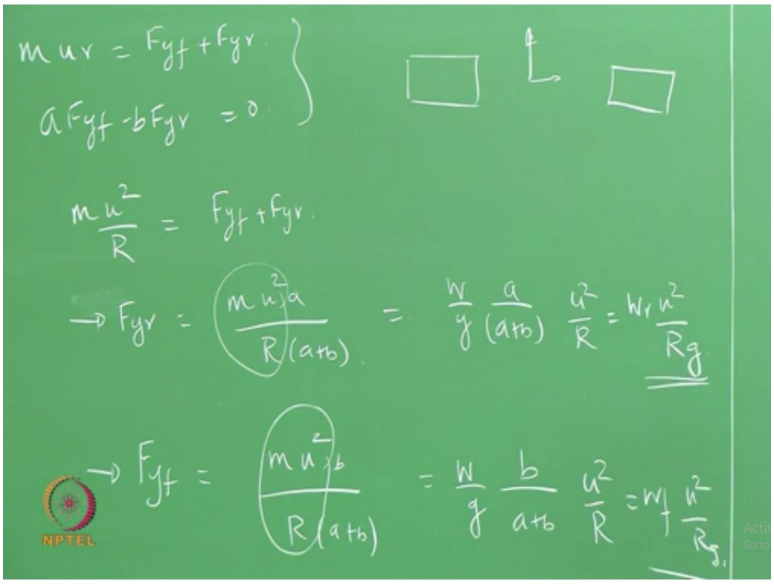

So since I said steady state, we do not have those terms, iz r dot, r dot does not exist because it is a steady state expression. So we have an expression of this form. These are the 2 equations. In other words, the governing equation now, differential equation reduces to these 2 equations because these r dot and we said the v dot it goes to 0, okay.

One of the key things that we found when we started writing down this equation is that, the 2 forces, that is force acting in the front and the other acting at the rear. Okay, remember that we were looking at the bicycle model and remember that we have one wheel or one tyre where the left and write tyre actually is collapsed. Okay, there is some confusion I said, frontend in the sense that front and rear has one tyre where the left and the right tyre are collapsed into one.

So the stiffness of this tyre is the sum of the stiffness of the left tyre and the right tyre and the same way it is for the front. So if you are given say a problem, the stiffness of a tyre, then you have to take it as 2 times that of the stiffness of the tyre. So that is just a small input. Remember that one of the key factors, the key conclusions that we made was that the front and the rear centripetal forces are distributed in the same fashion as that of the weight of the vehicle.

So we said Wf=W*B/A+B and so and it is distributed in the same fashion. S o Fyr can be written like this. This is what, we wrote, clear? And Fyf is written in this fashion. So in other words m u square R that is the term, that is like w, a/a+b and that is = mu/a+b. Okay, so you can type that as wr*u square, okay substitute again, you can write like that. That is a very straight forward expression. Why are we doing this we will know that in a minute.

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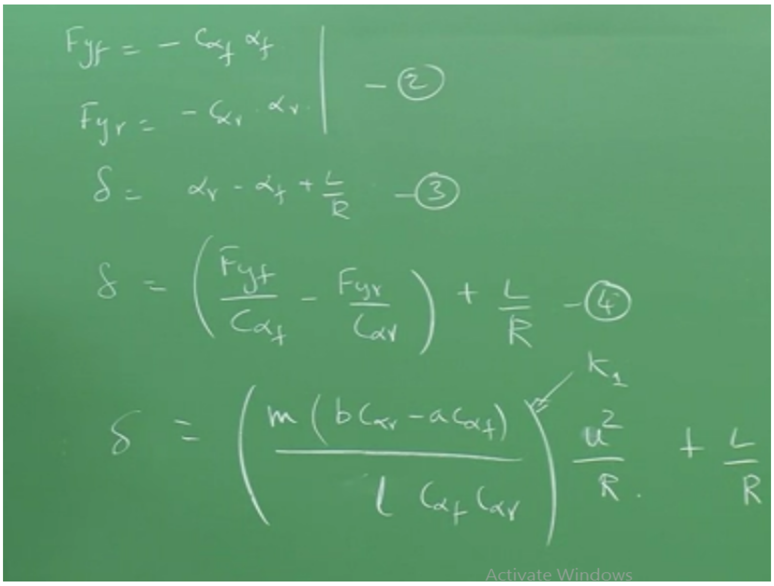

Now, this is also things which we wrote in the beginning of this derivation that the forces that are acting can be expressed in terms of the stiffness. So in other words we are using linear model for the tyre. Now let us substitute these expression here, all these, this one, because there is one, 2 and 3. Let me substitute that. Now I will write this as Fyf/C alpha f - Fyx/C alpha. Okay that will be the first 2 terms.

Now that - cancels out and so on. Even if you have written alpha f to be positive. You will get the same expression, okay, +L/R. And note that L/R is in radiance because we are replacing the angles. So if you want to put it in degrees, correspondingly you will make the changes. Okay. Now let me substitute that expression, this is what we have here from one, okay, on to the new expression which we would call as 4, right.

Substituted and write it down. We will get a very interesting part of that. You will get m into... I am substituting this into that expression. In other words, I am going to take here first these 2 expressions and then substituting with that, okay. Let us see what you get. Expand this. You will get bc alpha r, - ac alpha f. So that is what is the key factor here, right? Bc alpha f -, this is bc alpha - ac alpha f is a key factor here.

That will be divided by, you have a+b, so a+b which is l, a+b is nothing but the length of the vehicle l*c alpha f*c alpha r, okay? Multiplied by the u square/r is there, that is the term here,

+ let me call that expression 5 and spend a few minutes here. Look at this term here, let me call that term as K. It is a very close relative to K3. While it is just removed from K3 by that factor l. Go back and look at that. We had put L square there.

So factor l. Okay. So many people call this as an understeer gradient. Now look at that expression. Bc alpha r - ac alpha f is here, right? And what is your conclusion from this? That when the understeer gradient is positive, is high, okay? Then for the same u, very simple first expression, first things. For the same u you are going to apply more steering input. Alright. That is the first right, the beginning, alright.

We will also note that it is possible to express this in terms of w as well. We will come to that number later.

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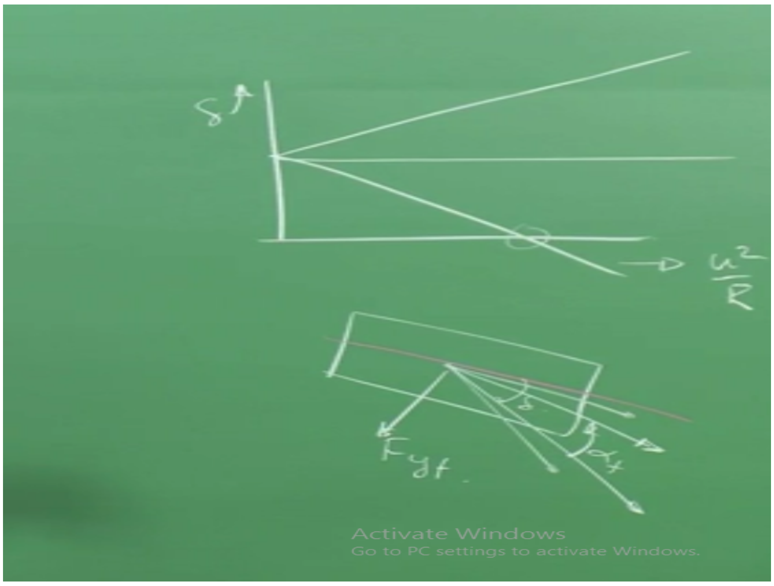

Okay, now let us go into the details of this. Let us say that I am plotting u square/r, do not be surprised if they say that understeer gradient is expressed in terms of per g value. In other words, you can also substitute for instead of m you can put a w, you can put a Rg. Okay, that is also possible that we will call this as k2, let us not worry about that right now. I just want to point out that it is possible to write that as well. Let us now look at this graph.

So this is for a steady state cornering. That is the result which we are looking at. Let us consider the simplest case. Understeer gradient = 0. That means that bc alpha r=ac alpha f. What happens, if I now plot this u square, this is = a constant which is L/R, the (()) (10:38) angle, that is a constant. In other words, when I take a turn whatever be the speed I will apply the same steering input, clear?

Okay, now let us look at the condition where k is positive. Yeah, bc alpha r is < ac alpha f, right? K is positive. Obviously, that is a straight line because I have plotted this as u square/r, okay? So what happens now? That straight line would keep increasing. That is the L/R. That straight line keeps increasing. So what does this mean? It means that if I have to take a cornering, in other words maneuvering with the same R.

If I now increase the speed of the vehicle, then I have to increase my steering angle. So in other words, the steering input that I give depends upon the speed of the vehicle and it increases with an increase in speed of the vehicle. So that is why we say that it is understeered. So I have to give more and more steering. If I now increase the speed, okay? Comes out very clearly from this expression. What happens if it is a oversteered vehicle.

In oversteered vehicle this quantity is negative. So when I increase u, since it is multiplied by a negative quantity, what happens to my delta, it decreases because obviously if you look at it mathematically it is just a straight line with a slope is going to be negative. So that guy goes now like this, right? There is one value of this u square/R at which delta that is required=0. In other words, you do not give any input to the steering delta=0 and the vehicle takes a turn, okay.

So it takes a turn. In fact, you cross that and come down, what happens? Delta becomes negative. What does it mean? It means that if you are going to take a turn to the right, okay you are cornering, okay, you give an input to the left. Okay, you give input to the left. It comes out very clearly mathematically that is what happens. That is what happens. I know, puzzled faces I see, but it is very simple mathematically, right?

Whenever mathematics gives a result like this, you look puzzled. You really want to know whether it is true or0. So you want to look at physics, what really happens. First let this get into our mind. Is this clear, okay. Why is this happening? Let us go to this my front tyre. Let me put that front tyre. Actually it is this alpha f and alpha which is playing the role of cornering. Those 2 guys are very important. That is why tyre is very important.

Look at that, tyre plays such an important role in this whole hanging characteristics, a very important. Let us for a moment look at this more carefully. Please note that b into my second expression here, a*Fyf-b*Fyr=0, right? Because that is a steady state. Okay, now coming back to this, remember my figure that was delta, right? And then I said that there has to be alpha f that has to be created in order that, that should exist in order that I develop what is called as the front force, clear?

So when alpha f now increases which means that my expression is going to become more and more positive. Okay, what happens it is going now up. Note that this is the direction of travel. Now if I now keep going up then I have to give more and more steering in order that I maneuver my vehicle to take a turn, right? So when alpha f is small, then already the guy is aligned towards that turn and delta needs to be small, right?

At one point of time what really happens is that this alpha f goes to the other direction. If it is like this then we get into lot more trouble of instability, right, because I have to now bring it back again. So heading becomes a problem. So that is what happens and in other words that is this role played by alpha f become extremely important, clear?

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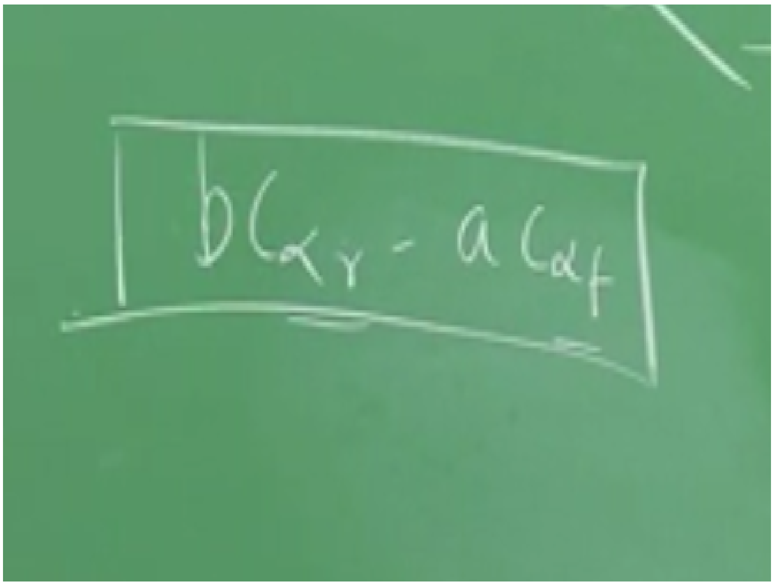

So the key factor here is this bc alpha r – ac alpha f. That is a key factor here. Now what is a and b. Remember that they are the CG locations. Imagine that or why imagine, you have so many front wheel driven cars. Most of the cars today are front wheel driven. So in a front wheel driven car, what do you think will be the distribution of a and b? Will a be small or b be small? a will be small. So all front wheel driven cars are understeer cars.

So the first factor, they will be more towards the understeer vehicle. The first factor that the vehicle manufacturer has to look at is what is this understeer gradient. Towards the end of this part we will look at what are the tests that are used in order to determine this. Let us not right now worry about this, but we need to know that understeer gradient is an important characteristics of vehicle for handling.

Do the manufacture produce an understeered vehicle or an oversteered vehicle. They do not produce an oversteered vehicle basically because the worry is, it can become unstable. You know it is very difficult to handle. Sometimes it can become counter intuitive, so all vehicles are understeered vehicle they are oversteered vehicles. It is that they are not oversteered, they are all understeered vehicles. Now the question is how much understeer it can be.

That is an important question. If it is, I want to be very safe and I wanted to be highly understeered, then tomorrow the newspapers or the magazines which does a review with their vehicle will say that this car is a damn square thing, it does not just turn. You need to apply so much of steering for this vehicle to turn, response is very bad and blah, blah, blah. You are in trouble.

So I cannot have so much of understeer that the guy who is going to drive the vehicle he keeps on steering. So that is a problem. So it cannot be a oversteer. It is just above the neutral steer. Okay, why is that people do not look at neutral steer? Basically because look at that quantity, it is bc alpha r and alpha f. Alpha f and alpha r, what are they?

They are the lateral slips stiffness as it is called, okay? Lateral slip because, lateral slip angle, lateral force is a slip angle, slope of that curve, so lateral slip stiffness. Depends upon so many factors. It is a tie up of so many factors and it depends upon the inflation pressure. If you are not maintaining inflation pressure definitely your handling characteristics will be affected.

Depending upon what is the inflation pressure how that varies and so on, they are getting affected. And a and b are not strictly are constant, for a car it maybe a constant for, even that we cannot say because there is a small difference can be there, how many people sitting, how they are sitting and all those things. But for a truck definitely, for a bus this definitely does not affect, okay?

So it cannot be straight away a neutral steer, so it will be slightly understeer, okay? So that is the importance of understeer equation. It is extremely important in rear engine cars, rear engine vehicles. Many of the buses are rear engine. So then you have to be very careful in looking at this constant.

Because the bus may not have that much effect because the passengers can be distributed and so on, but for a car, rear engine car becomes many of the electric vehicles for example small cars, very small cars are rear engine cars in which case you will get this vehicle to be over steered and hence that will be a problem. Then it comes to the fact that what happens to a formula one car. Formula one car, the driver just does not have the time to corner.

He cannot give so much of input, steering input in order that he corners the vehicle. So he would prefer it to be oversteered. Yes, this is important but these people are very well trained drivers, so they know how to handle it. So the vehicle will become a oversteered vehicle. One of the interesting factors, in India especially in the truck market is that they mix they tyres, or in other words they mix the radial tyres with a bias tyres.

Many times they use the bias tyres in the front, radial tyres at the rear and so on. That will affect the handling characteristics of the vehicle because usually the radial tyres have 2 times more lateral stiffness than the bias tyres. So mixing this is going to have an effect because of that factor. It can be a huge factor when it comes to the mixing. Okay, fortunately they have a radial at the rear, bias at the front. So the factor actually increases.

So it is not a problem but if it is the other way then you will get into trouble. So it is better that when you mix it, you be aware that it will affect the handling characteristics. Of course cars no one uses bias tyres, so it does not matter. But for trucks in India especially because of only 20%-22% of the tyres are only radial tyres yet.

In fact, some of the survey on my students did, we found that only about 6% of the trucks which we inspected recently they have radial in the front. Rest of them are only bias. So it is very important that we understand this factor. **"Professor - student conversation starts"** In case of a tractor trailer, how do you actually decide these things because when you are loading and unloading there is a huge difference.

Yes of course that is why I said this would vary. When it varies so you have to know the limits and accordingly adjust that. So it is not one factor. That is why the angling characteristics depend upon the load, how it is placed and all those things will have an effect, okay? So that is important. **"Professor - student conversation ends"**

Right. That is the key factor on understeer gradient. So as I said, the way it is defined is different. You can define this in w and G out and so on. So be careful, when someone tells you a value we will do some problems later. You want to be careful and ask him what exactly is your definition. I have got into trouble like that before. So you have to clearly know how someone defines the understeer gradient.

What is the formula he is using? So it can be k1 or it can be k2, for example you can put w here and take this g out and then write this whole thing.

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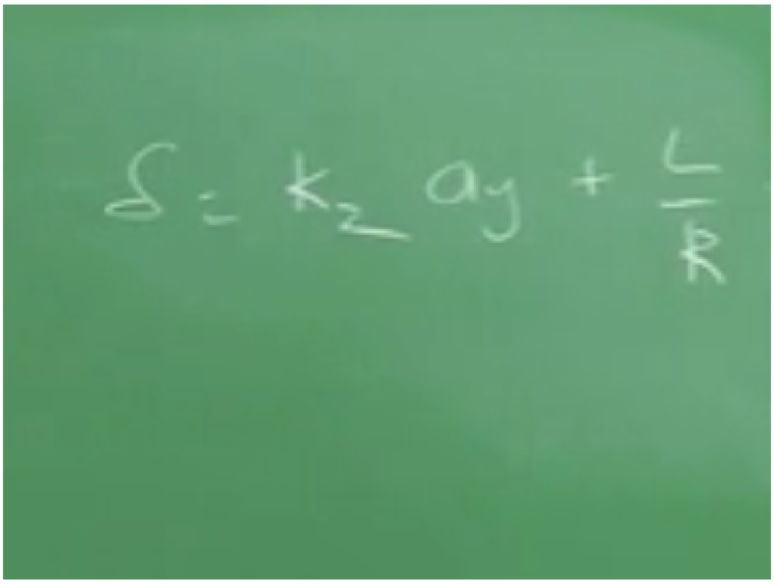

Now delta =, let me call that as k2 in that case, k2*ay+L/R where ay is the lateral stiffness expressed in terms of g. So raidence per G is what you would say is the unit. So you have to be careful as to how you are writing that. Clear? **"Professor - student conversation starts"** Actually user controls not steering prominently in that space. Of course, very important that user controls this. That is why he has to give an input. User control is that.

But this factor is not controlled by him. That is the whole thing. So when he presses, actually in other words which I should not say because we are looking at steady state, okay, so when the velocities are high, his steering input has to be higher. In other words, you take a turn at 40 km per hour and take a turn at 80 km per hour your delta is different. Is a factor which is very important to understand.

So note that the problem also is that, when you increase the speed, what happens to delta? It increases. That means that that much amount of steering input you have to give. So that is the time that you have to also consider when you look at understeered. Clear? Okay. Sir is the (()) (28:49) Yes. Because this whole, note that is how we have defined this model. So center of gravity location is the body centered coordinates we have placed it at the center of gravity location, right?

While using the (()) (29:05) of the front axial (()) (29:10) No, there you are just looking at the forces, we did not write down the equations. We are just looking at forces. Here we have put a coordinate system. We are looking at the equations based on these coordinate systems. Okay. Fine. **"Professor - student conversation ends"**

Okay. Now we will digress a bit. We will look at state space technique. Remember that he had expressed before we went into this expression on steady state we were looking at the stability from the state space perspective as well or in other words, we were writing it before we went into the stability in terms of state space. Remember that we had written this as what is the state vector, b and r.

Remember that we have written that in terms of b dot r dot=a*v r, okay, + some b*u b is ac alpha f/m, sorry, c alpha/m and ac alpha f/i and so on. So state space becomes a very important vehicle to study vehicle dynamics. So in fact many of the controls that are used are based on the state space technique and it becomes very easy for us to even understand what happens for a given input or in other how does the vehicle behave.

Quickly if you want to look at how the vehicle behaves if I give a step input or some other input, so that becomes extremely important. So let us now get into this state space and let us now derive the expressions for I would say brush your memory on state space. That will answer some of the question which were asked as well.

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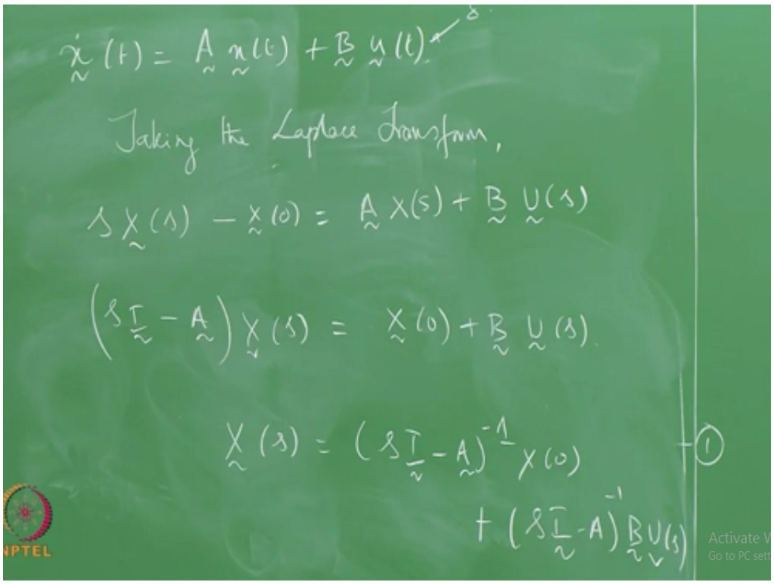

So let me write down the state space expression as… Note that when I put a squiggle at the bottom it means that it is a matrix or a vector, right? That is all. Remember that b for example we just now said, b is c alpha f/m and so in other words remember that b is c alpha f/m, ac alpha f/i, i is what we mean by i is that, because we removed that and u is nothing but the delta.

Okay, go back and look at that expression so that you can relate, a is the matrix which we wrote, you remember c alpha f + c alpha r and so on, that is the expression for a. So in other words I am just abstracting that expression and writing down this form. So what is the first step now? The first step is to take the Laplace transform of this expression, okay? So taking the Laplace transform.

So we are looking at the linear time invariant system, so a is it stays as it is, right? So sI - A, x s=x0 + b ux. Clear? So that x has dissolved, which you have studied = this. Clear? Now in the same way as we give in the traditional control system space, let us name.

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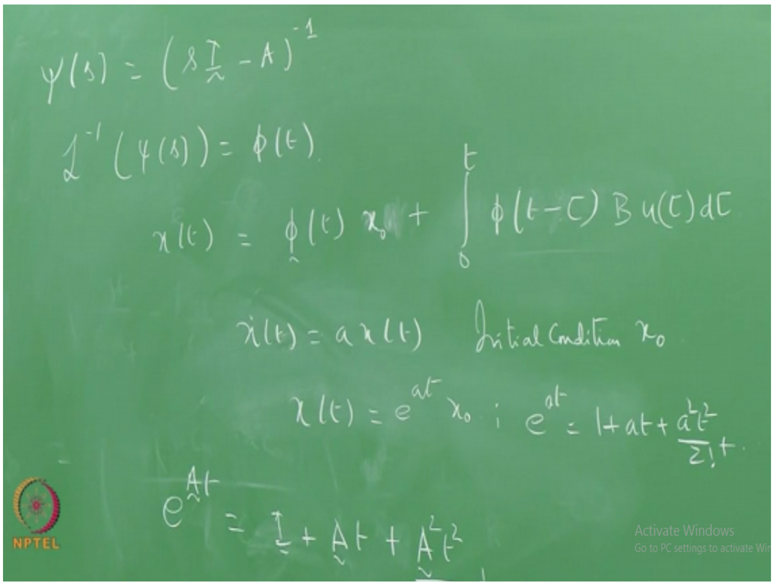

And name the inverse Laplace transform of psi x=phi t. I am just writing down a general frame work from which we will get into specifics, right? From this I can write down the expression for the xt, just taking the inverse Laplace transform of one, we can write down xt=phi ft x of 0. Sorry, I should not write this as not 0 x0, let me just. It is not of x. X0, that is xth t=0, that is the input.

Let me relive it as x0. X of t=0, that is what it is meant by x0. That is the initial condition, alright. So I can write it as x0 or xs, but as long as you understand it, fine. So I can leave it like this or I can put x0. But please understand that, that is an initial condition that I am, actually x0 is fine. So just note that it is the initial condition. So + phi of t B u of t.

Okay, now we will develop what can be the expression for psi of s and phi of t as well. Let us develop that expression. So how do you develop those expression? Is this okay? You deliberately wrote that we wanted to check how much of Laplace transform you guys know. Is it okay? What is a Laplace transform of multiplication? You have heard about this term convolution?

Yes, you would have seen in your earlier classes that convolution is given by a star. So if I say phi of, phi 1 of t convolution, phi 2 of t you can see in minutes what it is. Laplace transform of the convolution term =, in Laplace domain the convolution becomes a multiplication. So that would become s*phi 2 of s. So what is this term now? B is a constant of course. So it is a psi of x when I put. This is a multiplication.

So what should I get here? I should get a convolution. I should get a convolution. So be very careful when you write Laplace transform. So many people would leave out this term, x of 0, okay? We call it x of 0 I am very comfortable with that. So x of 0, so that x 0 of s is not there, x of 0. So basically we leave this out. When I say Laplace transform immediately x0. I know x0=x of s, that is one more term.

Like that you just cannot blindly write it like this and you have to look at when it is multiplication, you have to write down the convolution term. How do you write the convolution term, 0 to t, phi of t - dou, b u dou d dou, that is the convolution term. So that is the inverse Laplace transform., okay. And I will not say this again and you have to make sure that you remember this. So now we will develop a specific.

I suggest that you go back and revise your Laplace transform, maybe look at the Laplace transform table once more because we may be using Laplace transform when I give an input, different input we will be using that. So you should be knowing how to convert it. So get back to your control scores and look at these things once in order to understand this, okay? So I want to develop this and let me look at a simple differential equation over e which is a first order differential equation and write down that expression.

So x dot of t=a of x t with an initial condition of say x0. What is the solution for this equation? X of t, e power a t, okay, x0. Now let us define a matrix which I would call as e power at which I would call as e power at. In the same fashion as I would define e power at when a is a scalar quantity. What is e power a t? 1 + a t + a square 2 square/2 factorial + a cube t cube/3 factorial and so on.

So let me define e power a t to be like that, that is now identity matrix + a t + a square t squared/2 factorial and so on. **"Professor - student conversation starts"** Sir we should not worry about of u of t term, right, x0 of t is…No, that is the expression. If this is the expression, I have not come back to this. Just wait a minute. I have not yet looked at it. We will. This is only to tell you like what if of a t looks like.

Before you edit the same, can you (()) (43:39). Yeah. So that is the velocity at which delta becomes 0. Remember that we had k1*a y for example or m*bc alpha r - ac alpha f divided by lc alpha r*u squared/r + L/R. This term, that is what is there in the bracket, multiplied by u squared/r, that term if it becomes negative and beats this L/R then only it becomes unstable. So the point where both the terms are the same, that is the point. Clear?

Sir in that layer also like, after editing (()) (44:43) that means you are going on straight road? No, this please note that we are cornering. So imagine like this. I have a oversteered vehicle. I keep on increasing the speed. It is say for example it is a round turn which you see in the street. So for example if you are going round, so you start with a velocity. Then you are steering input. Then you go around, then you increase the velocity for a moment steady state, do not get confused in that. Let us say that you increased the velocity.

The steering which you have given you have to now make it smaller. Now further increase the velocity. The steering input again you will make it smaller. At one point in time, you are not going to give any input to the steering and the vehicle will go around like one velocity, okay? You increase the velocity further, you will have to give the steering this side in order that you go in the clockwise direction.

So at that point at which you give 0 input to the steering is the critical velocity, clear? I suggest that there are number of videos in the YouTube. Go back and look at these YouTube videos where steering input is given for an understeered vehicle and oversteered vehicle and so on, it will become very clear. So you will have a course next semester on vehicle dynamics lab in which you will also be studying or will be doing simulations as well as some testing as well, right? **"Professor - student conversation ends"**

Okay., Let us get back to this, e power at. Why are we defining like this? The interesting fact is that this matrix has all the properties of e power a t, right?

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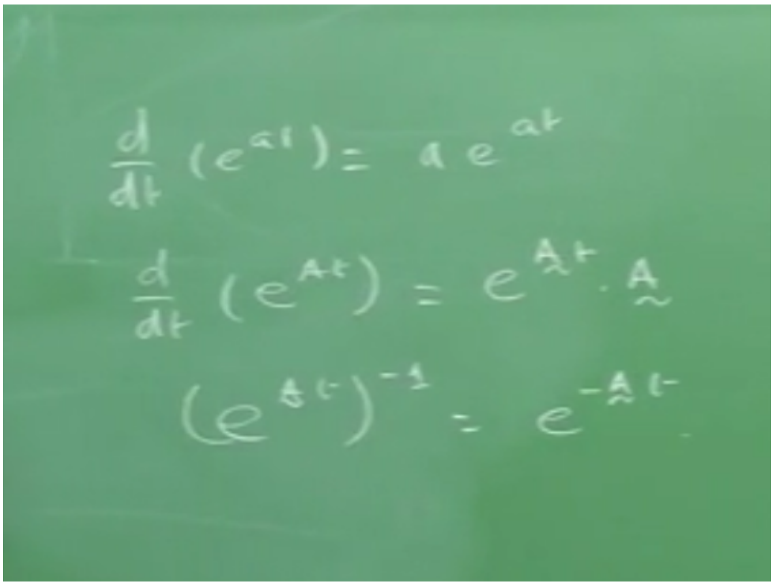

Now if I now differentiate what is d/dt of e power a t, a*e power a t. Okay, interestingly when you differentiate this matrix d/dt of e power A t= this, differentiate that and you will see that it is the same, right? Okay. So we are going to use this fact, okay, in order to understand what can be the expression for psi of x. And also the fact that e power a t, the inverse of e power A t = e power – A t. We are going to use that.

We will do that in the next class and you are going to substitute and given an expression. Clear? Any questions on this? Okay? Anyway we will continue it, derivation will take one more class then we will be completely on board, right? We will stop here. We will continue in the next class.