Nonlinear Dynamical Systems
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Lecture - 25
Popov Criterion Continuous, Frequency-Domain Theorem

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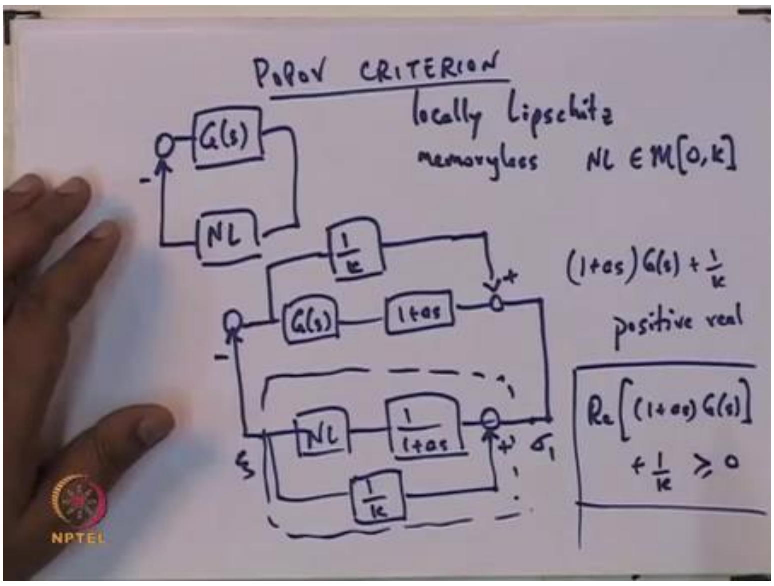

So, what we saw in the last lecture is the Popov criterion, and let me sort of restate what the Popov criterion is, so again we are looking at a situation where you have a non linearity, so a linear plant and they are interconnected in the following way and we want to talk about the asymptotic stability of this system and there are these conditions. First of all the nonlinearity is locally Lipschitz memory less, so locally Lipschitz something to do with the continuity of this function and memory less we have already seen that it does not depend upon the on the past. So, memory less, so the non linearity is all that and the nonlinearity of Nyquist also lies in the 0 k sector.

Then in order to talk about the asymptotic stability of all this, what we did was we constructed an equivalent a feedback structure, where what we did was to the non linearity, we put 1 plus s in series and we fed back 1 by k. And we had the plant a s. S, in series that we have 1 plus a s and this fed back negative feedback and here again the same game 1 by k put forward here and fed forward positive sign get through. So, we did this modification for the nonlinearity in the following way and so we do the modification for the linear part in that following way.

Then from that we have we showed that this new nonlinearity is still going to be passive that means in terms of this input, here I call it sigma 1. So, this whole thing is the new nonlinearity and the sigh is the output of the nonlinearity sigma 1 as the input and this nonlinearity is passive with respect to the input sigma 1 and the output sigh. But, then that linear plant if we can show that that is also a passive then the resulting system is passive. Therefore, this asymptotic asymptotically stable and therefore that is asymptotically stable, so the resulting linear plant that you have here is 1 plus a s times G s plus 1 by k.

But, this must be, this must be positive real if the, if the, if the interconnection is going to be passive, if that is going to be passive then this must positive real. Therefore, this must be positive real and this is the Popov criterion, of course this being positive real is like saying the real part of the transfer function 1 plus a s times G s plus 1 by k must be greater than a equal to 0 that is the, that is, that is the Popov criterion.

Now, how does one check the Popov criterion well as one can check the Popov criterion, so of course some other things need to be talk about? You see if this one plus a s times G s, this guy is plus 1 by k this is going to be positive real than 1 plus a s times G s this transfer function. So, let us just look at the transfer function G s and make some comments about the transfer function G s.

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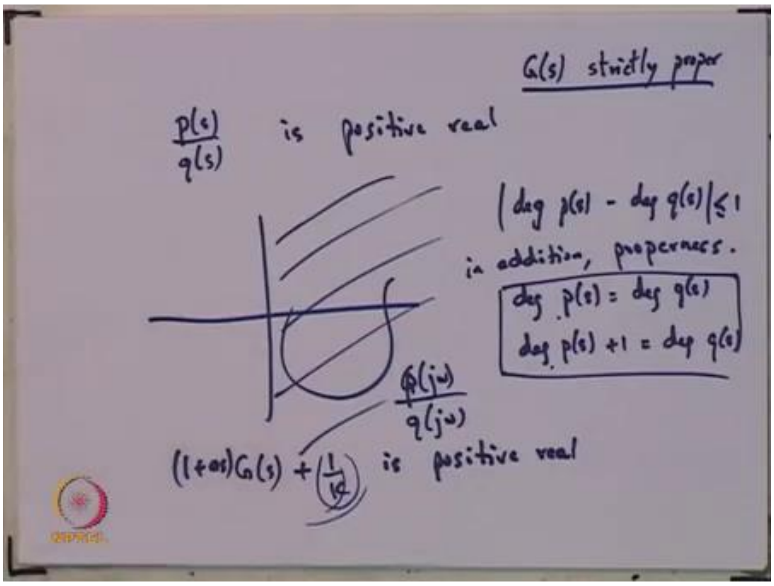

Now, you see given a transfer function p s by q s, if you say that this transfer function is positive real. Then what we are saying is that the Nyquist plot of this transfer function is going to p of j omega by q of j omega. So, the Nyquist plot is going to lie in the first and the second quadrant, sorry the first and the forth quadrant that means its real part is going to be positive. But, if it had to be positive then we can say something about the degree of p and q we can say that the degree of p, p of s minus the degree of q of s.

So, that means the difference in degree the modulus value can at most be 1, so relative degree can at most 1. Now, when you are looking at transfer function typically you are looking at proper transfer functions that mean the denominator polynomial as a degree which is greater than equal to that of the numerator polynomial. So, the two situations that you could have then for positive real functions is the degree of p s is equal to degree of q s or degree of p s plus 1 is equal to degree of q s. So, in addition if you put the properness condition, now the Popov criterion says that given a transfer function G of s 1 plus a s times G of s plus 1 by k is positive real.

Now, if these wholes transfer function had to be positive real that means t the degree of the numerator divided by the degree of the denominator must be having a difference. They must have equal degree or the numerator must be one degree smaller than the denominator. Now, this 1 by k portion is something that we can forget because this being positive 1 by k being positive we can take it off.

So, you can claim that one plus a s times G s should be positive real or shifted positive real that kind of things, therefore we can conclude that G of s for this to be positive real, this whole thing be G of s strictly proper. So, G of s must be strictly proper because you are now multiplying the numerator by one more degree in s, and the resulting thing continues to be proper therefore G of s must be strictly proper.

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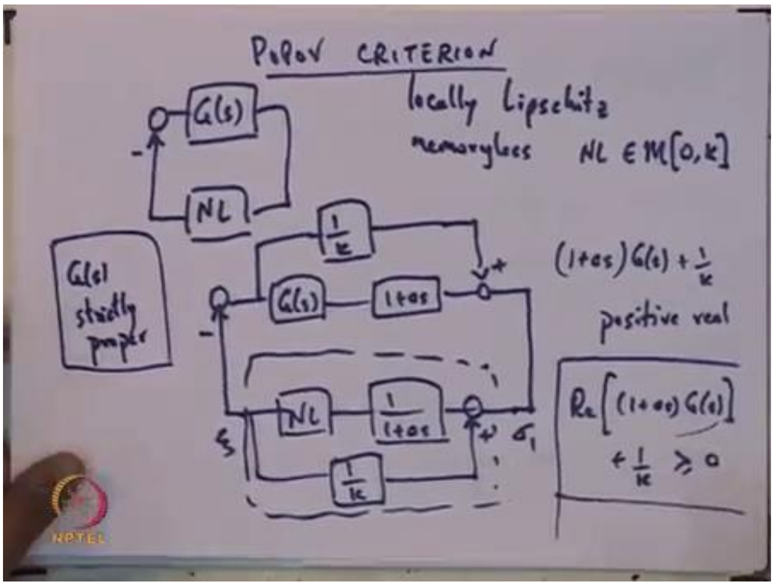

So, going back to the Popov criterion you always have this G of s interconnected to the nonlinearity the nonlinearity is in the 0 sector, and this G of s has to be strictly proper. Now, how does one check whether these conditions satisfied well the best way to check whether that conditions satisfied is by evaluation.

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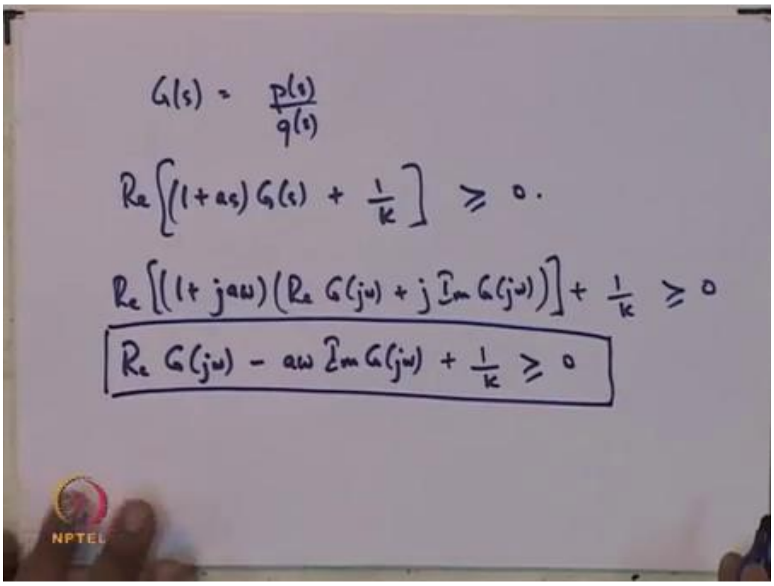

So, let us assume, let us assume that you have this transfer function G of s which is let us say p s by q s we did not we did not bother about that we want to check that 1 plus a s times G s plus 1 by k, the real part of this whole thing is greater than equal to 0. So, the real part when you substitute s equal j omega, so that is like saying 1 plus j a omega times the real part of G j omega plus j times the imaginary part of G j omega plus 1 by k must be greater than equal to 0. But, it is the real part of this thing, so one could evaluate the real part of this, so the real part is going to be the real part of G j omega.

So, the real part of G j omega then this product is going to be imaginary than you have a, this product here that that gives you minus a omega times the imaginary part of G j omega plus 1 by k is greater than equal to 0. Now, how does one determine whether the given transfer function G this condition is satisfied, so we do the following we do the following.

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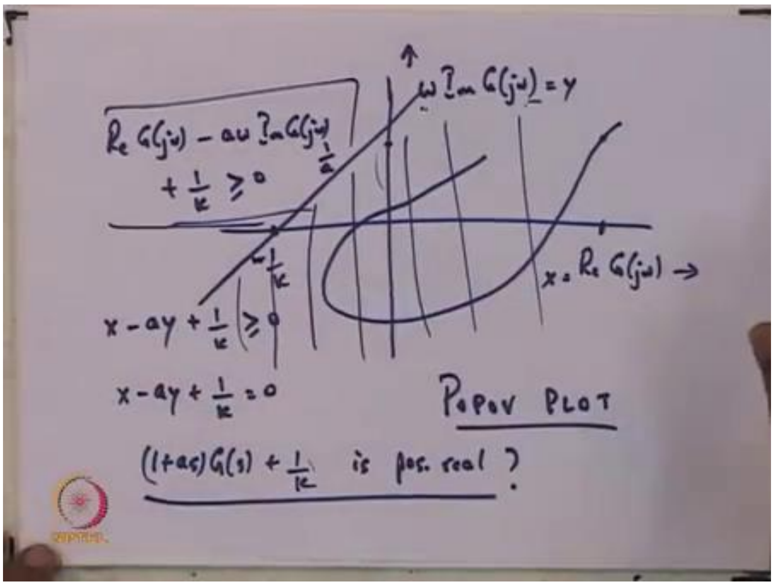

So, we make the following plot on the x axis you plot the real part of G j omega and then the y axis you plot omega times the imaginary part of G j omega. So, at any omega you evaluate the real part of G j omega it has some value let us say something and omega times the imaginary part G j omega it has some value. So, these two will give you some point there, now for each omega you evaluate this points and you plot this thing such a plot is called the Popov plot.

Now, what we need to check on the Popov plot is that the real part of G j omega minus a omega times the imaginary part of G j omega plus 1 by k is greater than equal to 0. But, this, now with the axis this showing the real part f G j omega and this action showing the imaginary part G j omega, this is essentially an equation of I mean if you if I think of this axis as y and this axis as x. So, then what we are saying is x minus a times y plus 1 by k is greater than equal to 0, so of course if you look at the equality you get this equation which is, which is x minus a y plus 1 by k equal to 0. So, this is equation of what line this line is going to pass through the point minus 1 by k and it is going to have a positive slope and the slope.

So, you have a line like that and the slop of this line is one by a, now x minus a y plus 1 by k greater than equal to 0 is essentially this portion of plot. So, in order to check whether a given transfer functions is going to satisfy the Popov criterion, what you have to really draw is the Popov plot. But, where on the x axis you have the real part of G j omega on the y axis you have the omega times imaginary part of G j omega for example the Nyquist plot. Now, you would be plotting real part of G j omega and imaginary part of without the omega, so you get the Nyquist plot.

But, this is not the Nyquist plot, but the Popov plot, so all the x, you have real part of G j omega on the y, you have omega times the imaginary part G j omega. But, you get the Popov plot and because you wanted to check 1 plus a s times G s plus 1 by k is positive real this is what you wanted to check. So, this 1 by k, so minus 1 by k you take as intersect of a line which slope 1 by a and then the Popov plot that you have plotted must be such that it should lie to the right of it. Now, that means if lies to the right of it of course it satisfies this particular any quality and therefore this transfer function is going to be positive real.

So, the way you check whether something is positive real or not is by drawing the Popov plot and you have the line. But, then you check whether it is positive I mean it is lying on the appropriate half space, so we have now seen how one uses this Popov criterion through the Popov plot. Here, you can get the idea whether they the resulting systems actually asymptotically stable or not, now initially when a lot of these results were arrived at. But, in those days itself for proves linear matrix inequality is had made an appearance, but at that point in time may be because the competition power was not that high.

So, these linear matrix inequalities stayed restricted to the, to the theory in the sense that they use these linear matrix inequalities in the proves of theorems and so on. But, was not practically use it was considered not very efficient, but nowadays of course there is lot more computing power and linear matrix inequalities have come in a big way. Now, people do use the linear matrix inequality, so what I will do right now is I would show how a lot of these results like the circle criterion and Popov criterion and so on. So, come up in the context of linear matrix inequalities and about a relationship between them.

But, of course this whole thing centres on a very central theorem which has proved by Yakubowich sometime in the early sixties. So, let me, now motivate how this comes about, so what we will do is we will assume that we given these states base equations of a linear system.

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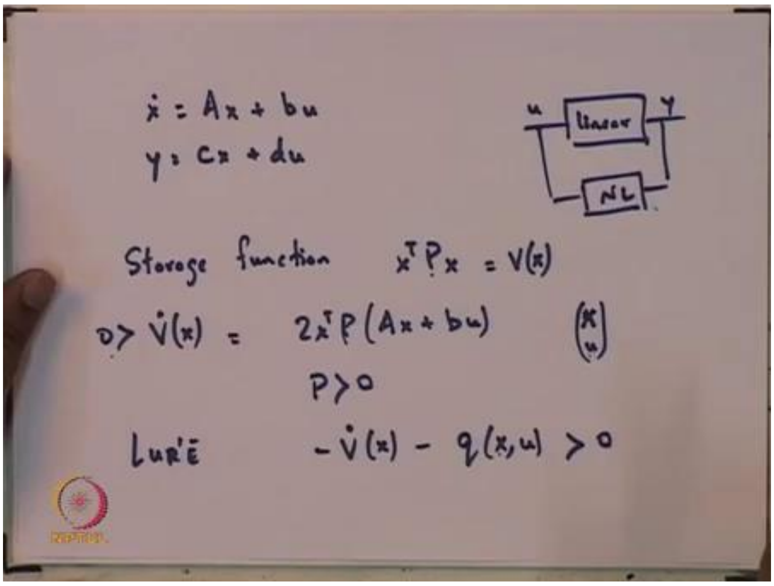

So, you have x dot equal to A x plus b, b u lets say and y equal to C x plus d alright, now this is the linear part, so you have the linear part. So, the linear part and you have the input you have the output, now you are going to connect a nonlinearity to this thing how we connect it is right now not very important. But, we are going to connect a non linearity to this linear part, but ultimately what one wants to do is find a storage function for the net system.

So, let us make the assumption that the storage function or the bipolar function is dependent upon the states of the linear part and let us say it is something like x transpose p x. But, of course if this had to be a bipolar function this p had to be positive definite, but so the question is whether one can find a positive definite p such that x transpose p x this is positive. Now, if we think of this as v of x and you evaluate v dot of x then this V dot of x must be negative and if that is true and this is true.

But, that means p is a positive definite matrix and its derivative is negative over the protectorates then what you really have is an asymptotic. Now, I mean you can conclude asymptotic stability, but if you evaluate this V dot x I mean in this particular case 1 could write this V dot x as something like 2 x transpose p times A x plus b u, I mean. So, of course there is something here something there, but I have just put them all together. So, this is the expression that you get for V dot and it is in terms of these states of the linear system and the input u and so what you want to really do or what you really find is in this x u space.

So, the space of states and input u want this expression to be negative for all values of x and u, this expression must be negative and for all values of x and the p matrix that you have must be positive definite. So, this expression must be negative and p must be positive definite, if you can do that then you can say that the net system is asymptotically stable. But, of course when I write such an expression down I have written this purely in terms of the linear part. So, you might wonder how is a nonlinearity going to come in at all, but you see when we write out the equations to for this x and u.

But, this x and u are not independent of each, instead they are related to each other and how are they related to each other well this u is a function I mean u is really coming from the non linearity. This nonlinearity uses the y as the input and so this u is actually dependent on y which in turn of course is dependent on x. So, really this x and u are not completely independent, but they are dependent and somehow that dependence has to come in. Now, the suggestion that has come about is that instead of asking for this V dot x which is, this expression to be less than 0.

Lure suggested that you should, Lure suggested that instead of so showing that v dot is less than 0 is the same as showing minus V dot x is positive definite. But, instead of showing minus V dot x is positive definite, Lure said that minus V dot x is positive definite minus some quadratic expression. So, let me called q which depends on x and u, so this quadratic expression it depends on x and u and this quadratic expression is in some sense. Now, a relationship is given by the nonlinearity and this resulting thing must be positive definite. So, what I am trying to say is that instead of asking the question for all x s and u s you must find a p which is positive definite such that for all x s and u s this expression is negative definite.

But, of course this expression is dependent on x and u, so it might not turn out to be negative definite and you might not get a positive definite p. But, this x and u when you just ask that question you are thinking of x and u as being independent, but they are really not. But, are dependent on each other through this nonlinearity and so instead of looking for this expression to be negative definite or the negative of this is to be positive definite.

So, what one does is, one looks at that expression and some more expression and this expression depends on a nonlinearity and this net thing should be greater than 0. This is in fact what comes out in the in the frequency theorem, so may be, so I would, I would state the frequency theorem right away, so what the frequency theorem states is the following.

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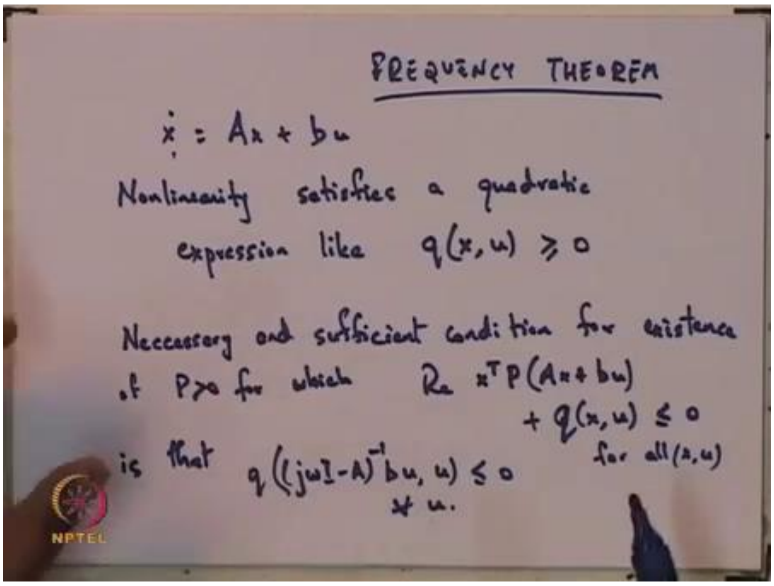

So, again consider states base equation A x plus b u let us say and let us say that we have a nonlinearity and the nonlinearity satisfies a quadratic expression like q x u. So, the nonlinearity satisfies this quadratic expression like q x, q is greater than equal to 0 for all x and u that appear on for the nonlinearity. Then the necessary, the necessary and sufficient conditions for existence of such as p that we are looking for existence of p is same.

For which that expression that we were talking about that is the real part of x transpose p a x plus b u plus G times x u being less than equal to 0 for all x and u this for the existence of such a p. So, the necessary and sufficient condition is that sorry I am calling it G a I am calling it q, so q x u is that q of for x I would use this equation and for x I would write down j omega i minus a inverse b u. Now, u this quadratic expression is less than equal to 0 for all u and if this is true, so if this condition is satisfied, so this is, this is also called the frequency theorem.

So, what the frequency theorem is saying that the necessary and sufficient condition for existence of positive definite p, this is a p which is positive definite for which the real part of all this is less than equal to 0 for all x and u. But, of course here, now we think of this x and u not as time signals, but as frequency signal, so in the frequency space you could think of these things. Then in that case, this thing is satisfied if you just take the quadratic part and for the x you substitute j omega i minus a inverse b u which is what you would get from the linear equation of the linear part.

But, his expression must be less than equal to 0 for all u this is called the frequency theorem and this theorem was in fact proved by Yakubowich. So, I am, I would not give a proof of this theorem, but we can straight away see that the necessity will always be there. So, I mean this condition that this thing is less than equal to 0, for all u this is a necessary condition and one way to see that is, see if you if you are thinking about the linear part, I mean the linear plant.

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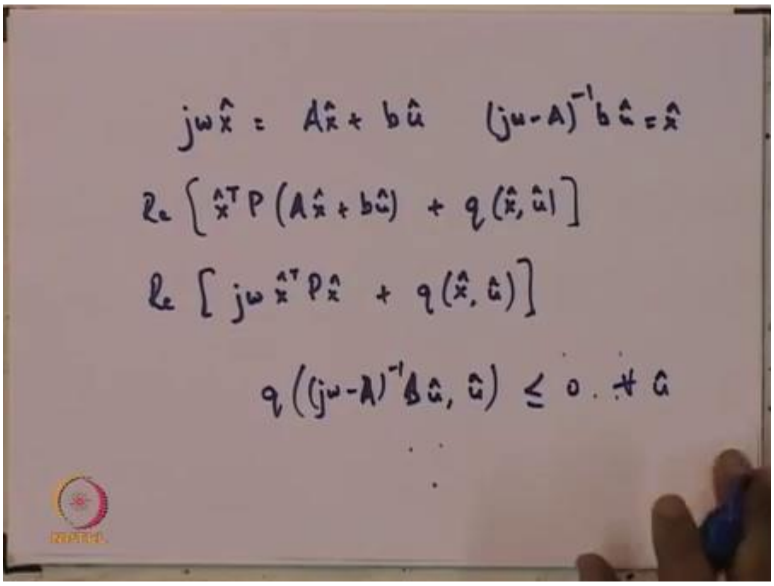

Now, you know you can always write down j omega x equal to A x plus b u, of course here let me just put hats because we are talking in terms of frequency. In fact, even in the even in the previous expression I could, I could put just hatch just to differentiate the fact that this u and this x hat we are thinking of is really in the frequency domain. So, we have this, now if you are going to look at the expression, so if you are going to look at the expression the real part of x transpose P. Then you have A x plus b u plus q of x u and we want this to be less than equal to 0, but out here because a x hat plus b, sorry all the hats are there, because A x hat plus b u hat is equal to j omega x hat, so this is the same as the real part of j omega x hat transpose P x hat plus q x hat, u hat. But, you see this expression here is a purely imaginary part, so when you are looking at the real part this does not appear. So, it is just the real part of this and the real part of this and for x hat no using that equation we can substitute and you get j omega minus a inverse b u hat that is equal to x hat. Now, just substitute that in here and whatever you have would be q of j omega minus i inverse minus A inverse b u hat, u hat this must be less than equal to 0 for all u hat. So, you see the necessity of this of the condition that we are talking in the frequency theorem is clear immediately.

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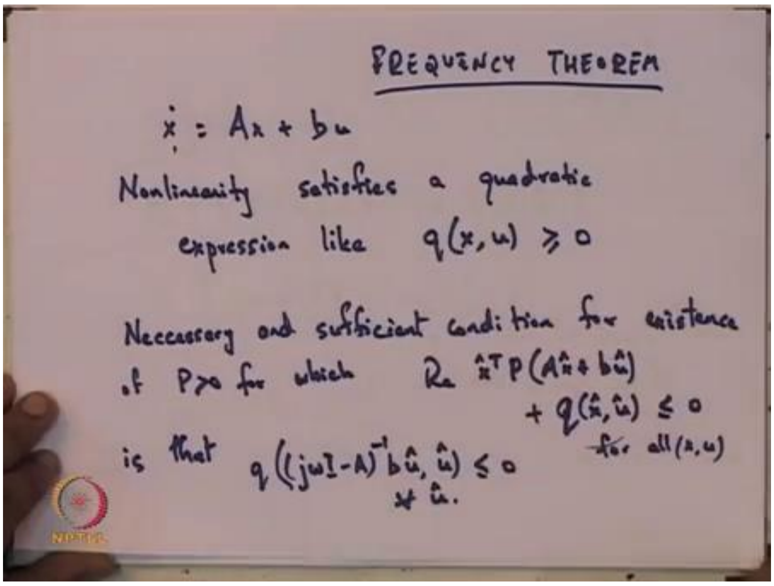

So, that mean you just assume the linear part and just the take it in the frequency domain and just substitute in here this expression will become a purely imaginary expression. So, you just left with this expression and that is less than equal to 0, of course while stating this I mean we wanted.

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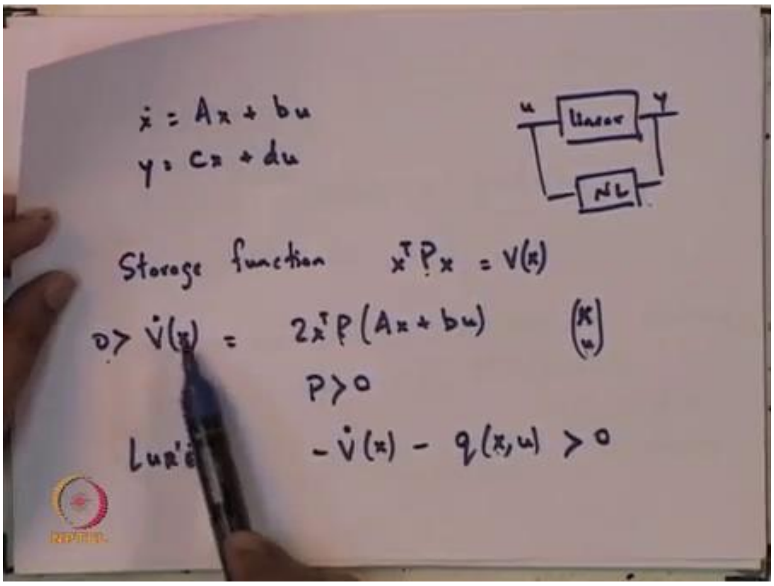

We wanted to find out this V dot is less than 0 and this expression is really a time domain expression and we move from a time domain condition to a frequency domain condition.

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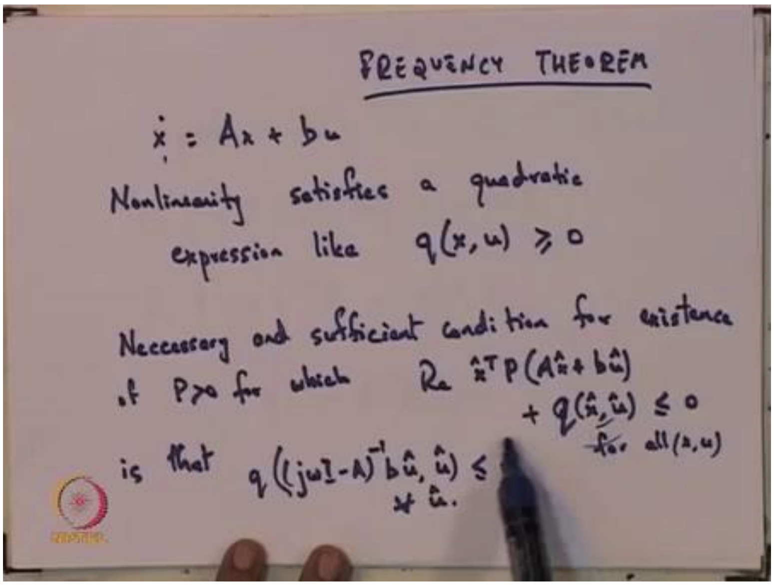

So, the frequency domain condition essentially was that this expression that you get, so as I told you earlier it was suggested that to try to find this to be strictly less than zero is not possible because there is a relationship between x and u. So, you add an extra term x and u and so you want those whole term to be greater than 0 and this you can just convert this into the frequency domain. The expression that you would end up with is this expression and the condition necessary and sufficient condition for this to be true is that you just take the quadratic satisfied by the nonlinearity.

For the x hat, you get the expression coming from the linear hat and substitute in that and this resulting thing must be less than equal to 0. Now, we can we can see immediately that this particular frequency theorem in some sense gives us all that we saw using the circle criterion and loop transformations and so on, so here is here is an example of this of this those particular result.

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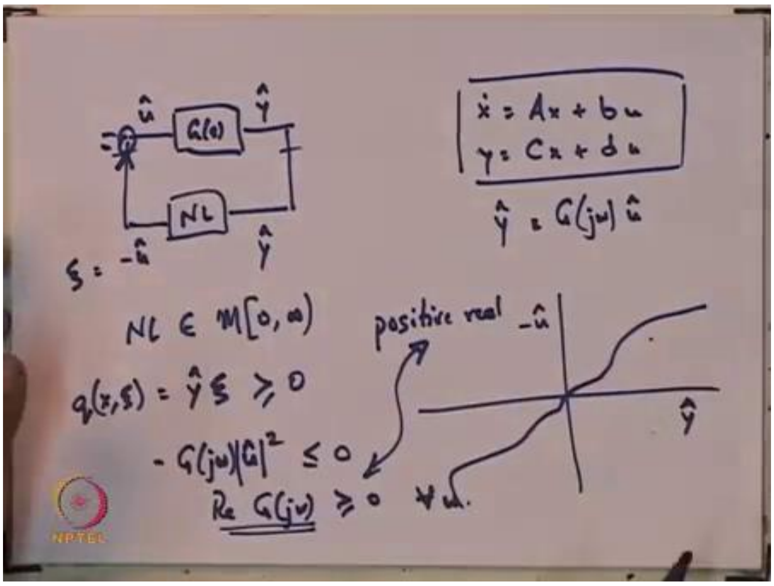

So, suppose we look at the linear plant x dot equal to A x plus b u y equal to C x plus d u, let us say or this is of course the states base. The states base equation one could also write the frequency domain equations and then you could just say G j omega u, G j omega u G j omega u hat G equal to y hat. So, think of this as the linear part and you have u hat, here you have y here and now suppose you are going to attach nonlinearity and something like this. But, let us not be bothered about whether its negative feedback or anything like that if you are going to have an interconnection likes this.

Then what you will have here y hat and what you will have here is u hat if it was a negative feedback. But, of course you would have had minus u hat here, now this nonlinearity let us assume this nonlinearity is a passive nonlinearity that means it belongs to the 0 infinity sector what that means is its either in this quadrant or this quadrant. So, it is like some something like that and here what you have this, the input output diagram of the nonlinearity. So, the input of the nonlinearity I really y hat and the output of the nonlinearity are going to be u hat or minus u hat depending upon how you want to see it whether it is a negative feedback or not.

So, just to avoid any complications, let me just assume that this is in fact minus u hat, so that you have a negative feedback and so on. So, if I call that u hat, I will have to call this minus u hat, so you have u hat then, now a quadratic expression which is satisfied by such nonlinearity. Now, one quadratic expression let me call me those minus u hat sigh then the quadratic expression satisfied by this thing this nonlinearity is input multiplied by output must be greater than 0 because that is precisely what happens in this quadrant and this quadrant. So, this means I could write that down as y hat sigh is greater than y hat sigh is greater than equal to 0. So, this y hat sigh is really it plays the role q x i that they are talking about in the frequency theorem.

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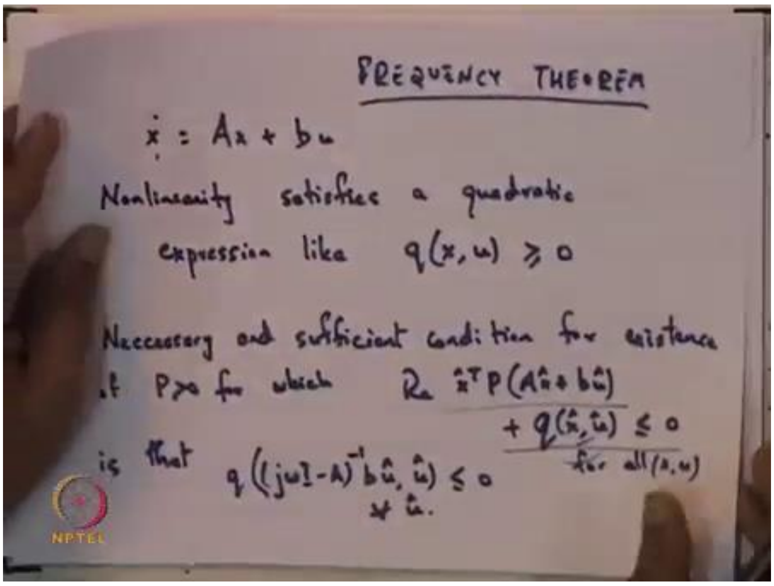

So, in the frequency theorem we said there is nonlinearity satisfies a quadratic expression like that, so that quadratic expression is here. Then further frequency conditions that the necessary and sufficient condition for existence of this P is that the real part of this must for which the real part of this is less than equal to 0. Now, that is simply you just take q, that means whatever is the transfer function that comes from the linear part and that must be less than equal to 0. So, if you, if we look at this expression, this is a quadratic expression and in this quadratic expression this y hat is something that comes from here. So, for y hat I could just write down G j omega u hat, so what I will have here is G j omega u hat and the sigh is really minus u hat, so its u hat squared so G j omega u hat squared.

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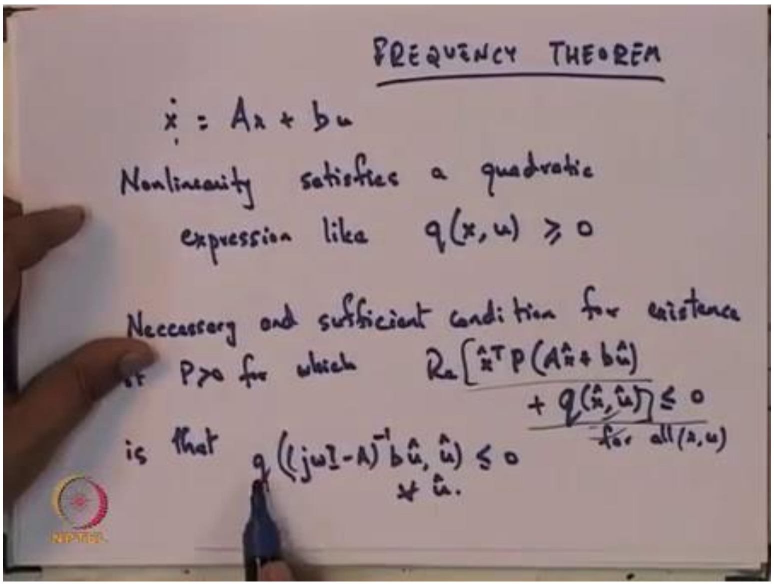

This expression that we are talking about is q this and this expression is essentially G j omega u hat with a minus sign because the sigh is really minus u hat. So, this is what you get and the frequency theory says that this must be less than equal to 0, now this being less than equal to 0. Well, the u hat square is anyway positive is the same as saying G j omega must be greater than equal to 0 for all omega, but of course G j omega, of course is a complex expression.

So, what we are really looking for is that the real part of this must be greater than equal to 0, but the real part of this being greater than equal to 0 is in fact our definition for practice real. Now, we do know that if you put a positive real function along with a nonlinearity which has, which is in the 0 infinity sector than that was our basic theorem for that was that was the basic passivity theorem you see. So, you arrive at the same conditions that means u the linear plant will have to be positive you arrive at the same condition by just using this frequency theorem.

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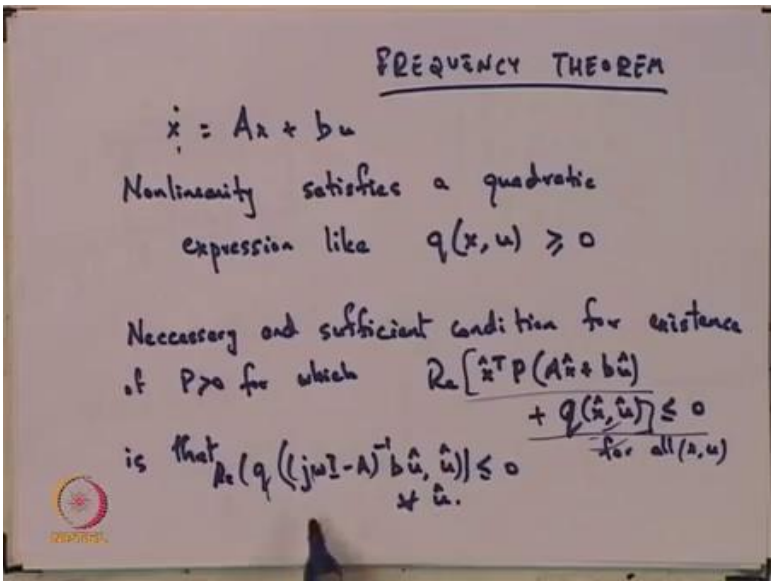

So, just to read rate the frequency theorem it says that if you have this in order to want this have this expression to be less than equal to 0, it is enough to substitute for the x. So, I mean whatever is the quadratic expression satisfied by the nonlinearity for that for the x part you substitute from the linear equation and you had.

The resulting quadratic must be less than equal to 0, actually perhaps we have to say the real part of that because after all this expression would be a complex expression. So, it is less than equal to 0 for a complex expression really does not make sense, so the real part of it must be less than equal to 0. Now, we can we could also look at other cases and you would immediately see that all the results that we had obtained by loop transformation and so on they just fallout from the frequency theorem.

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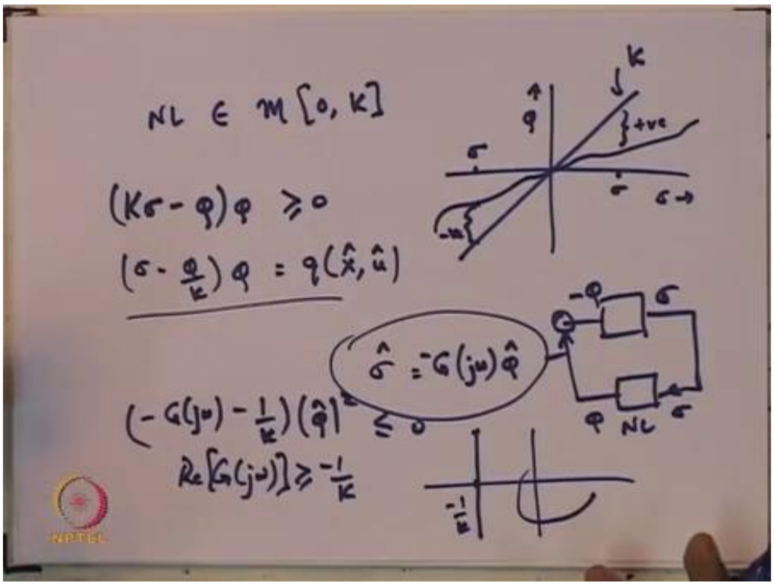

So, let us, now of course use the same linear part, but the nonlinearity let us assume is in the 0 k sector, now if it is in the 0 k sector. So, what that means is here is a line with slope k and so the nonlinearity lies in there, now if the nonlinearity lies in there then if you think of. So, let me call the input of the nonlinearity sigma and the output, let me call it phi then clearly this expression this expression k sigma minus phi, this expression k sigma minus phi. So, for this particular sigma what you will have is this length here which is positive, so if this multiplies phi.

So, phi is positive and k sigma minus phi is positive if sigma is negative, the sigma was negative then what you would have is phi would also be negative the output will also be negative. Here, k sigma minus phi this quantity here would also be negative, so this expression is going to be greater than equal to 0 for all nonlinearities that lie in this sector. So, this will, so I could rewrite this as sigma minus phi by k times phi and this is going to be the quadratic that is satisfied by the nonlinearity. So, I could say that this is q of x hat u hat is the quadratic q x hat u hat, now if we are thinking of the linear part and here is the nonlinearity with the input to the nonlinearity being sigma and the output being phi.

Now, you connect this up with a negative feedback, so you have minus phi here and you have sigma as the output. Then using the fact that sigma hat is going to be equal to G j omega times phi hat with a negative sign, now if this is now substituted in there. Then we would have affectively done the done the substitution us by the frequency theorem, so if you do that what you end up with is minus G j omega minus 1 by k times phi hat squared this should be less than equal to 0. But, this is same as saying G j omega must be greater than equal to 1 by k and this is in fact the result of course the real part because this, so this translates to the real part of this being greater than equal to minus 1 by k.

This in fact, is what we earlier saw that the, so here is minus 1 by k and you have this line and the Nyquist plot of G j omega should lie to the right of it. So, we get that straight away by using this quadratic and using the frequency theorem, so similarly of course when you have you know the circle the circle criterion will also come straight away from the frequency theorem.

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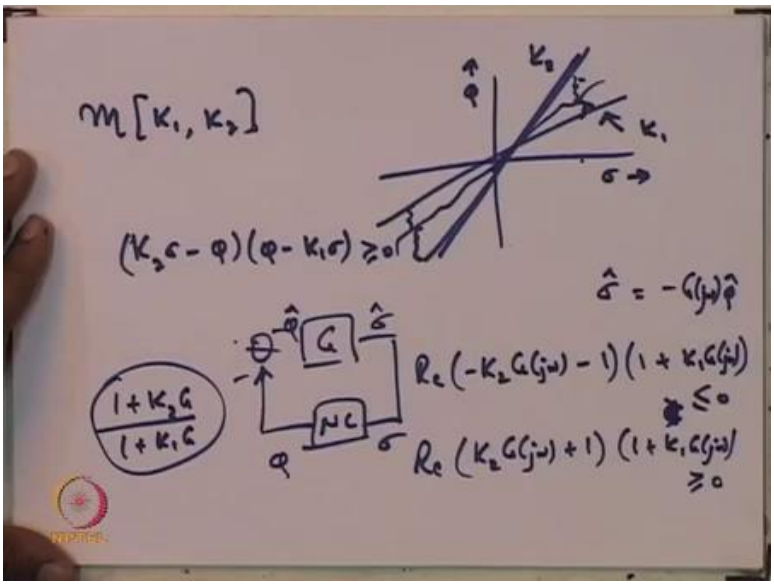

So, in the circle criterion for example we are looking at nonlinearity which lies in between these two slopes, so let us say this is k 1 and this is k 2 and we are interested in the nonlinearity lying between the sectors k 1 and k 2. So, the one way you can give a quadratic expression for such nonlinearity is the following, so it is k 2.

So, if the input of the nonlinearity is called sigma and the output is called phi then we have k 2 sigma minus phi, so k 2 sigma minus phi, this quantity here when sigma is positive or this quantity here when sigma is negative. Then this and then this multiplying phi minus k 1 sigma, so phi minus k 1 sigma is this quantity here, so when sigma is positive both these quantities are positive. So, their product is positive and sigma is negative both these quantities are negative and so their product is positive.

So, this is the quadratic that is going to be satisfied by the all the nonlinearities and if this is satisfied by all the nonlinearities then taking the frequency domain. So, I mean after all the interconnection was this is linear path and you have the non linear part and thus is the input of the nonlinearity. This is the output of the nonlinearity and you are connecting this up with negative feedback, so that is minus phi hat and sigma hat. So, just substituting this is minus G j omega times phi hat it be substitute that into this expression what we end up with is k 2 minus k 2 times G j omega minus 1 this.

Now, you can pull the phi out and then the other expression that you have is 1 plus k 1 G j omega this whole thing the real part of this must be greater than equal, must be this must be, sorry less than equal to 0. But, then these negative signs you pull out and you will get k 2 G j omega plus 1 times 1 plus k 1 G j omega, the real part of this must be greater than equal to 0. But, earlier from the circle criterion what we got is 1 plus k 2 G upon 1 plus k 1 G, this must be positive real, but that of course translates to this. So, as you see this frequency theorem is very powerful theorem and all these results that we talked about using circle criterion Popov criterion and so on.

So, all of them will fall straight into this category by using this frequency theorem and of course the way, the way Yakubowich arrived at the frequency theorem is because of several developments by other people like Lure who suggested this other modification. So, what exactly one would seek to be negative definite and so on, of course there is a lot of other associated literature which I have not got into.

For example, there is a case where you could be attaching several nonlinearities to a linear plant and it turns out that for each of these nonlinearities, you have a quadratic expression of the nonlinearity. Finally, when you are going to check for v dot being less than equal to 0 where V is the bipolar function kind of thing.

So, what you do is each of these quadratic which is satisfied the various nonlinearity, so each of those quadratics you add and these net expression you want to be less than equal to 0. Now, this is what is called s procedure and so there is a lot of such associated literature which I am not going into. So, I mean the set of lectures that I was planning to give as far as this course is concerned is sort of over and the rest of the course is handled by professor Madhubelur.

So, that is all I have to say thank you very much.