Nonlinear Dynamical Systems

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Lecture - 18
Positive Real Lemma Proof

So, in the last lecture I had started talking about positive real lemma, now I did not give the complete proof of the positive real lemma. In fact, what I had said was when you have the positive real lemma, it gives if and only if conditions, one set of conditions is in terms of matrices of from stage page representation of the system. The other set of conditions is a frequency domain condition, which deals with the fact that given transfer function is a positive real and things like that.

What I showed in the last lecture was that when the set of conditions that is satisfied by the matrices when that set of conditions are satisfied. Then, passivity takes place in the sense that I showed I mean we already had discussed in last class this thing about assisted being passive is equivalent to the existence of a storage function. The supply minus the rate of change of the storage is equal to dissipation, which is the strictly positive function.

In the last lecture, I showed that when the matrix conditions are satisfied, then they do get some other matrix which is positive definite and that actually stands for the dissipation function. So, today what I will do is I will start first with positive real lemma and I will give the complete rules of the positive real lemma and then we will carry forward with the rest of stuff, so let me recall the statement of the positive real lemma.

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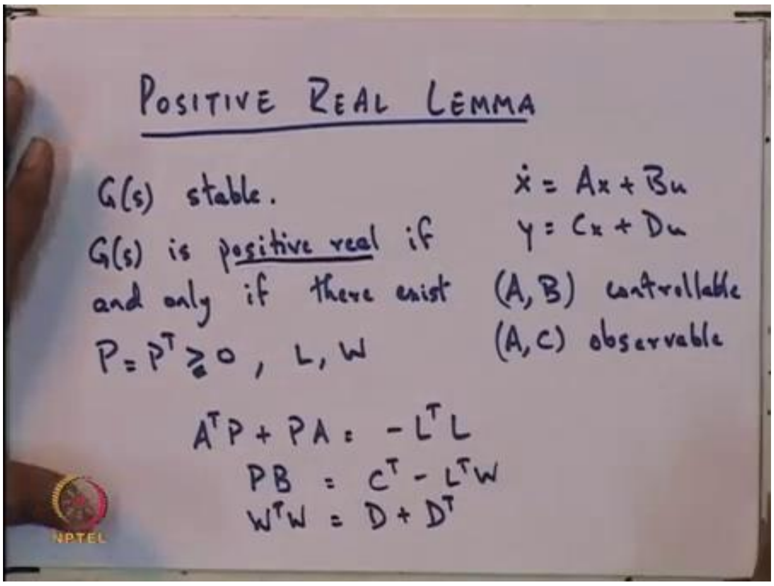

So, the positive real lemma, so let G s be a transfer function and let us assume that there is a minimal state realization of this transfer function, which is given by x dot equal to a x plus b uy equal to c x plus d u. Of course, because it is a minimal state representation, what that means is A B is controllable and A C is observable, now further we assume that this transfer function this stable. So, G s is stable, then the statement says that G s is positive real if and only if there exist there exist three matrices.

The first one P is a symmetric positive semi definite matrix, in fact most of the times we talking about positive matrix the symmetric positive matrix P and two other matrices L and w such that the following three equations are satisfied. The equations are a transpose P plus P a is equal to minus L transpose L the statement essentially says that using this positive definite matrix P and the state matrix say you end up with this Lyapunov equation. The resulting thing is something which is negative semi definite, now because L transpose L is going to be positive semi definite and the minus sign will make it negative semi definite.

Then, P B be in this matrix b from the statement representation is equal to C transpose minus L transpose w and the last equation is w transpose w is equal to d plus d transpose. So, of course there is also this, what we mean by positive real and what I had said in the earlier lecture is that.

For the time being, at least we will we will consider function to be positive real if a Nyquist plot of that particular transfer function lies in the first and the fourth quadrant. That means the real part the real part of the Nyquist plot is always positive and I had also said in the last lecture that of course this definition of positive real may not necessarily be the definition that you see in all the books of in the definition of positive real includes the fact that it is already stable and so on. What does inter cases are I will come to that a bit later and I will explain and why I mean of course, there is no agreement as to what exactly the definition of positive real is.

I will try and explain so that it is clear to you what the various notions of positive real that exist are and how they all related and within the epsilon neighborhood of one another. Now, in the last class in the last lecture, what I did show is the following that suppose we assume that there is A G s and we assume this is the minimal state representation. We assume that there are matrices P and L and w such that these equations are satisfied. Then, what it means is that the transfer function results in existing system which is passive, now the fact that passive is equivalent to positive real is something that I have talked about, but is not being completely proved.

So, in some sense, what I showed yesterday along with what I will show today that means these two conditions that this is equivalent to this should also prove that when you have either these matrix conditions or the fact that something is positive real satisfied. Then, you have a passive system, now let me begin the proof, so in the beginning let me assume, so I will prove this way, that means I will assume that there exist P L and w such that these equations are satisfied.

I will show that G s is stable and its positive real that means Nyquist plot is in first and the fourth quadrants. Now, if you look at this first equation this is the Lyapunov equation, now what we saying is using this a you write down the Lyapunov equation and with a positive definite matrix P, you are ending up with something which is negative semi definite. Now, it is well known that this is only possible if the matrix a is Horvitz, so if the matrix a is Horvitz, then the resulting transfer function is stable, so the first equation already shows us that G s is stable.

So, all that we have to do further is show that the transfer function that you get is positive real that means its Nyquist plot lies in the first and the fourth quadrants, so let us start trying to do that, so for that let me first write down what G s is.

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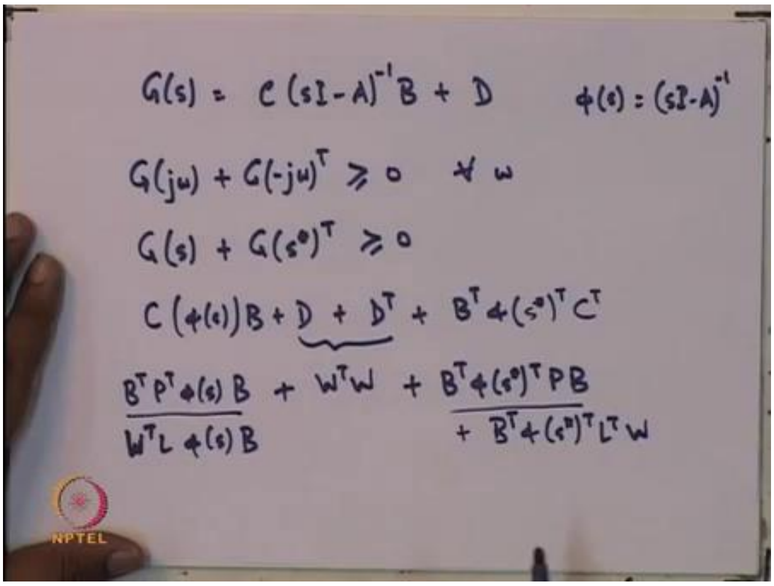

So, G s is really c s I minus A inverse b plus d, let me use the symbol phi of s for s I minus A inverse. Now, what I want to show is the following that g of g omega plus g of minus j omega transpose is greater than equal to 0 for all omega. If I show this I would have shown it for the Nyquist plot, but in fact what I am going to do is something more general, then what I am going to show is g of s plus g of s star transpose is greater than equal to 0. This is roughly what I would be showing, now how to show this, so I will write down the expression for this and I will write down the expression for this, but I will use phi of s instead of writing s I minus a.

So, this expression is c phi of s b plus d plus this expression will give me d transpose plus b transpose phi A star transpose phi S star transpose c transpose and now what I will do is I will make use of some equations that we already we have in positive real lemma. So, in the positive real lemma, we see the d plus d transpose w transpose w and we assuming these things are satisfied. So, for d plus d transpose I will substitute by w transpose w, so for this one I can write w transpose w and there are these other two terms. What I would do is I will use this particular second equation for c transpose, I will substitute P b minus L transpose w.

So, if I do that this particular expression becomes b transpose phi s star transpose times P b that is one expression plus I will have a have one more expression, which will have b transpose phi s star transpose L transpose w. Now, similarly, just like what we did for c transpose I can substitute for c it will be the transposes. So, these two guys will appear as transposes here, but instead of phi s star transpose I will have phi s for those two terms.

So, maybe I should just write them down, so I will get another term B transpose P transpose phi s band one more term which is W transpose L 5 s b. So, I get these phi terms, now out of these phi terms let me concentrate, so there are these phi terms that appeared. Let me concentrate just on this one and this one, so the let the other three terms be as they are I will just concentrate on these two.

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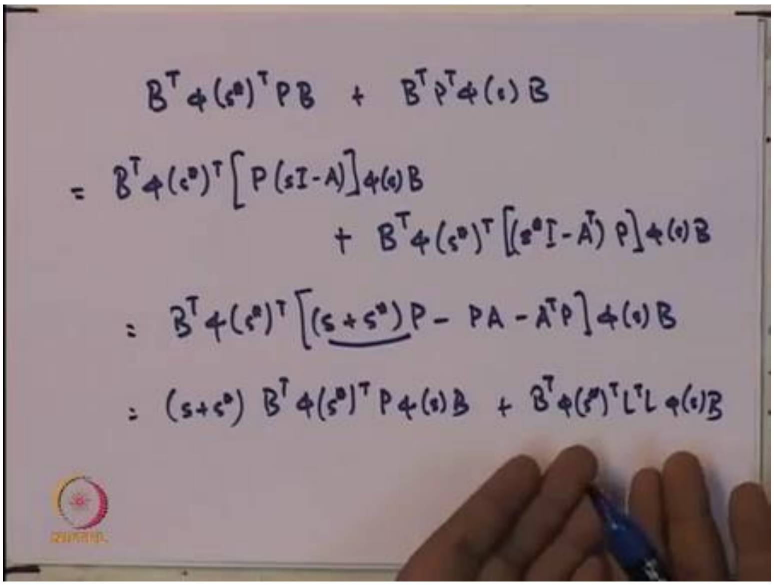

So, just concentrating on those two, I have B transpose phi s star transpose P b plus b transpose P transpose phi s b, now I am going to do some simplification of this. So, what I would do is this particular expression, I can write it down in the following way b transpose phi s star transpose and now inside I introduce p times s I minus a and remember that we have already said that this phi of s is the inverse of s I minus a.

So, these two are really inverses, so I am effectively writing this down, but I have written the first three terms down here and then these two actually cancel followed by b and I do the same kind of thing for the other term also. So, sorry probably I should not have done it for this one, but I should have done for this one I guess phi s, no it does not matter.

Now, for this one I can write plus, so that one is same as this one plus here I write down b transpose I have the phi s transpose and then I have s star I minus A transpose, which is really the inverse of this. So, these two can cancel then and then I have p, so this p transpose because P is asymmetry matrix transpose is a same as P, so I am just putting p here and then I have phi s b.

Now, if you look at both the terms, both the terms have b transpose phi s star transpose in the left side and phi s b in the right side. So, what is inside can just put in together and so then what you have is b transpose phi s star transpose. Now, putting the things inside together, you gets plus s star times p minus P a minus a transpose P times phi s times b. Now, we again go back to the positive real lemma the first equation in the positive real lemma say that a transpose P plus P a is minus L transpose L. So, we can substitute that in there, so if you substitute that in there, then this particular expression can be written as this particular term will give me s plus s star these are just scalars.

So, I can pull them out b transpose phi s star transpose P phi s b, so I have just used up this much and then the other portion. So, P a plus a transpose p from that first equation that should be minus L transpose L, so I will substitute that minus L transpose L and therefore, with b transpose phi s star transpose L transpose L phi s b. So, these two terms we picked up and we end up with these two terms, this is a term which is sort of symmetric if you like, but multiplied by s plus s star and this is also something which is symmetric, but what you have is that L transpose L.

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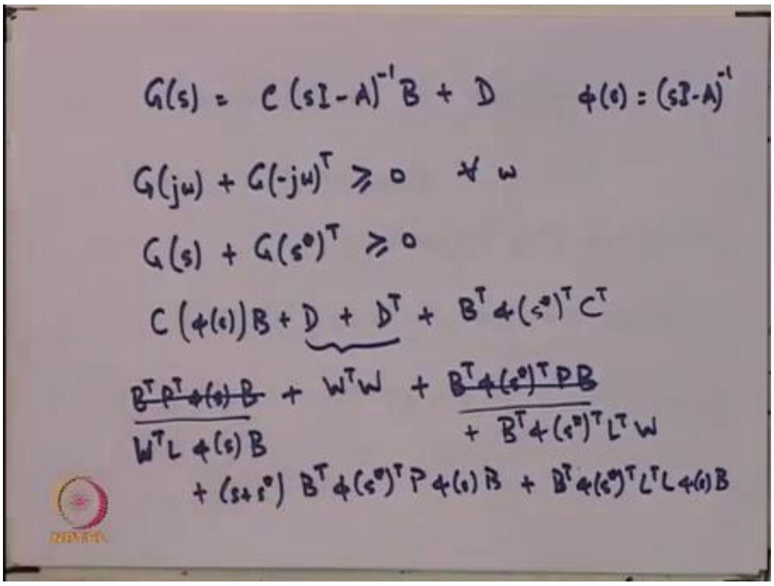

Now, let us go back to that previous thing where you had these phi terms and we picked up these two terms and did manipulations to get what we have there. So, what will I do is into this slide, I will cancel these two terms and I will add the two terms that we have got. Now, the two terms that we have got now are s plus s star times b transpose phi s star transpose p phi s b, that is one term.

The other term is b transpose phi s star transpose L transpose L l transpose L phi s d, now let us forget this particular, let me now not think about this one term. Let us look at the other four terms and if you look at the other four terms you see this w transpose appearing in two of the terms and w appearing in two of the terms. So, you can write this as a sum of squares, let me just write that as the sum of square in a fresh slide.

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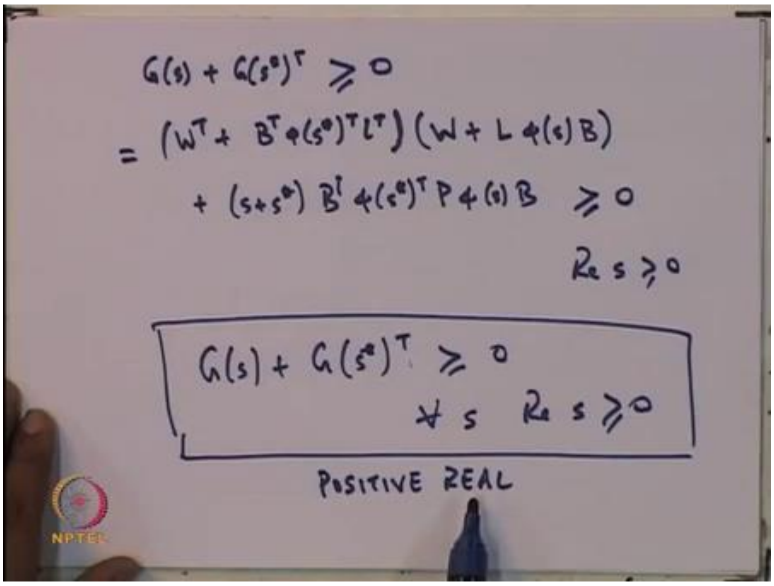

So, as the sum of squares what you get is W transpose plus B transpose phi s star transpose L transpose multiplying W plus L phi s b. So, if you multiply this out you will get w transpose W transpose L phi s b transfer phi s star L transpose W and then b transpose phi s star L transpose L phi s b. So, you would have got this, this, this and this these four terms, now these four terms and then the one other thing that we have is s plus s star b transpose phi s star transpose P phi s b.

So, this is the full expression that you will get and this is the expression mind you when you started out with g s plus g s star transpose this is equal to this whole expression. Now, if you look at this whole expression this really a square, so if this is a square this will always going to be positive and if you look here p the assumption was that P is a positive definite matrix.

So, whatever is this thing this is something acting on a positive definite matrix and as a result what you have here is something positive and if you assume that s is such that the real part of s is greater than 0, then s plus s star this is going to be positive. So, this whole thing is going to be positive, so what we can conclude therefore, is this is greater than equal to 0. So, effectively we have shown that this is greater than equal to 0, so what we have done?

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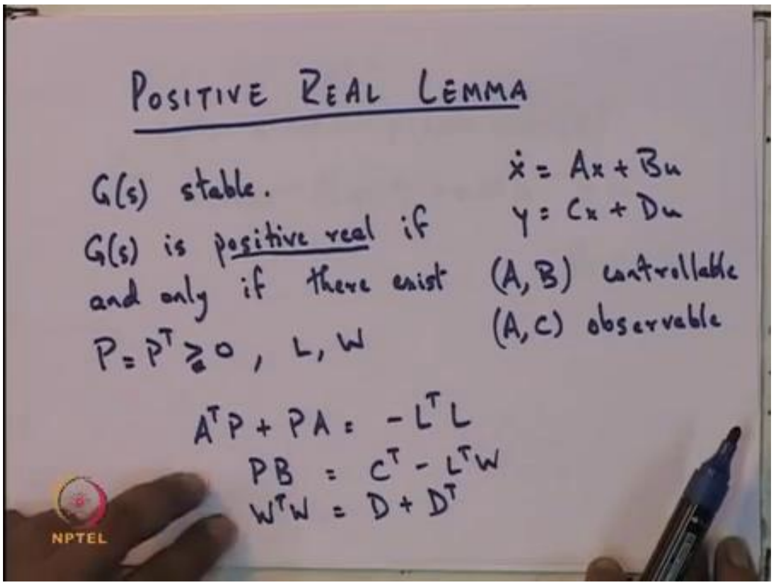

Just now, we have started in the positive real lemma we have started with this assumption and we have gone ahead and shown that this G s is positive real. In fact, what we have shown is whatever I have been using as the definition of positive real, it is a something slightly more than that. So, what we have really shown is if you look at this slide, it is clear what it is that we have really shown, we have shown that G s plus G s star transpose is greater than equal to 0 for all s such that the real part of s is greater than 0. In fact, this should ideally be taken has the definition of positive real, the definition of positive real, in fact rather than use S greater than equal to rather.

Then, use S in the imaginary axis, which means you are looking at whether the Nyquist plot is in the first and the fourth quadrant. Then, use that is this definition that means what we are really saying is that the whole of the right half plane should map under this map G s plus G s star transpose. When you are taking this map from the complex plane to the complex plane, a whole of the right half plane should map to something which is in the first and the fourth quadrant. In that case you can call it positive really a and so if you use this definition of positive reality, then in the positive real lemma in the original statement, you do not have to insist that G s is stable because it is it has to be stable if this condition has to be satisfied.

You know these are season that is no general agreement about what is the exact definition. So, we will leave it at that so what we have now effectively shown is one way of this argument that means assuming that these equations are satisfied. We have shown that this, now we want to show the other way that means if you assume G s is stable, then G s is positive real you want to show that these equations are satisfied. Now, in order to do this, I will have to invoke some other generic theorems that that are known, none of them is the spectral factorization theorem.

Now, as we go along, I would talk about this spectral factorization theorem, this spectral factorization theorem is a theorem which plays a central role in other fields also not just in control theory. So, that is something that we require and I would also invoke a lot of a things that we know about realization theory, that means given a transfer function how do you realize state space representation, so let me start.

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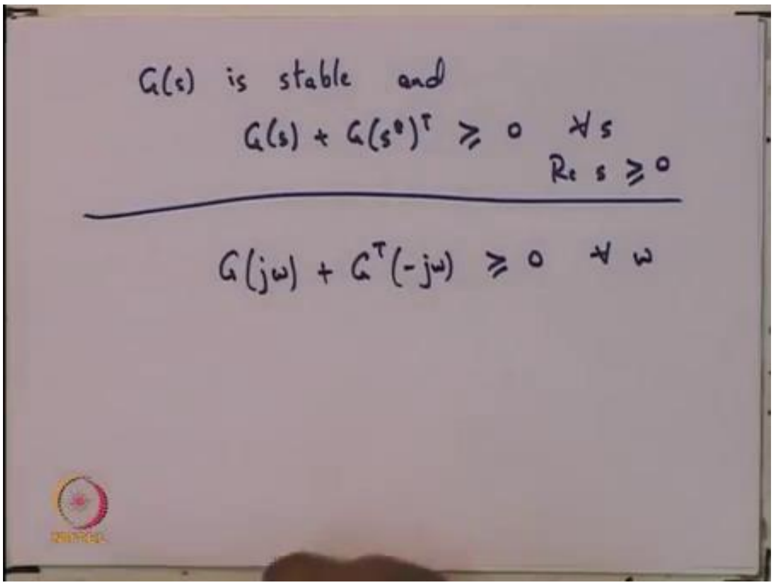

So, we are assuming G s is stable and G s plus G s star transpose is greater, then equal to 0 for all s such that the real part of sis greater than 0 greater than equal to 0. So, this is our assumption, let us look at this particular thing, now what this means is if you specifically evaluate this particular matrix on the imaginary axis.

Then, what this tells us is j g j omega plus g transpose minus j omega is greater than equal to 0 for all omega, now what this means is that evaluated along the imaginary axis, this particular quantity on the left hand side is greater than equal to 0.

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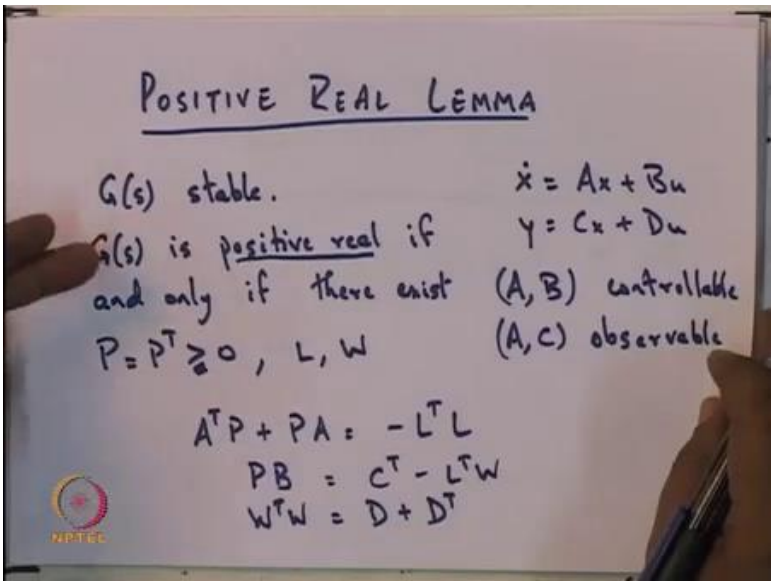

Of course, even though I have only been talking about transfer functions these things also hold for matrices of even though I stated this positive real lemma, but before that I was only talking about transfer Functions which are single input single output. We can carry out the I mean the positive real lemma as it stands as valid even for G s is which are not single input single output they only constrain is that this G s must be multiple input multiple output. So, they are square matrices the number of inputs is equal to the number of outputs and G s is positive real, well there is a definition of positive reality for the matrices which I have not given, but I will give as soon we finish the complete proof of positive real.

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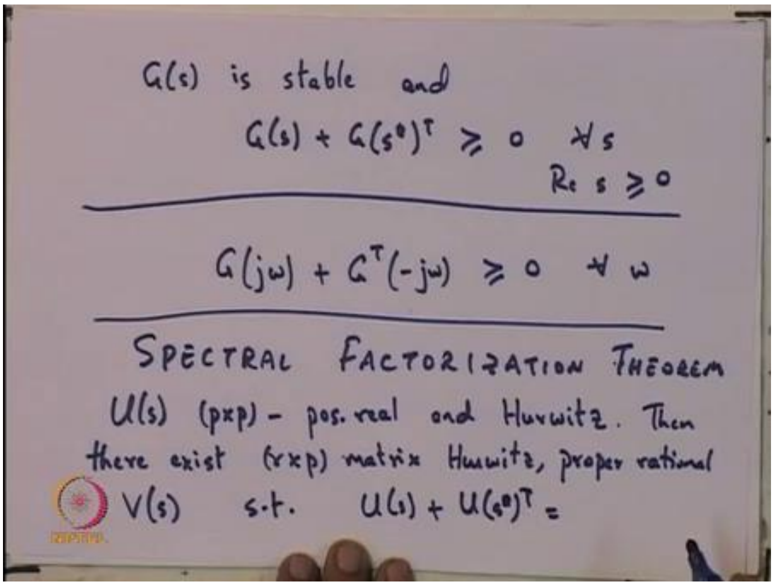

So, whatever I have been showing holds even for the matrix case it is not just for this scalars case even though earlier I have only talked about this scalar situation, the single input single output situation. You could also look at the multi input multi output situation and exactly the same kind of proofs goes through. So, the exact definition of what positive real is formula I input multi output matrices transfer functions, which have matrices I have not yet given. The definition is such that it simplifies to exactly the definition that we had, I mean the kind of definition that we had about the Nyquist plot that kind of definition or about the right half plane mapping to the first and the fourth quadrant.

Well, the same kind of thing does appear, but that I would give just after finishing proof of this positive real lemma. So, let us finish the proof for the positive real lemma, so coming back here, this is what it means and at this point I would invoke the spectral factorization theorem. So, what does the spectrum factorization theorem say, well the spectral factorization theorem says the following. Suppose, you have us, which is let us assume this u s is a P cross P matrix which is positive real, what we mean by a positive real matrix is something that I have not yet defined.

I will define just after this after the after the complete proof of the positive real lemma, but just assume I mean you could listen to this portion by just thinking of u s as a scalar single input single output case, that would be good enough.

So, you assume u s is a P cross P positive real and Hurwitz, that means stable then there exist there exists r cross P matrix which is an r cross P matrix which is Hurwitz it is proper ration. Also, then there exists r cross P matrix that is Hurwitz proper rational, let me call it v of s such that u of s plus u s star transpose is equal to, sorry perhaps I will just use in a new sheet.

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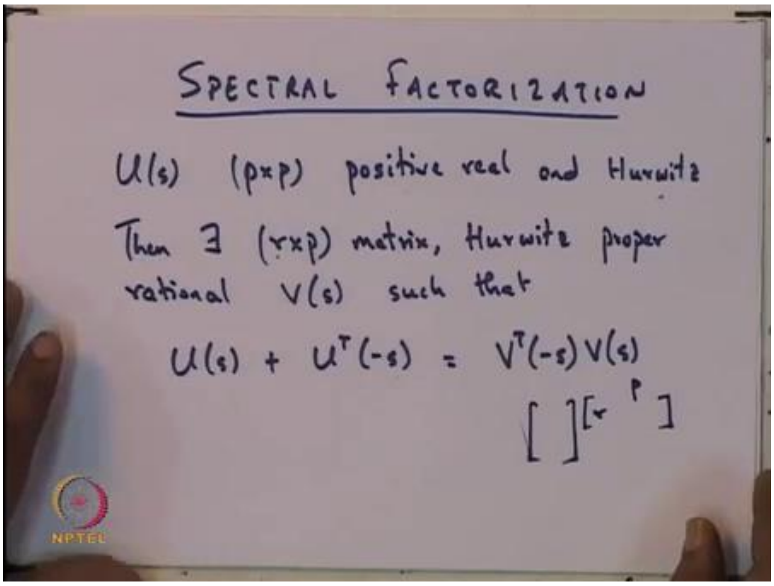

So, we are talking about spectral factorization, so suppose you have a u of s which is a P cross p positive real and Hurwitz matrix, then there exist and r cross P matrix, which is Hurwitz that means stable proper rational proper rational. Let us call this matrix v of s such that u of s plus u transpose of minus s is equal to v transpose of minus s times v of s. so, let me explain what is going on here, so we are talking about u of s, which is a p cross P positive real matrix positive real.

Well, I will give you the definition shortly, but it is a positive real u. So, I mean instead of this P cross P 1 could just think of 1 cross 1 and so this is just a transfer function and it is positive real Hurwitz, then there exist a R cross P well this R cross P. This is under P cross P this only appears in multi input multi output in single input single output of course, this r will be al P is 1 and R. Therefore, we want 1 cross 1, there exists a matrix which is Hurwitz proper rational call it v of s such that this matrix u of s plus u transpose of minus s.

The sum of this two matrices is equal to the product of v s with v transpose of minus s and the r comes in the matrix case essential because when you add these two matrices there are P cross P square matrices. It has a certain rank and that rank is this R and therefore this matrix v is not really a square matrix, but it is, so v is more like this. So, V transpose is like that, so the product of these two matrices is a square matrix, so this has P columns and r rows that is what v s is.

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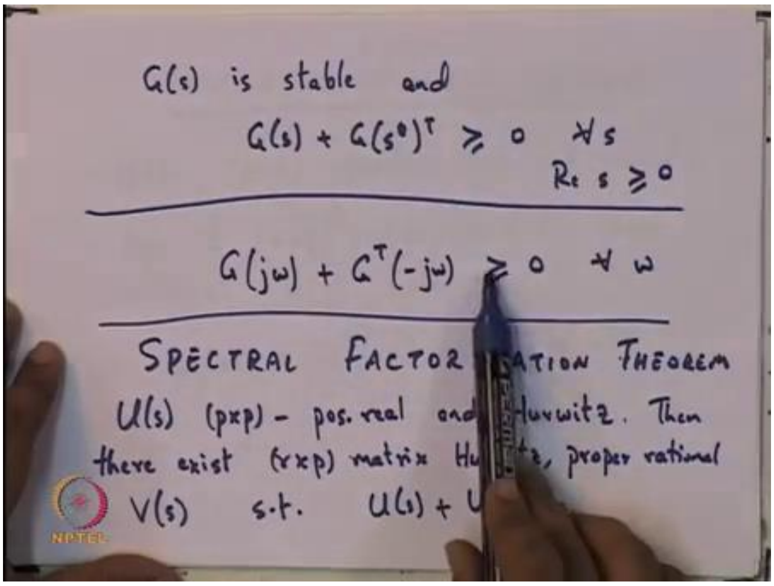

Now, if you see earlier we had said g sis stable and therefore, this thing holds, but you see g of j omega plus g transpose of minus j omega. So, if look at the imaginary axis you say that in the imaginary axis its positive definite if you think of this the matrices, it is positive definite matrix.

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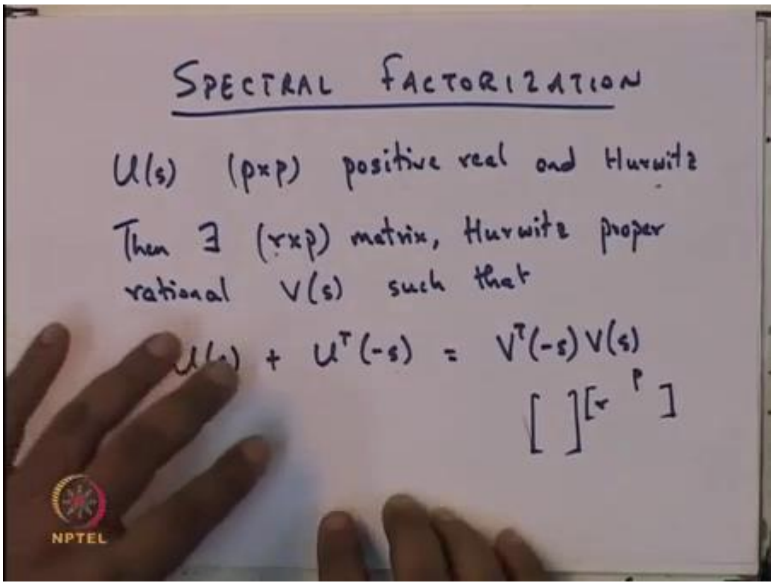

So, what would happen is when you look at this sum this sum will have roots which are on the left half plane and roots which are on the right half plane. Then, this b transpose minus s times v s is constructed by using all in this sum using all the roots which are in the left half plane you use that to somehow construct this v s. Then, the v transpose minus s comes automatically because of some symmetry which exists in this sum. Now, the spectral factorization is of course a very well known result and used widely for example, in communication theory. We will invoke this, so because we have started out with this G s which is stable and has this property, therefore now we can use this and invoke these spectral factorization theorem.

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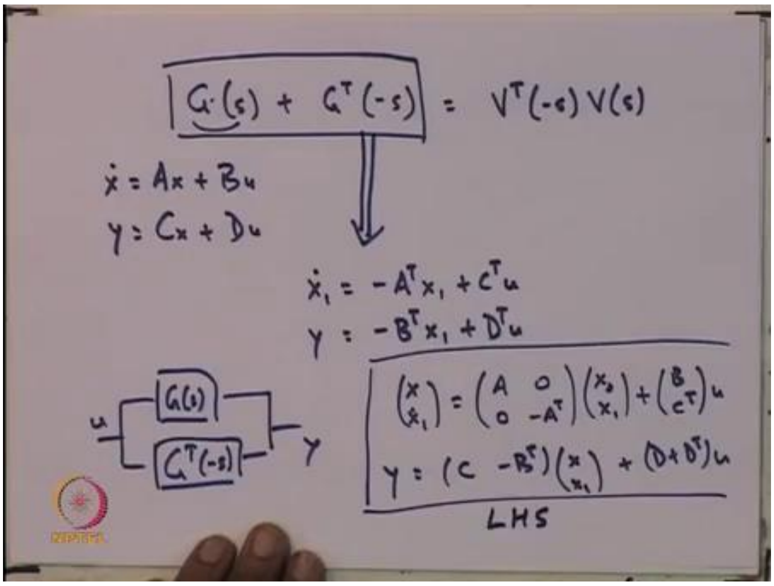

By invoking the spectral factorization theorem, we can say that G of s plus g transpose of minus s is equal to, let me just continue calling this V transpose minus s into v of s. now, what I am going to do is I am going to start off by looking at state representations of each of these matrices. Now, if you look at g of s we have already seen what the state representation of this g of s is, so it was x dot equal to a x plus b u y equal to c x plus d u. Now, if this is the state representation for g of s, then from here we can get the state representation of g transpose of minus s.

The state representation for g transpose of minus s would be let me call this states here x 1, so x 1 dot is going to be a minus a transpose x 1 plus c transpose u and y is going to be minus b transpose x 1 plus d transpose u. Now, here we have G s plus g transpose minus s, so it is as if these two transfer functions, they are adding up which is like, so there are two systems which are parallel and we you could think of them as this is g of s is g transpose of minus s.

Out here, you give u out here you get y, so if you now decide to look at the state representation of the full transfer function that you have on the left hand side, then that is given by x x one dot equal to a 0, 0 minus a transpose x 1, sorry x, x 1. What you have here would be b c transpose u and y will be given by c minus b transpose x, x 1 plus d plus d transpose u y. What we have got here is the state representation this is the minimal state because we have said that this is the minimal state representation for G s.

Therefore, this is a minimal state representation for G transpose minus s and therefore, this is a minimal state representation for what is on the left hand side. So, let me just call the this left hand side, so what we have is a minimal state representation for the left hand side in the same way let us consider a minimal state representation for the right hand side.

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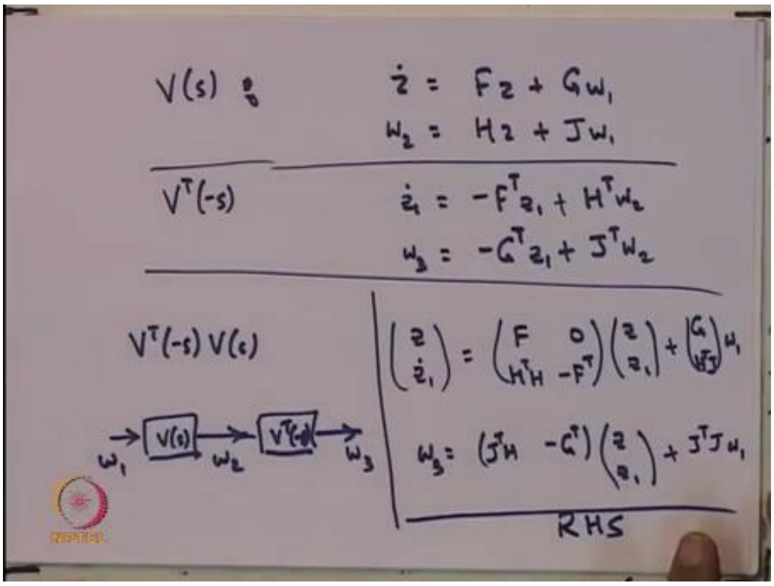

So, in order to construct a state representation for the right hand side, let us assume a state representation for v of s. So, let this state representation of v of s be given in the following way, so z dot is equal to let us say f z plus g w and so w is the input from this v of s. Then, let me call it w 1 and then w 2 is equal to h z plus j w 1, so if this is the state representation for v of s, then v transpose minus s has a state representation, where the matrices involved will be minus f transpose.

Here and here, you will have H transpose here will have minus g transpose and here you will have j transpose y, but on the left hand side we were looking at v transpose minus s times v s. So, this is like saying that w 1 is an input to v of s that way and the output of that is w two, but this w 2 is the input to v transpose minus s and the output to that. Let me call it w 3, so it is a series connection if you if you if you think about it and so I will appropriately use w 2 as the input for this for the representation of v transpose of minus s. So, the input I will use as w 2, let me call this state that one, I had call z.

So, we call that z 1, so z 1 dot equal to minus f transpose z 1 plus h plus h transpose w 2 and the output, let me call it w 3. So, w 3 is equal to minus g transpose z 1 plus j transpose w 2, now what is what was there on the right hand side the net thing, it is a series thing. So, I will have to put both these state representations together and I will end up getting a state representation for this thing, so the state representation I will get for this is going to be z 1 dot is equal.

So, f 0 z 1 plus G w 1 and for z 1 dot, I will get minus f transpose z 1 and here I will get a term because H transpose w 2, but w 2 is h z plus, so I would get here H transpose H and here I would get H transpose j. The output equation well w 3 is equal to would have minus g transpose, so this I have taken care of and then I have j transpose w 2, so putting that in there I will have j transpose h and z and plus j transpose j times w 1. So, this now is the minimal state representation for the right hand side.

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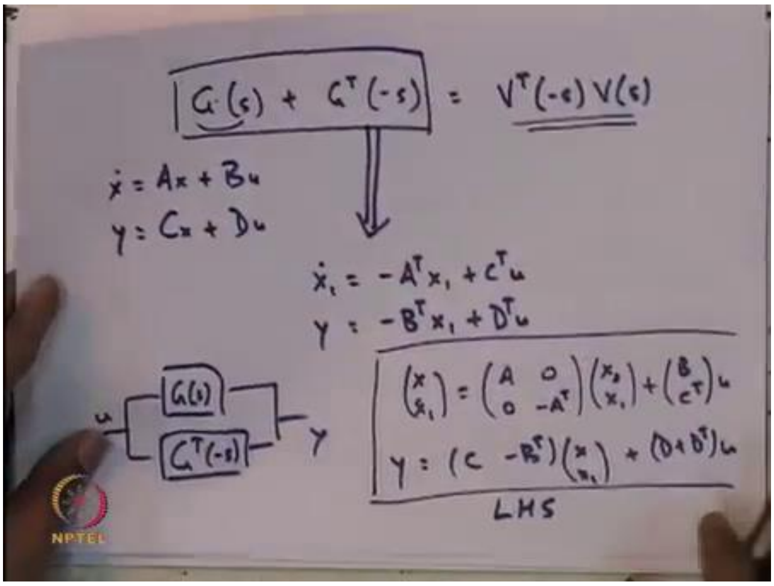

So, if you recall there was a minimal state representation for the left hand side that of this equation that we obtained and we have also obtained minimal state representation of the right hand side of that equation. So, this left hand side and the right hand side ideally if they are equal that means these two minimal state representations area similarity transform away. Now, what we would do is we would use some smart way of manipulating these matrices in such a way that we can show some relation between this and that.

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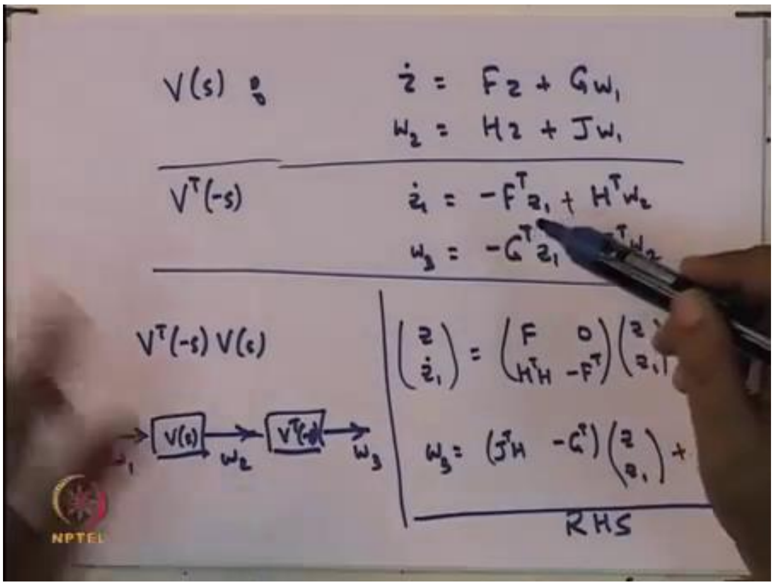

So, one other thing that I wanted to mention is that from the spectral factorization theorem we get that that G transpose s plus G s plus g transpose minus s is equal to this product v transpose minus S v. We can always take this v s to be Hurwitz and if v s is Horvitz, this matrix f is a Horvitz matrix. So, please remember that this matrix f is a Horvitz matrix, so first what we are going to do is we are going to do a transformation on this second state representation that we got and the transformation that we would do on this thing is using a matrix which will help us help make the system matrix. Here, we will convert the system matrix here into a diagonal matrix that means we will try and get rid of this H transpose h now the way we get rid of this H transpose is the following.

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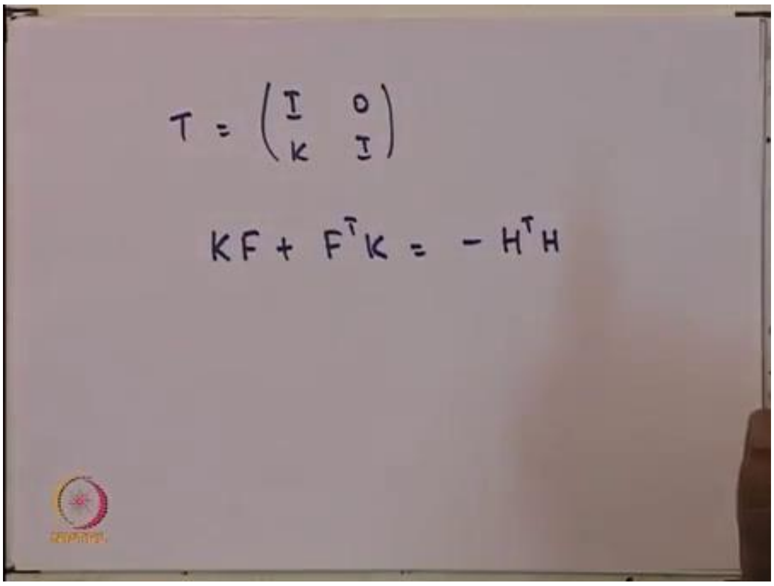

So, we will use for the transformation a matrix t which is of the form I k 0 I and this k is not any old k, but this k is a k that would satisfy the Lyapunov equation k f plus f transpose k equal to minus h transpose H, now what?

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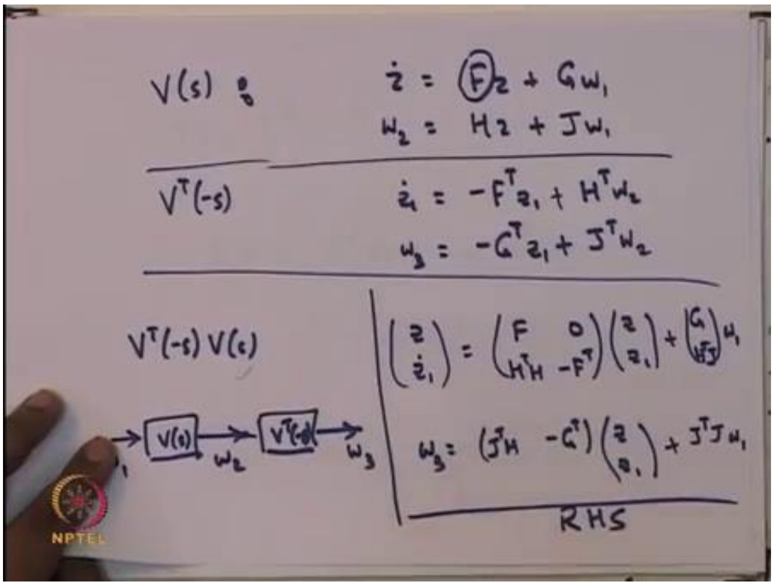

Let me just revisit this is the equation the minimal state equation that we got for the right hand side. We want to make this matrix diagonal in order to make this matrix diagonal also remember this f is a Hurwitz matrix.

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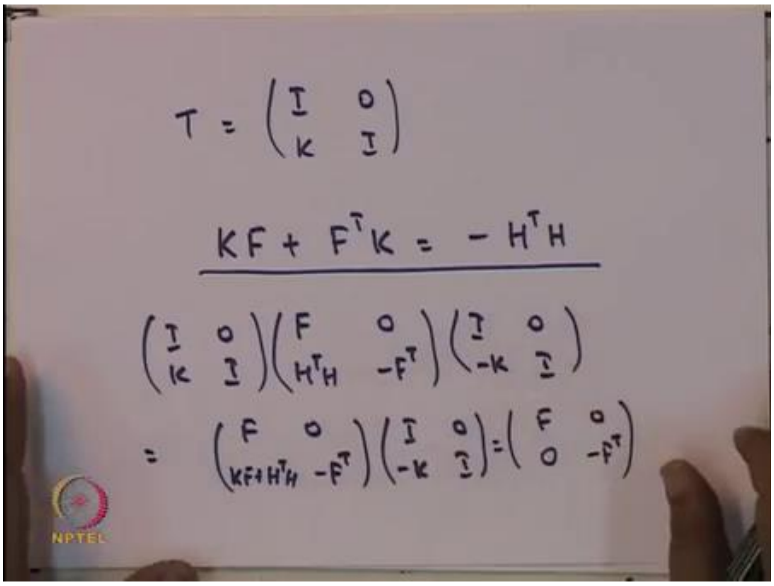

Now, if f is a Hurwitz matrix, this equation for any positive, I mean f is a Horvitz matrix if you write down this Lyapunov equation, what you have on the right hand side well h transpose h this is always going to be positive semi definite. So, with negative sign this is going to be negative semi definite and therefore, you will always get a solutions k for this k is the k that we use in the similarity transformation matrix. So, if you use this k in this similarity transformation matrix, so let us just look at what we would get we have H transpose h minus f transpose and we are going to use this.

So, I 0 k I and of course, the inverse of that matrix is I minus k 0 i so if you multiply this out multiply the first two matrix you get f 0 and this one when you multiply you get k times f plus h transpose H and the other one you get minus f transpose. Then, post multiply the I 0 minus k I, so when you multiply this and you multiply this you get f 0. Then, you have a this thing multiplying this thing, now if you see what happens is you have k f plus H transpose H plus f transpose k, but that is essentially this Lyapunov equations.

We get a 0 and then the last one gives you minus f transpose, so we have effectively manage to diagonalizable that matrix by using the similarity transform t, now if you use this t and you diagonalizable. Then, the kind of matrices is that you would get what I will do is rather than do the calculations I just write down what are the matrices that you would get. These are the original matrix is that you had and if you do this transformation using I k 0 I, then the system matrices is that you would get.

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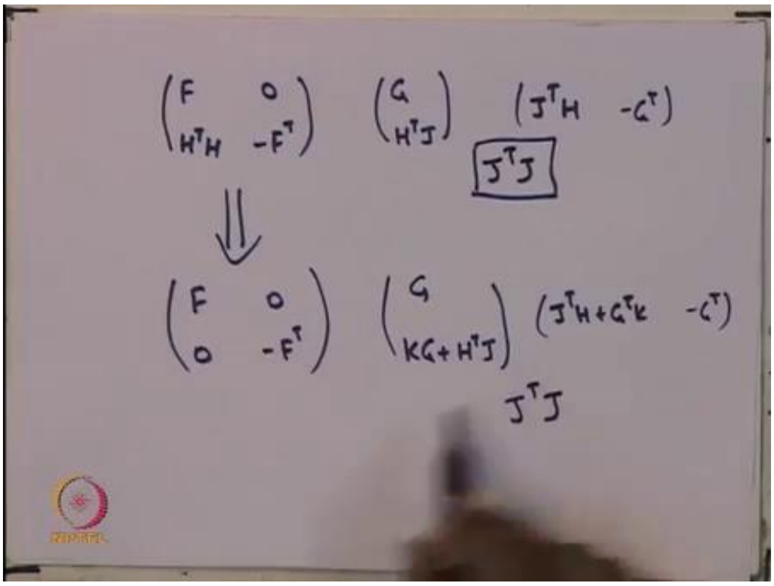

What I will do is I will write down the original system matrices and I will write down what is corresponding system matrices are so the original matrices this is the system matrix that you had an under transformation this goes to f 0, 0 minus f transpose. Then, you had the input matrix as G H transpose j and under this similarity transform H similarity transform under this similarity transform this will to go.

This will continue to be g and here you will have k g plus h transpose j then you have the other matrix being j transpose the c matrix or the observable matrix to be this and that will get transform to j transpose h plus g transpose k. Here, we will have minus g transpose will remain as it is and then the last one which was j transpose j the feed through matrix that will remain j transpose j, now this is the new set of matrices that you have.

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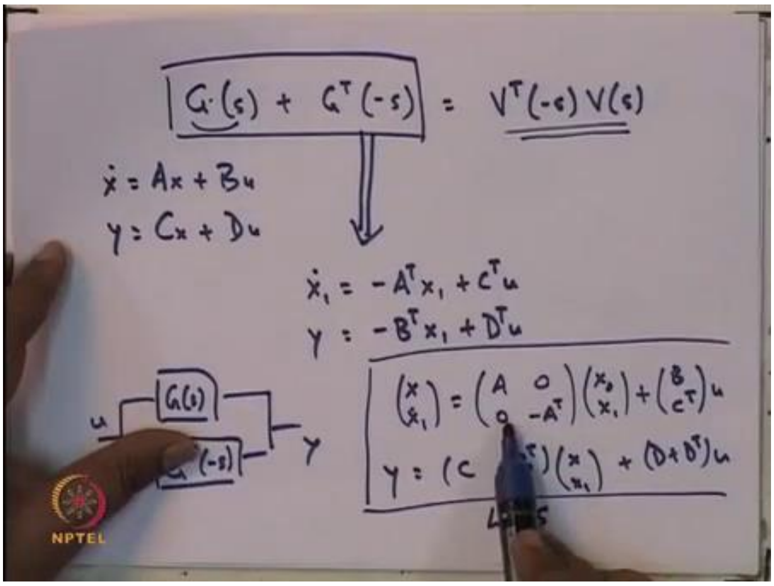

What will now show is that you see the right hand side at these matrices. And one can show that there is similarity transform that you can do on these matrices such that you get these four matrices.

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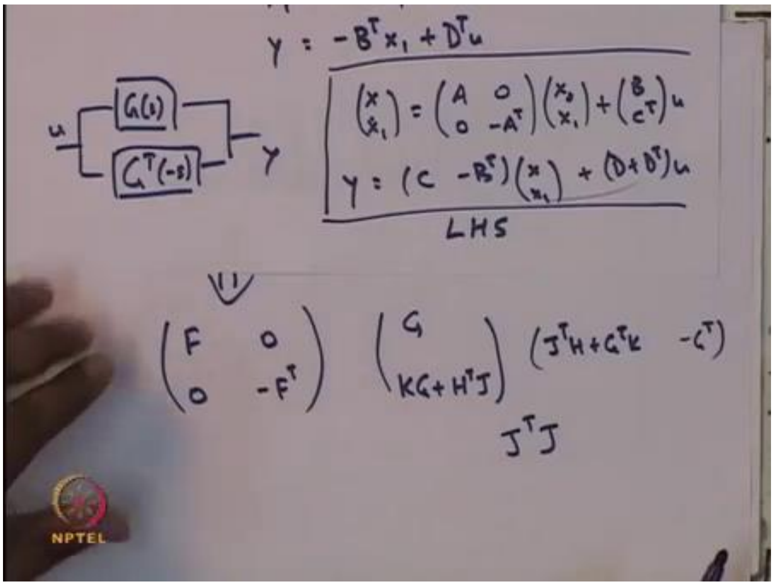

So, this is what we are saying this left hand side matrix this is the right hand side matrix there is a similarity transform which will take this matrix to here this matrix to here, this matrix to here and this matrix. Here, I am not going to go into the details of how to construct this similarity transfer, but I would just say that this similarity transform that you use you can show that it is a block diagonal matrix and so on. Now, it should be clear that once you do this transform whatever this gets transform, so suppose you have that similarity transform acting on this to make it this, then what about this gets transform by this will turn out to be equal to d plus d transpose, so this J transpose j is really same.

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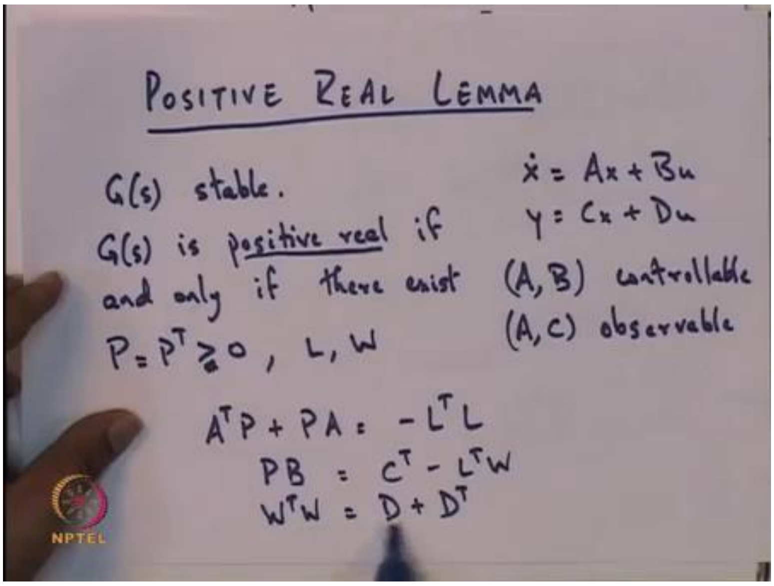

Let me pull out the positive real lemma this last equation says d plus d transpose is equal to w transpose w. This is really saying that this matrix is the same as this matrix, now when you do the other transforms, then from equating whatever is here to whatever gets transform.

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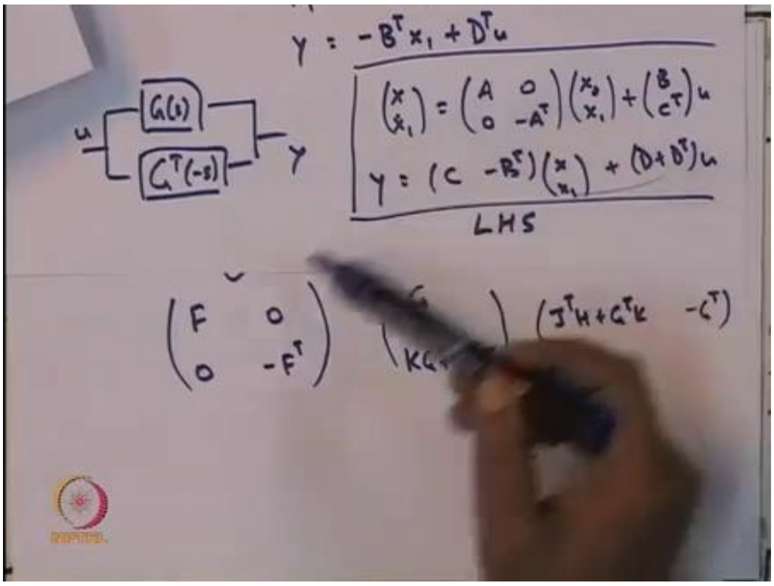

Here, you will get this equation and the other equation about L transpose L that you will get that you will get from the similarity transform that takes A to F.

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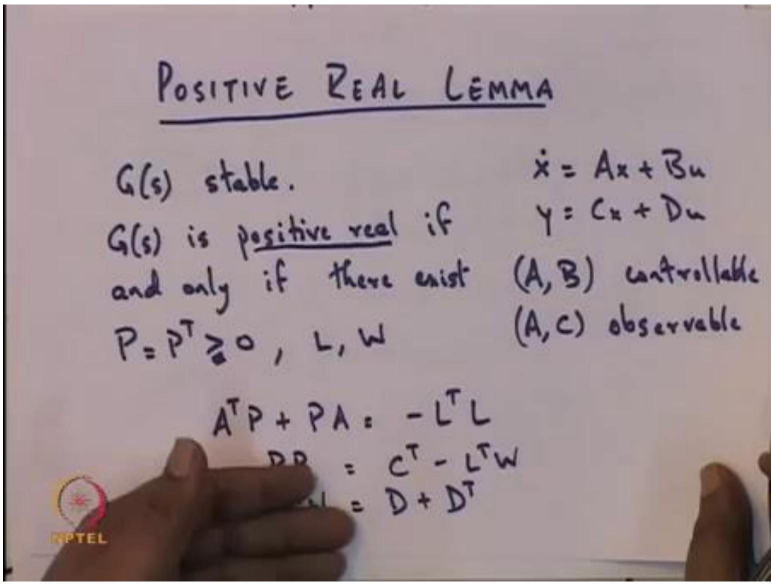

So, as a result what you have shown from this construction is by this method, I am showing this similarity transform between these two you have shown that these equations are going to be satisfied. So, that was the converse proof of the positive real lemma in the of it looks like an out of time and so let me stop this a lecture, but what I would do in the next lecture is I would start of by giving the definition of positive real for matrices.

For the matrix case we prove the positive real lemma assuming, I mean initially I had only given the understanding of what positive realness is for scalar single input single output system. Of course, it can be extended to matrix case the proof of positive real lemma because the proof was dealing more which states space. It really did not matter whether we were looking at the single input single output or the multi input multi output case, but what exactly the definition of positive realness is that I would talk about in the next lecture.