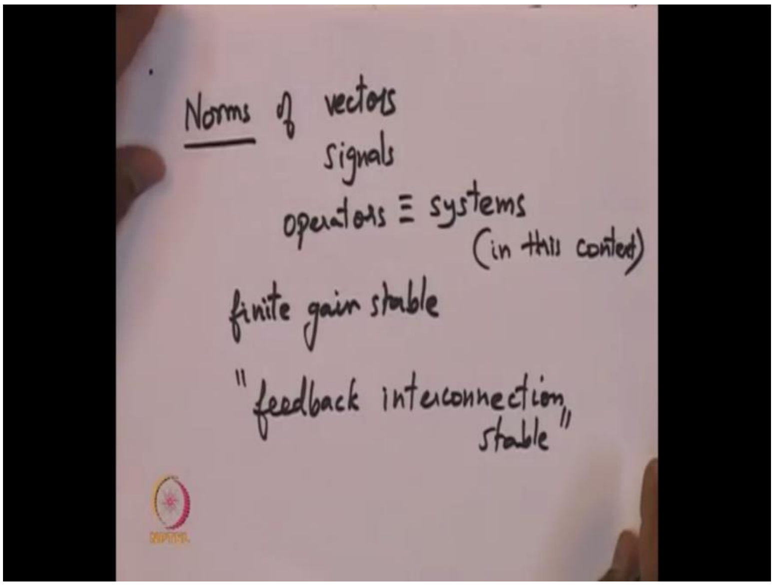
Nonlinear Dynamical Systems
Prof. Madhu N Belur
Prof. Harish K. Pillai
Department of Electrical Engineering
Indian Institute of Technology, Bombay

Lecture - 13
Signals
Operators

Welcome everyone to this lecture in non-linear dynamical systems. Today, we are going to cover about some.

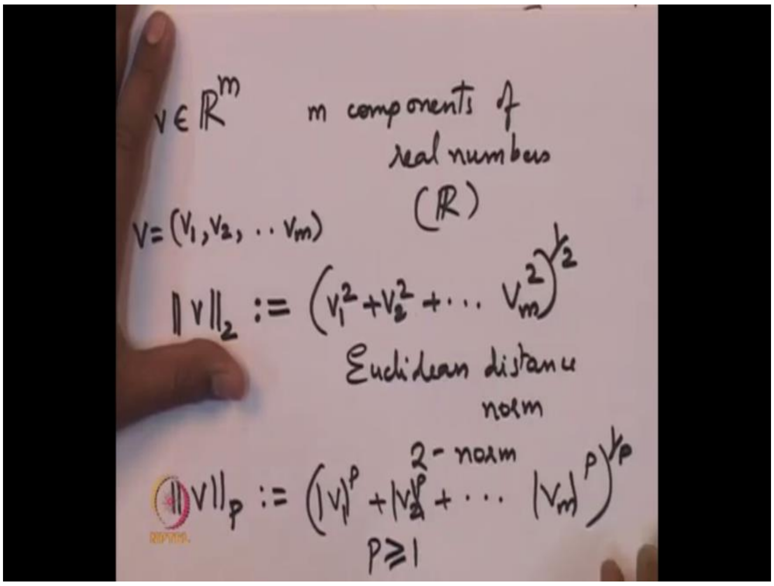
(Refer Slide Time 00:23)



Norms of vectors in particular, we are interested in signals and then norms of operators are in our context systems, of course this is not standard. There can be various types of operators, but we are speaking in this context in this context. So, we will also speak about eventually finite gain stable that is the objective for this lecture finite gain stable. We are going to the definition of, what does it mean for a system to be finite gain stable? For that, for defining that we need these other things also. And eventually we will speak about feedback interconnection being stable under what, under what conditions do we speak about a feedback interconnection is stable feedback? Interconnection is stable in order to define this, we need these concepts.

So, today's lecture is titled norms of vectors signals operators and illusion of stability of a system possibly non-linear. Of course this will also require us to understand linear systems better, because I personally believe that one requires are good understanding of linear systems before one goes ahead. Understands non-linear systems just like one should know, how to build a one floor building well before one goes ahead and starts building a 5 floored building storey building. So, we have vectors spaces.

(Refer Slide Time 02:16)



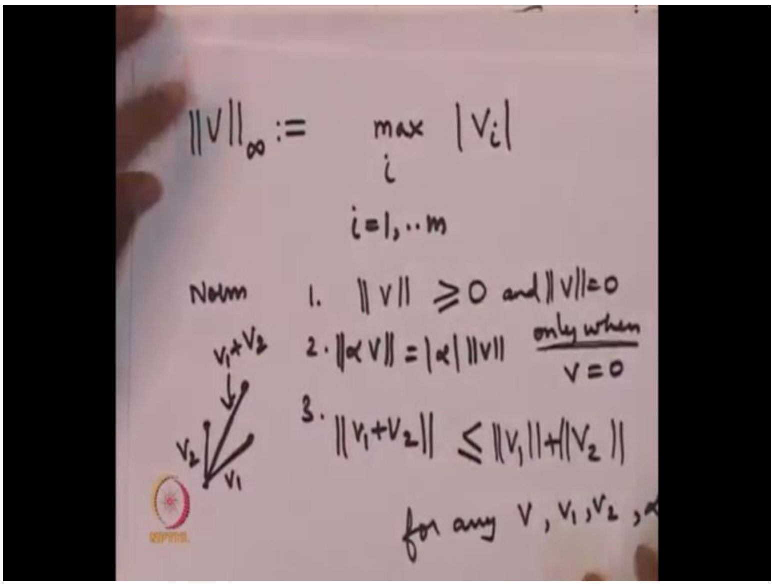
These are called vector spaces. There are various types of vector spaces, but these are the simplest, these are m tipples. There are m components, m components, m components of real numbers. So, real numbers are also called as R and hence m components are called R m 1, can think of is V 1 V 2 up to V m. So, one could choose to call write it as a row vector, like we have done here or one can write it as a column vector. In any case it is m components and each component can be chosen independently as any real number hence these all together constitute R m.

Now, one can speak of a norm for vector V in R m. Suppose, V is equal to this 1, can speak of V norm. So, norm we might have seen in the beginning of this course already. So, norm is required to satisfy a few conditions, but we can think of only start seeing some examples. So, 2 norm of V is defined as we use this symbol for define, when we have a colon on any of the 2 sides. It means, one is being define as the other since the colon is on the left side, the left side is being defined by what we will write.

Now, V 1 square plus V 2 square up to V m th component square and then together square root. So, this is the most common notion of distance, which we call as the 2 norm, which is also called Euclidean distance. This is our convention version of distance Euclidean distance or Euclidean norm as I said this is also called as 2 norm why because instead of the second power. Then, taking square root 1 could take any power 1 could take the p th power and still it will satisfy the notion of the norm V 1 to the p th power plus V 2 to the p th power plus up to V m th to the p th power, but then if p is odd.

This does not ensure that these are all positive quantities. So, we will take absolute values of each of this real numbers and we will add after taking, we are going to add after taking the p th power of the absolute values. If, p is 2 or if it is any even number then this absolute values is not required to be taken. They are all real numbers positive or negative and for even powers. They will always be positive after taking the even p th power and after adding them all. We will take the p th root, so for any p greater or equal to 1. This is how the p th norm of a vector V is define done can also take, so called the infinity norm.

(Refer Slide Time 05:04)



What is infinity and how is that related to p 1? Can check that if you have m fixed components and you go on raising p? You make p 2 3 4 5 and you take this absolute value, and then take the p th power, and then add them, and then take 1 by p th power.

Then, it will eventually convert to the maximum of the absolute value of the I th component where i varies from, i varies from what from 1 to m, when you say, max over i of the absolute value of V I, it will turn out to be the value that p th norm of V converges as p tends to infinity. So, this is i, which is same as saying where i ranges from 1 up to m as take different values of i from 1 to m, and for each of this cases. Look at the absolute value of the i th component of V and look at the maximum over these m components of the absolute value. And that is defined as the infinity norm and how is it related to p. It also turns out to become equal to V p as p tends to infinity.

So, this is another notion of norm, all these norms have a notion of distance they all satisfy our convention feeling of distance. So, norm, so why do we not take p less than strictly less than 1, because a norm of a vector has to satisfy 3 conditions 1 2 and 3, what are those conditions? For any vector V, the norm has to be greater than or equal to 0.

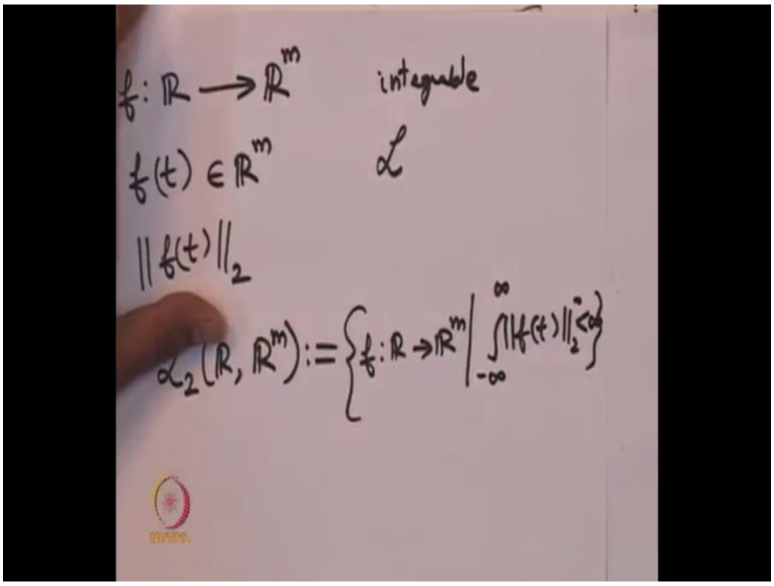
We are not, we are not comfortable with a distance, which is equal to minus 2 any norm. If, somebody says, this is a norm of this vector. It had better be a positive quantity and if the norm is equal to 0 that is only, when only, when the vector V is equal to 0. If, somebody tells that look here is the vector whose norm is equal to 0 that should happen only, when V is equal to 0, but for the 0 vector is obviously happen because of the second rule that we will write. If, somebody writes, if somebody scales a vector that is nothing but the norm of the same vector multiplied by the absolute value alpha, is a scalar.

It is a real number for what we are doing in this case. So, if somebody multiplies a vector by 5, then get scaled by 5 or by minus 5 means, you just multiply it scale by 5 times in reversal direction. Both of this should just result in magnification of the vector by 5 times independent of plus 5 or minus 5 that is why the absolute value. So, this automatically means that when V is equal to 0.

Then, the norm is equal to 0 that is why only when is playing a role in item 1 and the third one is, so called triangular inequality. This cannot be greater than V 1 norm plus V 2 norm. All these 3 are satisfied for any V, V 1 V 2 alpha for any vectors V 1 V 2 and V and for any real number for any scalar alpha. These 3 in equal to have to be satisfied, this one just says that the norm cannot be a negative quantity, and it is norm is equal to 0 only, when V is equal to 0. Second one says, about scaling by a scalar alpha and the third one speaks about the, so called triangular inequality, what is a, what is triangle and what is inequality? About it this is vector V 1 and this vector V 2, then V 1 plus V 2 by the, so called parallelogram rule.

So, the length of this cannot be greater than length of this plus length of this and that is what is triangular inequality this one is V 1 plus V 2. So, these 3 together are required for any notion of norm and the p th norm satisfies all these 3 conditions, but that for that we required p to be greater than or equal to 1. So, how do these norm play role? We are actually concerned with signals.

(Refer Slide Time 09:13)



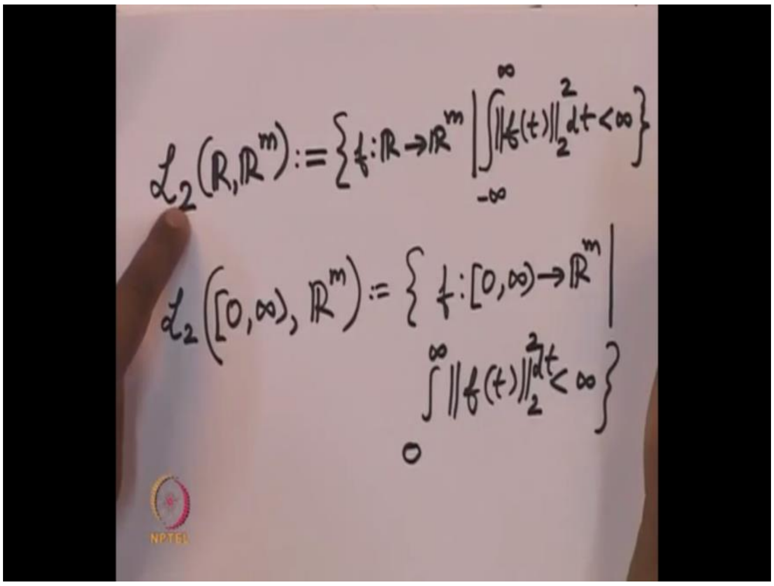
So, consider function f that some real numbers, for any real number, it has n component sin what does it mean, f the independent variable R means you, we in our course because we are dealing with the systems course this independent variable. We like to think of is time one can also of think of it is space, but in any case there is only one independent variable f of t at any time t f of t is a vector in R m.

This is the meaning that f is a map from R to R m. So, we can speak of that vector's norm because of it is 2 norm. Now, we are going to be interested in looking at all, so called square integral functions, you take any function f R to R m. So, this integrable, it is said of all set of integrable function. It is also called as L l, I think stands for Lebesgue thanks to one friend of mine called Ajay, who told me that L that we use, so often in L l

2 L l p. All that L stands for Lebesgue, the person who came up with a concise a very systematic definition of integration after I man.

So, L 2, we are going to define now L 2 is a, we are going to say, what is the set? It is the set of, we will define it as a set of all functions from R to R m such that, such that some property satisfied, what properties from minus infinity to infinity just take the 2 norm, this should be finite. Let me write this slowly and clearly. So, what is the L 2 norm defined and what is the L 2 space defined? It is defined as…

(Refer Slide Time 11:28)



Consider the space of all functions from R to R m, which all functions will be taken put into the set. Take all those functions from R to R m and check. Take all those f for, which this vertical bar should be read as for which it can also be read as such that all those f from R to R m such that the 2 norm. This 2 norms stands, because at any time t, this is a vector in R m integrate.

This from minus infinity to infinity integrates. This in respect to, this should be a finite value, this finite value itself. Of course we will very soon define as the L 2 norm of the signal f, but if this norm is finite. If, this after integrating, if you get a finite value then that f, you will pick and put include into the set L 2. So, what is L 2 the set of all set signals it turns out and this will be a vector space, what is vector space? About it you take any 2 function f.

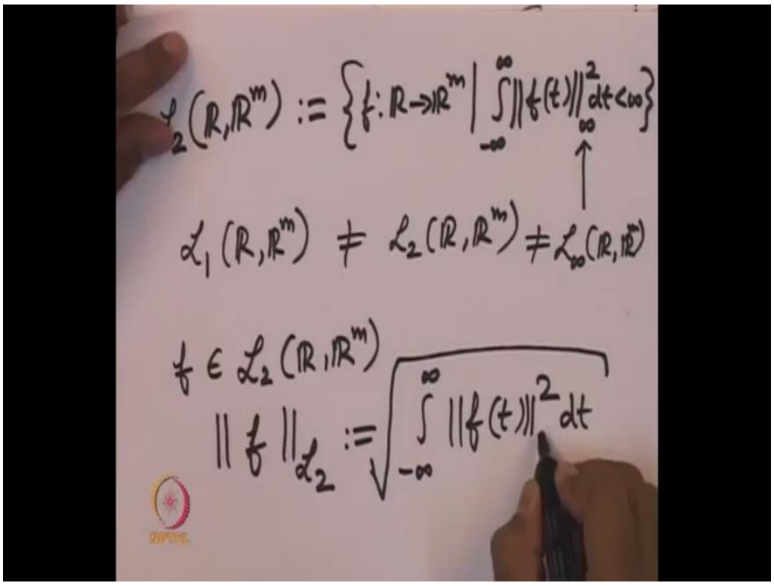
Here, you add them that will also continue to remain L 2, if you take any function f and you multiplied by scalar. It will also be continued be in this and also the 0 vector is there in this in for this set the 0 vector is a 0. The signal with which is equal to 0 for all time t and, so this is what make L 2 R to R m into a vector space. Of course you may not be interested in R. You may be interested in only functions that take values from 0 to infinity to R m.

These are that the function we considering here, the domain could be different. This one is the co domain, the vector space in which takes its values at any time t, but t itself need not be varying from minus infinity to plus infinity, but it could be varying from only 0 to infinity. This is defined as set of all f from this domain to R m such that integral.

Now, we will take not from minus infinity, but only from 0 to infinity of this f of t d t. Sorry, this is, if this is less than infinity, we will go and put the function f inside this. So, notice that this 2. Here, is different from this 2? Here, this 2 here refers to f of t as a vector in R m and there you take the Euclidean norm, but sorry there is one important thing that I missed, what is this 2?

Then, this 2 refers to that you take the norm and you take the square here. Also here I should have been putting a 2 here. So, this 2 up here refers to this suffix below this l. Let me see, if I missed the 2 in the previous slide. Here, also I had only half written this is slide number 4. Here, also needed 2, this 2 here refers to this. So, let me just quickly speak about and these 2’s do not have to be related what precisely.

(Refer Slide Time 14:54)



We can speak of L 2 space from R to R m, F take all those functions f from R to R m such that integral minus infinity to infinity of f of t. This norm, we can also take as a infinity norm, but still you have to take the second power d t as an infinity. I hope this is clear that if you are taking L 2 spaces here, then the power should also be 2. Here, in the, in the super script after taking the infinity norm of the vector f of t at any time t.

You take the second power and you integrate from minus infinity to plus infinity, and that should be a finite value that should not be infinity. The reason that people often skip writing this is because the set that you get here is eventually the same. The set of all function that you take will eventually be the same whether you take infinity below or you take 2 below t equal to 1 or any other greater than or equal to 1.

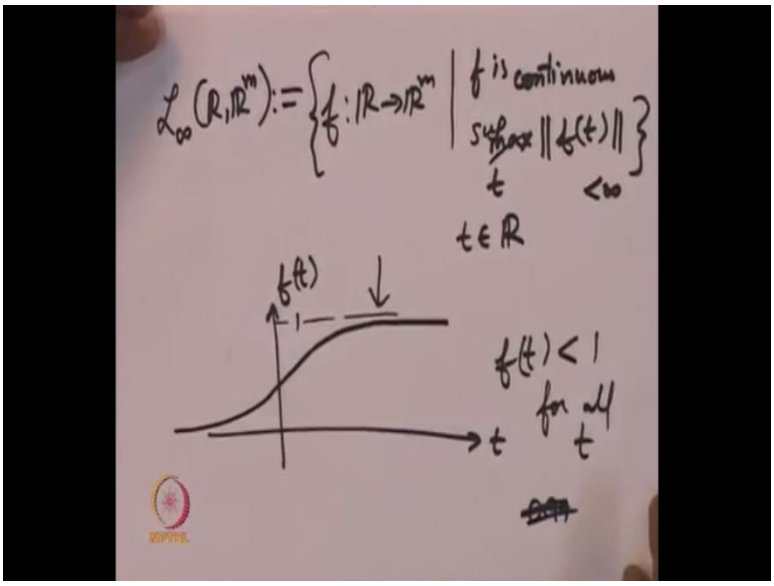
Whatever norm, whatever norm you take in R m that norm does not decide what functions come into this set L 2? What certainly decides is the fact that you took the second power and still it is finite? So, clearly well it is not obvious, but it is important to know that L R, R n is not equal to L 2 R to R m nor it is equal to. We will quickly see some example s L infinity, we are yet to define, but this L 2 or infinity refers to what you put in the power after taking the norm of f of t. That norm of f of t, we have indicated here that is a norm in R m.

These sets are not the same, the reason that people always skip the specifically, which norm it took in R m is because there is some important, very important result saying that all these norms in finite dimension vectors spaces, are equivalent because of that it turns out to not matter. which norm you take here. Of course, the actual 2 norm of L 2 norm of the function f does depend on which norm you take, but as long as you do consistently. It will not matter much in all over arguments.

So, let some function f be inside this L 2 R to R m. This already means that square integrable. This condition that we have written here, means that is square integrable. The square integrable whether it is a square integrable or not that particular statement does not depend on whether you take infinity here or you take 1 here or 2 here. It only depends the square refers to the power 2 up here. So, you take f here, then this f we will define, it is L 2 norm, L 2 norm is defined as integral from minus infinity to infinity f of t to d t.

We have taken already the square here; we are going to take square root here. So, notice that I have skipped writing it whether it is, whether I take 2 or infinity. It does not, it does matter what the L 2 norm of f is, but it will we have to only do it consistently and for this course. Why do not we just stick to the 2 norm in R m, one can also speak about the, one can also speak about the infinity norm L infinity norm.

(Refer Slide Time 18:43)



So, that for that we have to define what is L infinity R to R m? That is defined as the set of all f from R to R m in which for the purpose of the argument, why do not we say, continuous f is continuous is not required. But then I will tell, why it does make a difference eventually over all t of f of t max of this is finite?

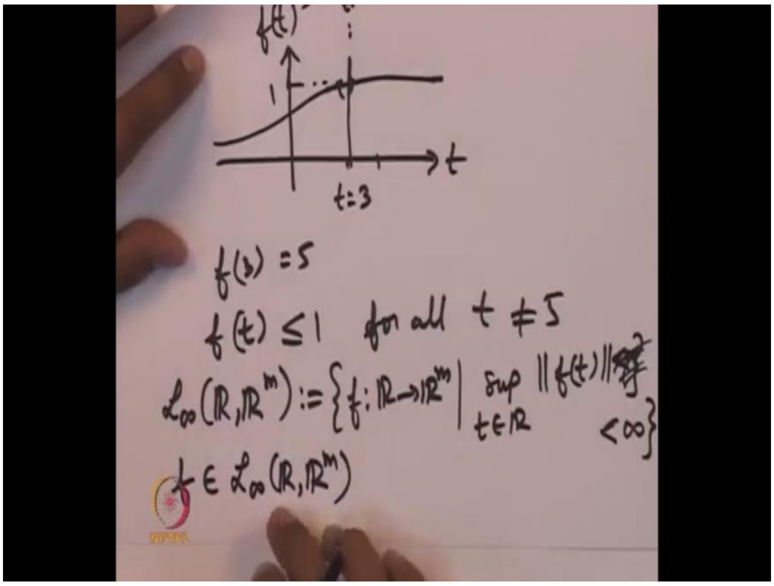
So, this is what is this is set of all the functions whose maximum value is finite, but then this max is taken over what t varying over what range t inside R. So, notice that this R is not a finite set, it is not a bounded set, it is not to be précised that a compact set because of that this max may not exist. There are little concerned that, what a bounded set? What a compact set to be précised? The maximal exist, but over R this strange problems. So, we need to correct this max to sup. So, I will tell, you what the difference is?

So, suppose here is the function f of t wave start off goes on increasing and it is saturating, and it is saturating since it is saturating to 1, but for no value of t. Does it become equal to 1, f of t is always strictly less than 1 for all, for all t, but at t 10 into infinity it reaches 1. So, we want to know, what is the, what is the lowest value? Which is above all a value of f of t? Now, you take all values of absolute value of f of t and look at the smallest value above all of them, that is called as a sup that is not strictly above, but greater than or equal to.

So, in this case it is equal to 1 that sup is equal to 1, but sup is always strictly greater than f of t, it is never equal to f of t for any value of t why because f of t is strictly less than 1 for all t. So, here in this example, the max is never attained the maximum value is the maximum does not exists why because is 0.99. The maximum value no, it goes and exceed 0.99 eventually. So, there is no maximum value, but the supremum exists supremum is equal to 1.

So, this is the important settled difference between sup and max and over compact sets over a set over, which we are looking for the maximum, if that set is closed and bounded, then the 2 turn out to be equal. Then, the sup is equal to max that time the maximum is attained to be precise then the supremum is attained. Now, important statement is, now coming back to, why we assumed f is continuous? Now, we want to come back to, whether L infinity, whether the infinity L infinity norm turns out to be the limit as p tends to infinity of L p norm. There it turns out that if it is not continuous.

(Refer Slide Time 22:08)



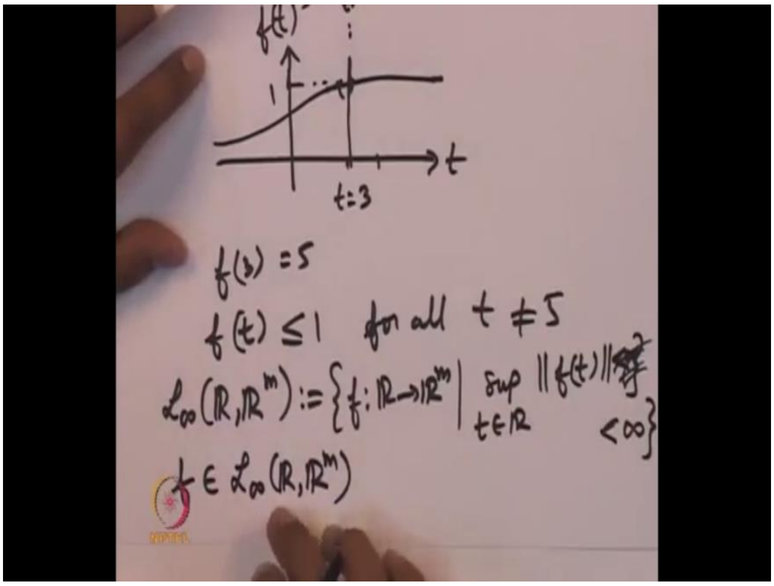
Then, here is an example that looks at this function. Let us say, which is equal to 1 eventually, but only for one value only for t equal to 3. It take some different it is this is equal to one here, but for t equal to 1, this equal to 5, clearly the graph is not to scaled, but f of 3 equal to 5, otherwise f of t is less than or equal to 1, all t not equal to 5. So, look at this graph, there is a small hole, here only for t equal to 3. It turns out to take value equal to 5, but for all other values of t it is less than or equal to 1 only for one value of t equal to 3, it is value 5.

So, one can ask below this, how much area is covered under this point area, is 0 thickness of this area is equal to 1, thickness is 0 and height is 5, so height into breath. So, area is equal to 0 because the breath is equal to 0, thickness is equal to 0. So, you must say, how much area is covered under this point, under this point area covered is 0 for the area to be non 0. You need at least, you need to be, you need equal to you need f to be equal to 5 for at least some width, and only one point. If, it is equal to large value, that cannot change.

The L infinity norm that is indeed it turns out to be the case, if you take the power p and you let p to tend infinity. So, let me tell a slightly mode correct definition not slightly mode, this is indeed a very correct definition, but this is when f need not be continuous. So, L infinity consists of the set of all points like this and at 21 point it cannot matter, it cannot become infinity, any way sup here notice that here we are not requiring it to be continuous sup over all t in R or of f of t norm as long as the supremum is less than infinity, as long as the supremum is less than infinity.

You will take all those f and put it inside the set L infinity, what I have written here is less than infinity. So, this is what constitutes the set L infinity, but if you have a f inside L infinity, what is it? Is norm till now for every t it turned out to just be its L p norm and, but for L infinity? It will turn out to be slightly different; we need that essential supremum, which I will come to in the next slide.

(Refer Slide Time 25:14)



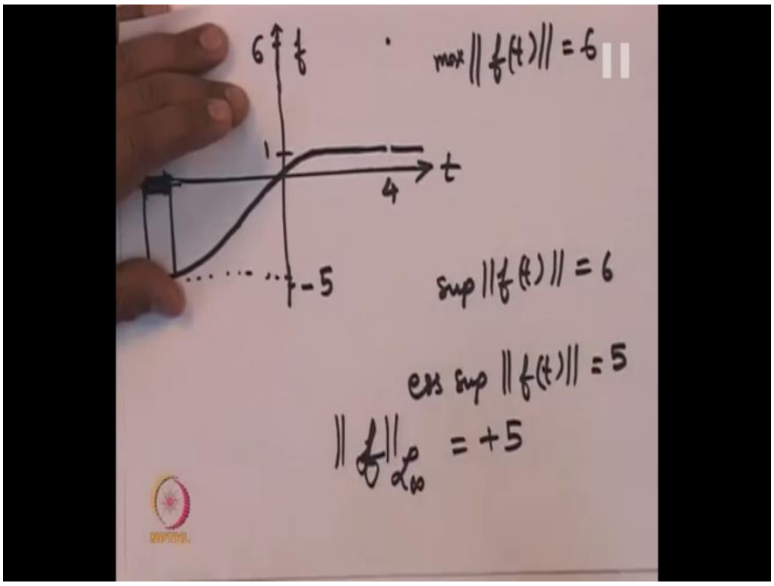
For any f in L p for p greater than or equal to 1, but not equal to infinity, what is it? Is L p norm it is defined as integral from minus infinity to infinity? Why minus infinity to plus infinity, because we wrote the domain of f is equal to R of the norm of f of t. The p th power for any time t f of t is a vector in R m which norm to take in R m that is supposed to write subscript. We have decided to not write why because we going to stick to the same norm for all our arguments. You can the 2 norm, there you are not forced to take the p th norm in R m for that purpose.

So, we will integrate only, when p is less than infinity after taking the p th power, but now that we are defining this as a L p norm. So, this is already a finite, if you are taking the L p th norm. This is already finite; you will take the p th root of this value for any f in L p. We know only that this integral is infinite only because the integral is finite. We have decided to put the f inside the set. Now, you take that f and compute this value.

You know, it is a finite value that we will define after taking the 1 by p th power will be the p th root that is defined as the L p norm of the function f, but for L infinity norms. This we are going to say supremum over all t in r of f of t, f of t is norm, f of t is norm, which norm we will take again. It does not matter as I said, but we prefer writing this essential supremum, what is this essential supremum? This full form stands for essential; this is what I am saying here, is what I studied several years ago.

Thanks to my teacher, Prasad H Banavar for this essential supremum stands for that. This f should have reached this value over at least over at least some small interval. It cannot be equal to this value at just one point. So, this is t f of t and at one point, if it is equal to 5 that is t, equal to 3 and essential supremum will be equal to 1 and not 5. The essential word ensures that this supremum value. There are many values of t close to this supremum value, very close orbitally close and one qualifies for this number, but for f of t equal to 5 f of 3 equal to 5. There is only one point t equal to 3. That is said essential ensures that. So, this for this particular f of t L infinity normal infinity of this particular function turns out to be equal to turns out to be equal to 1. So, let me take a concrete example…

(Refer Slide Time 28:35)



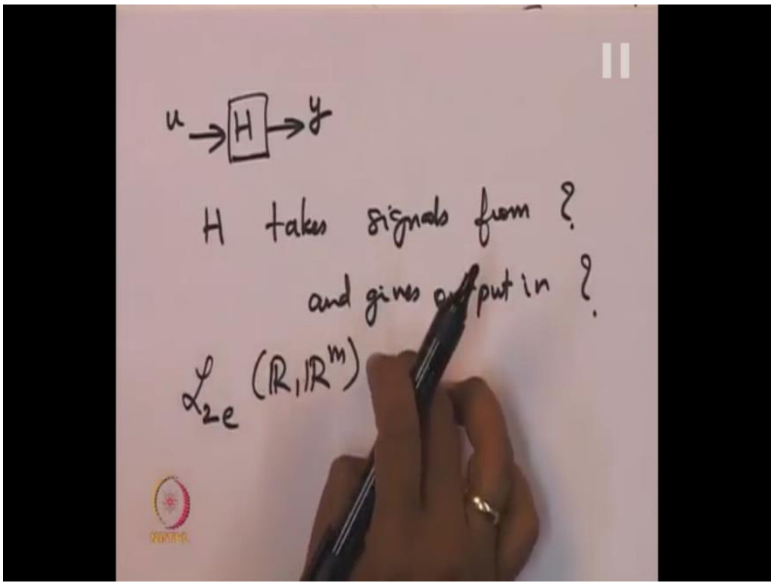
So, here is the function 1 minus 5 at one point is equal to 6 at t equal to 4. So, here is the function f, which is never equal to minus 5, but it seems to be saturating to minus 5 on t 10 into minus infinity. It grows like this for t equal to 4 alone only for t equal to 4;

suddenly it becomes equal to 6. This is where continuity would have helped us, but for t equal to 4, alone into sudden jump to value 6 after that it seems to be saturating to 1.

So, f of t max after taking the absolute value max is equal to 6, sup f of t also equal to 6 sup, and max over mode over t 10 varying from minus infinity to plus infinity, but essential supremum of f of t equal to plus 5 plus 5. We are taking the norm is the essential supremum attained for any value of t, it is converging to minus 5 as t tends to minus infinity. It is never equal to minus 5, for no value of t. It is minus 5 still the essential supremum is equal to 5, because it is for enough values of t to be precise to those who are inclined for a set of measure greater than 0.

For, this length strictly greater than 0 it has come very close to minus 5. So, when you take the norm it will become plus 5. So, that is why the essential supremum indeed is equal to 5. So, f L infinity norm is equal to plus 5 and this will indeed be equal to the value when you take L p th norm of fee of f and let p tend to infinity. That is the best part and if f L continuous then the, then this 6 automatically get ruled off for continuous functions each value is attained over the set of measure non 0 not necessarily attained, but it comes very close to it over a set of measure non 0. That is what, that is how continuity helps these, this is only about norms of signals. Now, we have to also see, we want to think, so all our operators…

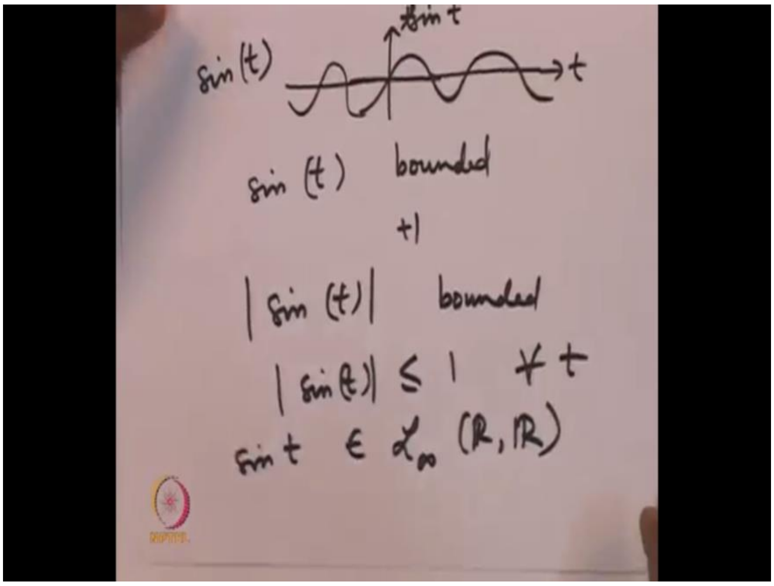
(Refer Slide Time 31:38)



All our systems that take input u output y and that is the operator H. So, H takes signals from where is the question and gives output in are the input and output in L 2, are they in L infinity? That is the question we want to answer for that purpose. It turns out that we have to extend our spaces L 1 L 2 L infinity L t, we all have to be extended.

So, we will quickly see what is the meaning to L 2 e R to R m? This extension is, this e for extension is coming because our systems take signals from not L 2 necessarily, but L 2 e and give you signals possibly in L 2 e. If, H is not stable, we eventually want to say that even though u is in L 2 y might be an L 2 e. That is the purpose that we are going to define this external space as soon as we define this L 2 e space. We will see some examples of, what is an L 2? What is an L infinity? What is L infinity, but not in L 2 etcetera. Through the examples we are going to see now.

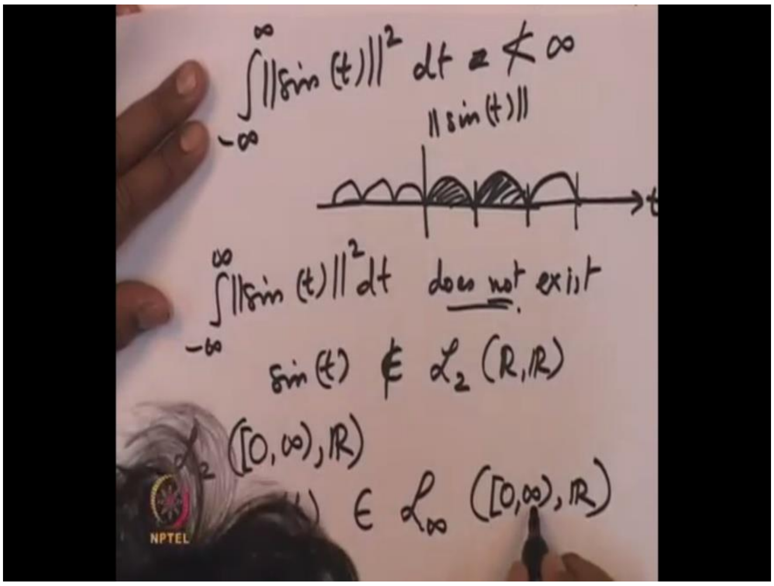
(Refer Slide Time 32:58)



So, consider sin t is, this is how, this is how sin t looks. Is this bounded as t tends to infinity? Is this finite? Yes, many is there a value that is greater than or equal to every value of absolute value of sin t, sin t bounded, what is the maximum value plus 1? What is the minimum value? We are not supposed to see, whether sin t is bounded. We are supposed to whether absolute value of sin t is bounded then luckily that value actual value is also equal to plus 1, sin t is lesser or equal to 1 for all t.

So, that is why sin t is in L infinity, what is the range from R to R at any for any value of t? Sin t gives you a real number of course; it may not be giving all real numbers as it is output as it is range never the less any values, sin t gives is the real numbers. So, sin t is the example of L infinity, is it in L 2? That is what we will see now…

(Refer Slide Time 34:30)

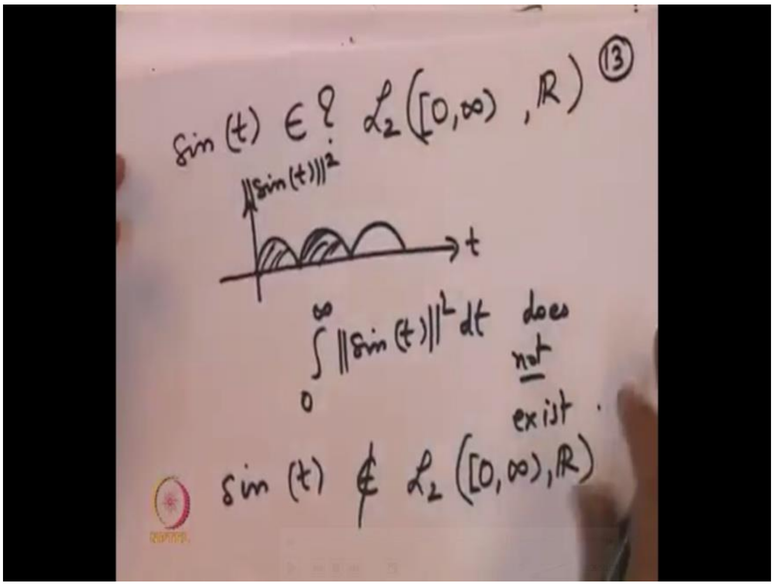


Can we integrate, can we take sin t 2 norm second power d t from minus infinity to infinity. You have sin t, this is absolute value of sin t because sin t is actually a real number. It is a scalar 2 norm, it is Euclidean norm is, norm is nothing but absolute value. So, for each period there some area under this, so over this when you integrate from minus infinity to plus infinity, this is not less than infinity, it becomes unbounded this integral does not exist.

We will say, minus infinity sin t d t square does not exist is of course; in Laymen term we say, the integral is equal to infinity that is not correct mathematical language, will say that integral does not exist. So, we will say, sin t is not that, so we are not allowed to take this function put it into L 2. It is not an L 2, but now we can ask what if it chopped? What if it stopped at some for the purpose of L 2 e? We prefer considering L 2 0 to infinity only R plus 2 R. The same signal sin t, sin t is also in L infinity from 0 to infinity to R, why because over this range? Also it is bounded in fact it is bounding from minus infinity to plus infinity.

So, over a subset also it will of course be bounded. So, this it is inside this, so we are interested in extending this space actually, because we are interested in only future e being chopped future being extended. So, this sin t, which was then L infinity R to R is also in L infinity 0 to infinity to r. So, this sin t unfortunately not in L 2 0 to infinity to R also it is not in that. It is neither in, so we ask a question…

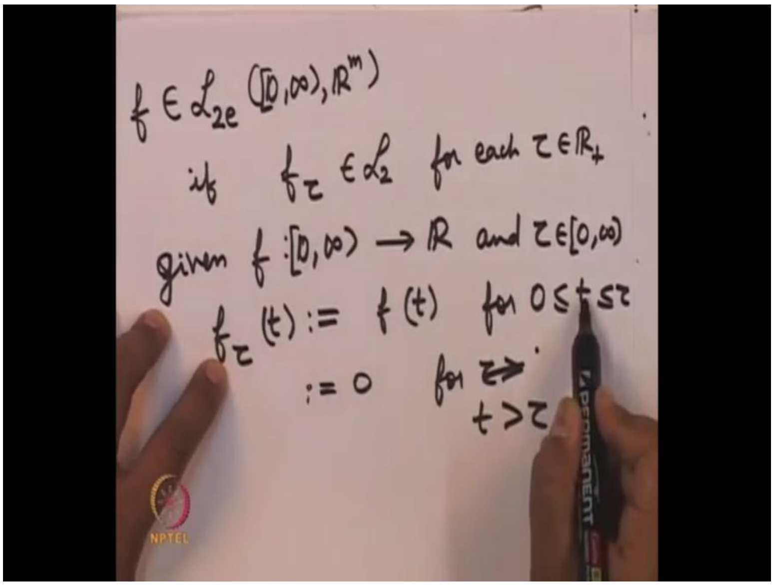
(Refer Slide Time 36:59)



Is sin t question mark? Is it in L 2 R to R? They only answered the question as no, what about 0 to infinity? Answer to this also no, what is this sin t square? Now, we are going to integrate this, the area under this is all for each period, it is some non 0 positive value. When you integrate from 0 to infinity 0 to infinity of sin t square d t, this does not exist, does not exist.

So, we will say, sin t does not belong to L 2 0 to infinity. Of course, if you make on this side of from infinity instead of infinity. If, you will take of finite value, then sin t will belong that brings us to the extension. So, what we are going to do? Is we are going to say that our signals sin t etcetera. Do not really belong from 0 to plus infinity; they are always up to some finite value. We never ask as t tends to plus infinity, what happens whether the norm, whether the total integral is finite or not is not a question, ask for so, for so much in the future. So, what we will do? Is we are going to say…

(Refer Slide Time 38:43)

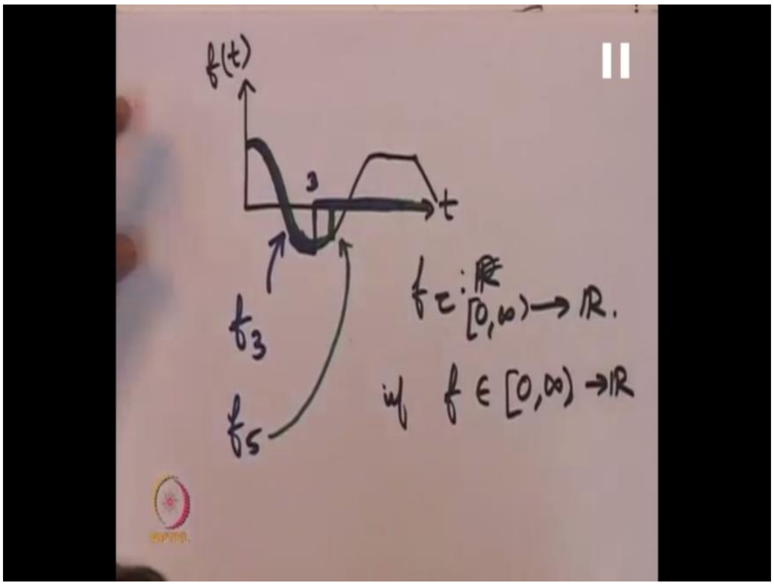


Some f is set to be in L 2 e 0 to infinity to R m. If, f tau, we are going to define what is this f tau is in L 2. For each tau in R plus each tau in positive value only makes sense. Of course the question arises have already defined this f tau without telling, what is I have already defined L 2 e using f tau without telling what is this tau, f tau?

So, given f and we also take a tau also inside this range, it does not make sense to chop at negative values. So, f tau is a new function that is defined as is equal to f of t for 0 lesser or equal to t lesser or equal to tau, and equal to 0 for tau greater than, sorry for t greater than tau. So, what we have done, we have chopped whatever for all values of t greater than tau, we have chopped it to 0, this a definition of the function f tau, f was a already a function from 0 to infinity to R.

Somebody gives us the value tau, some positive value and we have used that value tau to define a new function f tau, f tau is also a function of 0 to infinity to R. It behaves like f as long as t is less or equal to tau for t is strictly greater than tau; it is just equal to 0. So, we will see a graph of such a function.

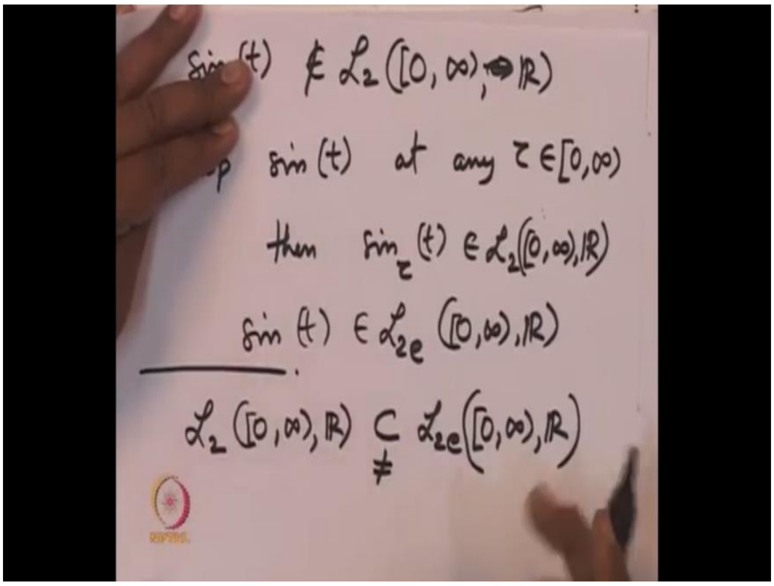
(Refer Slide Time 40:49)



So, for the purpose of this decision, we are considering f define only for non negative values of t, this how f of t looks. Suppose, let me take a different color pen. So, here is another this is for this blue 1 is equal to f 3, where we have assumed that we have chopped it at tau is equal to 3 again speak of f 5 as f 5. This is new function that is in green that gets chopped a little further from here onwards it becomes 0. This is a green 1, so we can chop it at different values of tau and once it is chopped, it is sent to 0 that chopped version is, what we call f tau.

So, f tau is also an element of from 0 to infinity function on 0 to infinity to R. If, f is in 0 to infinity to R, this is a meaning of a chopped signal. Now, we have concerned that there are many signals, which are in L infinity, but not in L 2. There are various types of signals, which are not in L 2 perhaps the chopped versions are that is our next thing that is what we see in details now.

(Refer Slide Time 42:39)



Consider sign of t, again this is not in L 2. We have already saw that it is not in L 2, which from this range from this to R. This is how it is called the domain co domain sin t is not in this why because it has, what we call infinite energy over this ranges, but for any for any chop. Here, chop sin t at any, at any tau in that range or whatever range it is defined chopped sin t at then again, then sin t chopped. We know that that is in 1 2 sin t is not too bad as soon as we chop it to any value of tau it comes into L 2 the chopped 1. So, we will see sin t though it is not in L 2, it is in L 2 e.

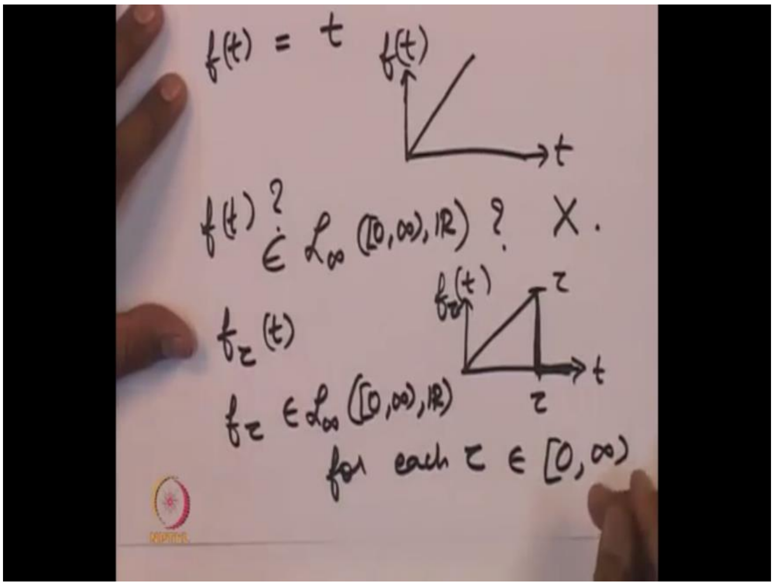
So, how is this? In extension sin t is already in L 2, but for every chopped version of sin t for every tau you chop it and it is in L 2. So, why do not we put it in not L 2, but L 2 e will extend the space L 2 to L 2 e where for every chop the function is in L 2. So, this is recall our definition, this is how we define. So, this is a 14 th slide, so we will say, f is in L 2 e from this domain to this co domain. If f chopped is in L 2, but for each tau, it is not that you chop it at some carefully designed tau. You can always chop it at tau equal to 0 and always come back to L 2. It will become a 0 signal no for each tau in R plus, you chop and it comes into L 2.

Then, you see that function f was not too bad, you will put into L 2 extended. So, one should note that what is not obvious? Is that this is genuinely an extension? What is it mean to be an extension L 2 0 to infinity? The extension word is justified because of this property L 2 e, if without chopping already the function was square integrable. Of course in for every chop also it will be in integrable that is why this one is contain inside this.

We already have an example to say that 2 sets are not equal sin t is the example for what is here, but not here.

Hence, this set is a strictly largest set of this in other words the proper subset of this is how extension is defined. Let us see some more example of L infinity extended L 1 extended etcetera.

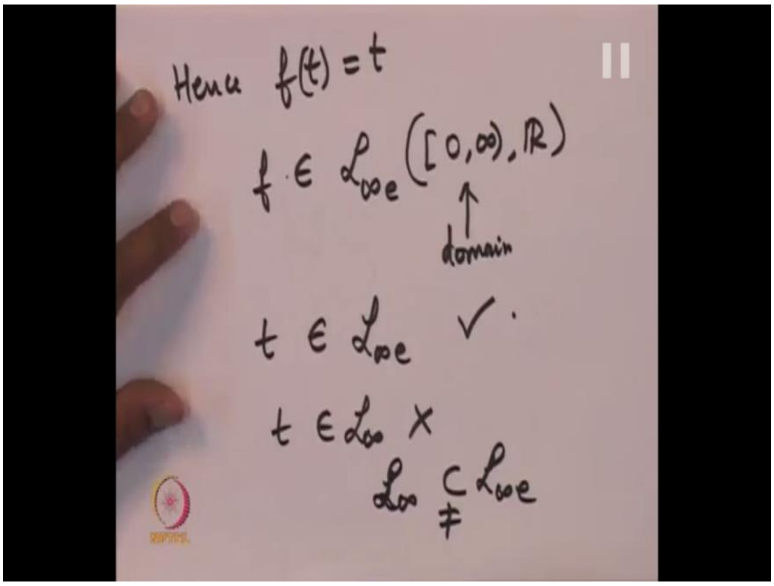
(Refer Slide Time 45:43)



So, consider f of t equal to t w. Here, we are interested in only for positive value of t. This is our so call ramp, this is a ramp signal. It is, if this f of t is it in L infinity answer is no, answer is no. It is not even bounded, it goes on becoming very large, but you chop f of f to any value of tau for whatever value you chop any f tau looks like. This f tau is also again function of t tau is some value is here, it goes on increasing till there, from there on it is 0 for each tau.

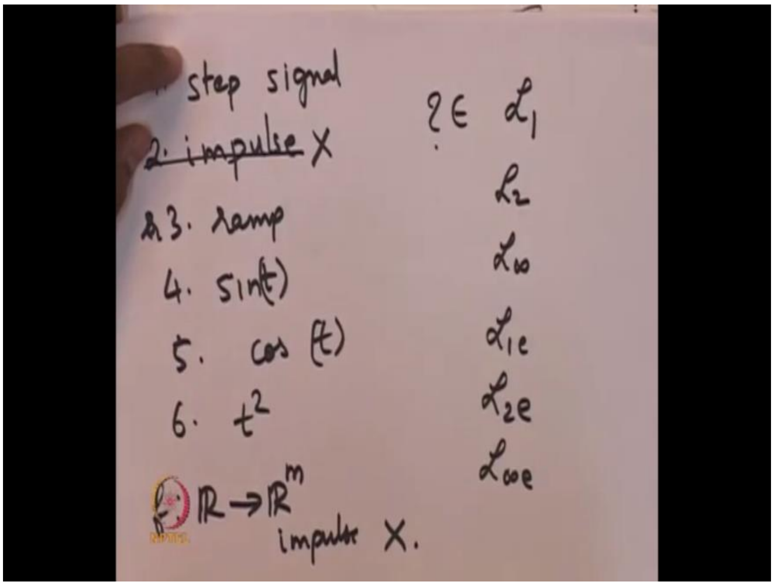
It is bounded, what is a value maximum value is tau? So, for each fixed tau for each finite value of tau it is bounded. Is it uniformly bounded? No, the bound depends on tau in that sense it is not uniformly bounded in tau. So, this f tau is in L infinity. Thank God, for each, for each tau in hence we will say, f is in L infinity e L infinity extended.

(Refer Slide Time 47:15)



Hence, f of t equal to t f is in L infinity extended, what is the domain and co domain? So, for now on anyway this is always going to be our domain and this co domain is often clear from the context. So, we will just say L infinity e, if the function is in L infinity e, but is t in L infinity. No, this is yes. So, it is not bounded what about this also shows again that L infinity is also a proper subset of L infinity e. So, we need this extended class of signal, all our signals live in this space extended spaces.

(Refer Slide Time 48:20)

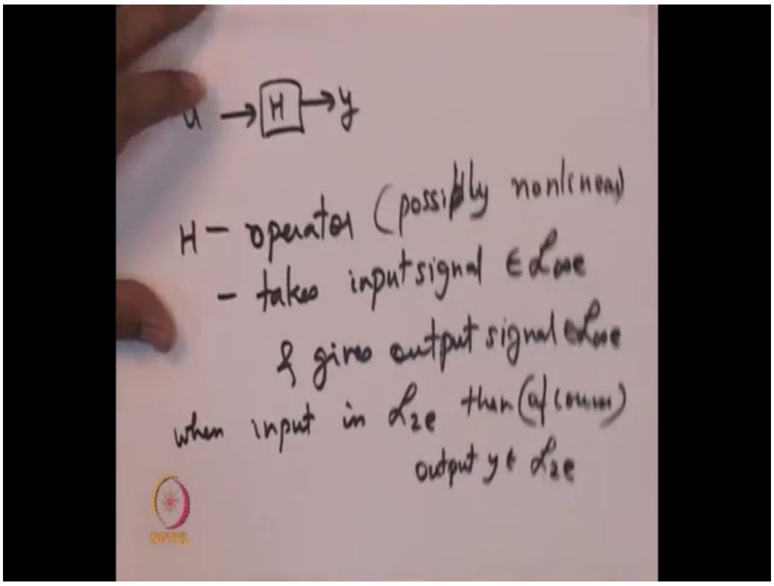


What about step input, step signal impulse ramp sin t 5 sin of t cos t sin t cos t are expected to not be different what about t square. So, these are all in watt check whether they are in L 1 L 2 L infinity L 1 e L 2 e L infinity. This is a extremely good exercise to try one self for each of these signals.

Fine, these are important signals step signal ramp sin t cos t square impulse is not a function, no need to check of this for here because impulse is not a till now we had been saying f is a map from signal itself Lebesgue integrable function. All over f sets functions is impulse of this type impulse. No, so please do not check for this if somebody says, is the impulse in L 1. It is not even a function is it in L 2. It is not a function, no question of checking whether impulse is in any of this sets because it is not even a function strictly speaking it is what we call a distribution.

Thanks to Shawns, who made this theory very precise impulse is not a function, it is a distribution. Of course impulse is a limit, is a limit of a sequence of functions, sequence of function each of which has area, under it is 1 an it is not equal to 0 for a smaller and smaller thinner and thinner interval around 0. It could be the limit of a sequence of function, but itself upon mode, upon the limit itself is the impulse distribution not a function.

(Refer Slide Time 50:40)

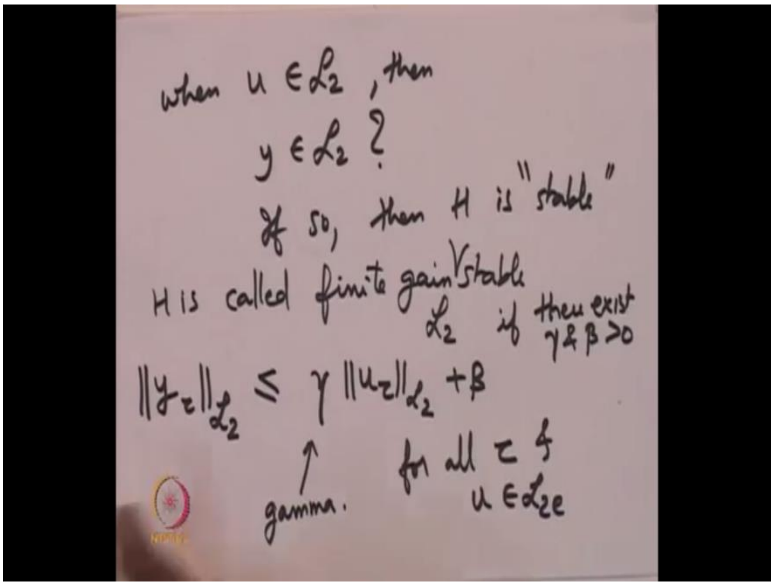


So, this brings us to operators our H, we want to ask, is this h stable? Is the magnification it causes to the input to give you output? Is the magnification bounded by some value that bound magnification for that we need these L 2 these norms. It is with respect with these norms speak about the bound.

So, H operator takes input signal and gives output. This input and output live where they live in some such extended space. You can say, L infinity extended an output also L infinity extended these extended spaces themselves, do not have a 2 norm. They do not have a infinity norm.

The operator possibly non-linear of course; we will see, what all these means? Possibly non-linear, what this all means in the case of linear system of course something, we will see in much detail. Now, we are going to ask when input in L 2 e then of course, then of course; output y in L 2 e. This is not too surprising, but what is good about?

(Refer Slide Time 52:29)



What we will define as stable systems is when u in L 2, then y in L 2 question mark. So, then H is, what we will call stable. Of course L is in a very general sense. This is input output stable, this is not Lyapunov stable. We are studied Lyapunov stable there, it was for auto normal systems, and here we are speaking of input output stability. So, this is the question we are going to ask in a very loose sense. We will say, if the u was in L 2 for every u in L 2, if the output is also in L 2. Then, in some sense we will say, H is good, y because in general u in L 2 e goes to while in L 2 e. If, you restrict the input to L 2 from L 2 e, when you restrict it to L 2 that does not mean y will also get very conveniently disrupted from L 2 e to L 2, but if that happens, then H is kind of good, then H is stable.

So, we will stop today's lecture with the precised definition of stable. So, H is called finite gain stable, finite gain stable that refers to L 2 stable. Let us say these introduces L 2 stable, if y tau L 2 is lesser or equal to some gamma. This is gamma u tau L 2 plus beta for all tau and u in L 2 e. What are this gamma and beta? If, there exists gamma and beta greater than 0 such that is inequality is true for every chopping tau and for every u in L 2 e. For every signal u that you give us input the corresponding output is bounded by such a quantity, in which the gamma and beta are some 2 positive numbers that cannot be changed when you take different tau in u.

Please notice the sequence of arguments; we will call H as finite gain L 2 stable. If, you can find 2 numbers gamma and beta both greater than 0 such that this inequality is true for every tau and for every u in L 2 e. So, this is easy to confuse this gamma. Yes, why is this gamma, while this is y. So, this is the definition of finite gain L 2 stable notice that of course gamma and beta are clearly not unique, if you have found one gamma and beta.

You can take a larger gamma and larger beta and still is in equality will be satisfied because each of these quantities are positive, this quantity is positive, this quantity is positive. You multiply them by a larger number and still this inequality will be satisfied. So, clearly gamma and beta are not unique, so greater than 0 as soon as, as soon as such a finite gamma and beta exists. You will say, this H is finite gain L 2 stable. So, in our next lecture, we will see what this means for the case that H is a linear system. That is the next thing, we will see in our next lecture. This ends today's lecture.