Nonlinear Dynamical Systems
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Lecture - 26
Describing Function Method

Welcome everyone to this lecture. Today we will start with a new topic called describing functions, this technique has widely been used since perhaps 1930's or 1940's or little later and it is a method for finding out periodic orbits.

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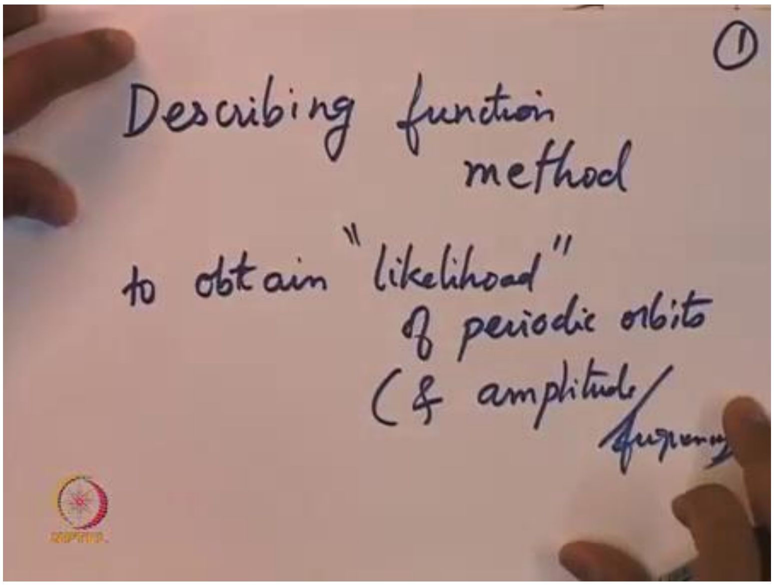

That is there we will see it is the approximation method, so the periodic orbit values the amplitude and the frequency those values are expected to be approximate. So, more precisely we can say to obtain likelihood, likelihood meaning we it is not a method that guarantees easily there are sufficient conditions for guaranteeing periodic orbits and also sufficient conditions for ruling out existence of periodic orbits. But, otherwise what we will study in this course is more about likelihood of periodic orbits and amplitude frequency.

This is something in continuation of what we what I told in the very start of this course that we need non-linear systems for sustained oscillations, for robust sustained oscillations. Keeping that in mind we will see an example of how non-linear systems can achieve this. And the technique used for finding the amplitude and frequency goes through this particular method called describing function method, this describing function method is not to be confused with another word.

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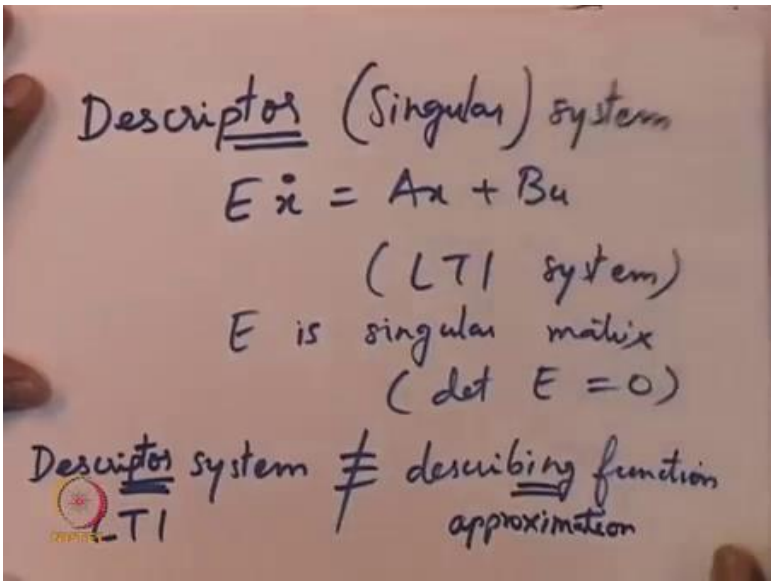

There is also something called descriptor which is also called as singular system, so this one is a linear system. So, we have all seen state space LTI system LTI system in which E is singular matrix determinant of E is equal to 0 this is descriptor system.

This descriptor is not the same as describing function, descriptor system is not the same as describing function when I started studying describing function many years ago that time many places on the net had interchanged these words. So, please read very critically whatever you read on the web one should go for established authors text books or papers research papers by established authors rather than relying on the net. If one wants to use the net of course one should because so many things there are all free that time one should read everything very critically.

So, please note the descriptor system is different from describing function. This is a linear time invariant system while this is an approximation method in non-linear systems. Approximation method for finding out periodic objects their frequency and amplitude. So, let us take a specific example let us begin with a specific example.

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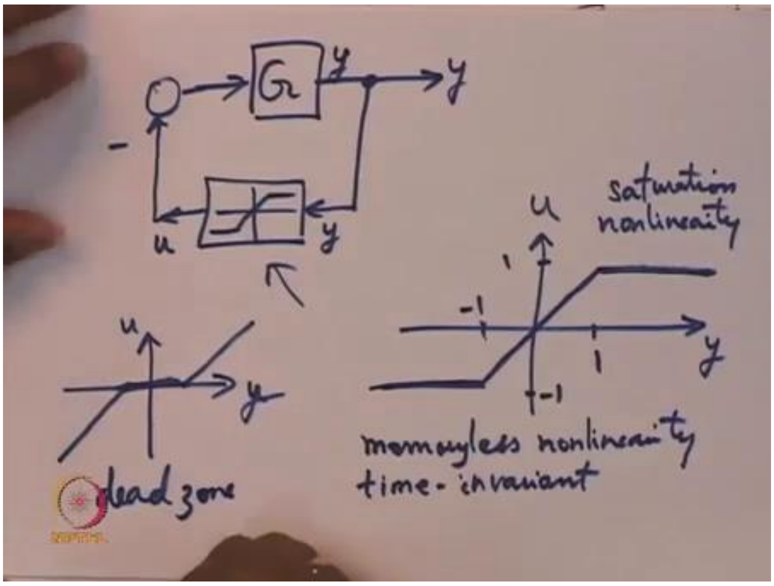

So, consider a system G a transfer function a linear system which has in the feedback path the saturation nonlinearity. This particular device suppose this output is input to this saturation nonlinearity is y and output is u then the graph of this is up to some value. It is linear outside some value it saturates we will call this saturation nonlinearity.

For making the standard we can make this equal to 1 this equal to 1 this is minus 1 this is minus 1. So, it is linear in the range minus 1 to 1 that time it has slope 1 outside that range it has saturated to that value plus 1 here minus 1 here. This is an example of memory less memory less nonlinearity what is memory less about it the output y output to this device output u depends only on the value of y. Now, it does not depend on the rate of change of y does not depend on the second derivative of y in other words the whether y is increasing or decreasing it takes this value.

Further this dependence of u on y does not depend on time explicitly yes memory less and time invariant. Many of the commonly encounted nonlinearities are like this memory less and time invariant nonlinearity. Another example is dead zone this is a dead zone for certain sorry because you have this you have this nonlinearities in the feedback path we prefer calling input to this nonlinearity as y and output is u. So, that output of the original system continues to be. So, this is another example of memory less time invariant nonlinearity this is called dead zone.

It can also be answered as play in the system this is more correctly called as dead zone nonlinearity. This is saturation nonlinearity these are extremely widely encountered in practice. So, let us if one wants to increase the slope if one wants to change the slope of this all one can do is just multiply this output u by a constant k. So, that slope for this range is equal to k, so that that can easily be accommodated let us see how it can be accommodated.

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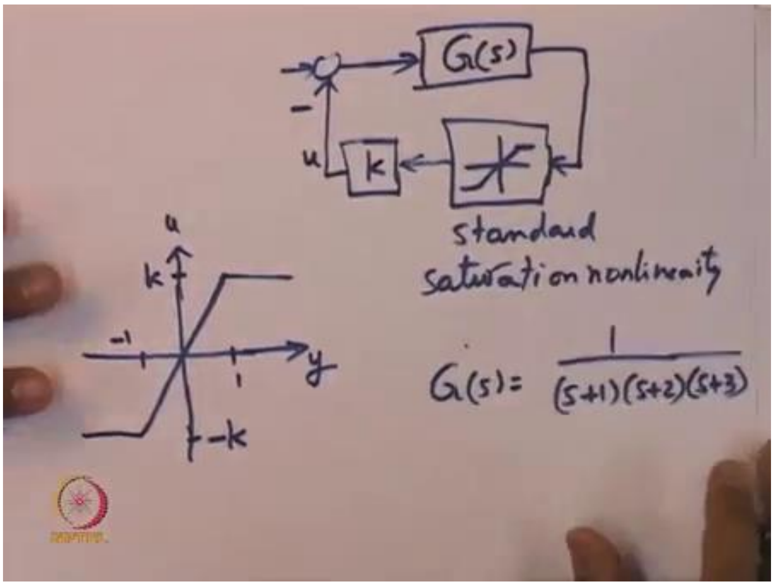

So, let us take this example of a G of S in the feedback path it has the standard saturation nonlinearity standard is not a very standard term I am calling it to say that it is exactly slope 1 in the range plus and minus 1. And then it is saturating plus and minus 1 saturation what comes as output goes through a linear block k. So, one can easily consider an amplification by k and then by such an amplification 1 can get any other slope. Also if you want slope of k in this range if you want slope of slope of value k 1 can easily make this change by that you can get this by just making this output go through a constant gain called k.

So, let us look into whether there can exist periodic solutions. Let us take an example G of S equal to 1 over S plus 1 S plus 2 S plus 3. Let us take this particular transfer function and see for what value of k 1 might expect periodic orbits for doing this we will use some linear systems theory. We will use development of Nyquist criteria Nyquist plot and Nyquist criteria for stability by which is all applicable for linear systems from that we will find the value of k which we will beyond that value of k we expect that there will be periodic orbits there will not be instability that is what we will calculate in detail now.

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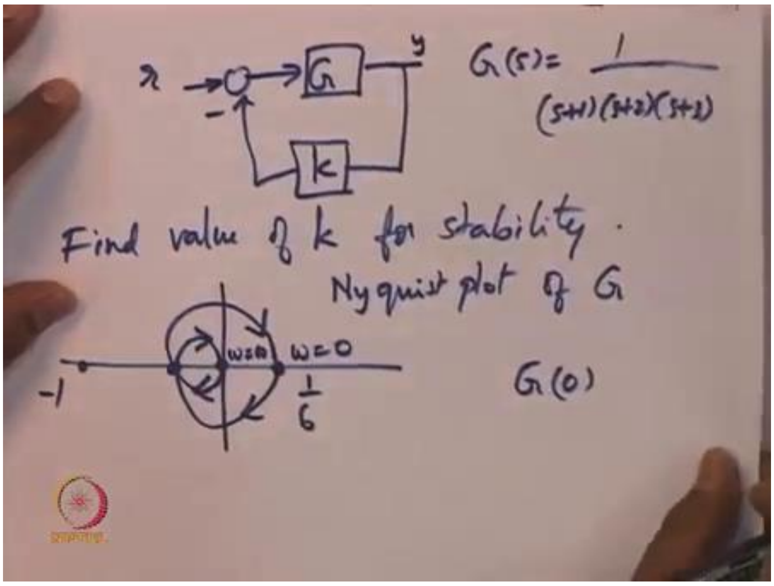

So, as I said we will first calculate for a linear time invariant system in which G of S equal to 1 over S plus 1 plus S plus 2 plus S plus 3 find value of k for stability this is the problem. So, what one can do is one can plot the Nyquist plot Nyquist plot of G Nyquist plot of G with this negative in the feedback will give and depending on the number of encirclements of the point minus 1. We can get lot of information about whether the closed loop is stable for unity gain and even if it is stable for unity gain for what value of gain in the feedback path it will become unstable.

It is only the loop gain that matters this negative sign is important this k could have been either in the feedback path or in the feed forward path. So, let us consider in the feed forward path now, so this Nyquist plot goes like this what is this value it is 1 by 6 why 1 by 6 because this where G of 0 where omega equal to 0 starts. This is omega equal to 0 then it becomes like this and this is where for omega equal to infinity and then from omega equal to minus infinity onwards it comes like this.

So, we are interested in calculating this point precisely where it intersects the negative real axis because for larger values of k we expect that till encircles this point minus 1. So, first let us find out whether it is whether the closed loop is stable of course it is easy to infer that the open loop is stable, because the poles are all nothing but minus 1 minus 2 minus 3 all the three poles are in the left of complex plane. That does not mean that the closed loop will also be stable closed loop is stable by that we mean the transfer function from r to y for that purpose we are interested in finding the number of encirclements of the point minus 1 by the by the Nyquist plot and the number of open loop unstable poles.

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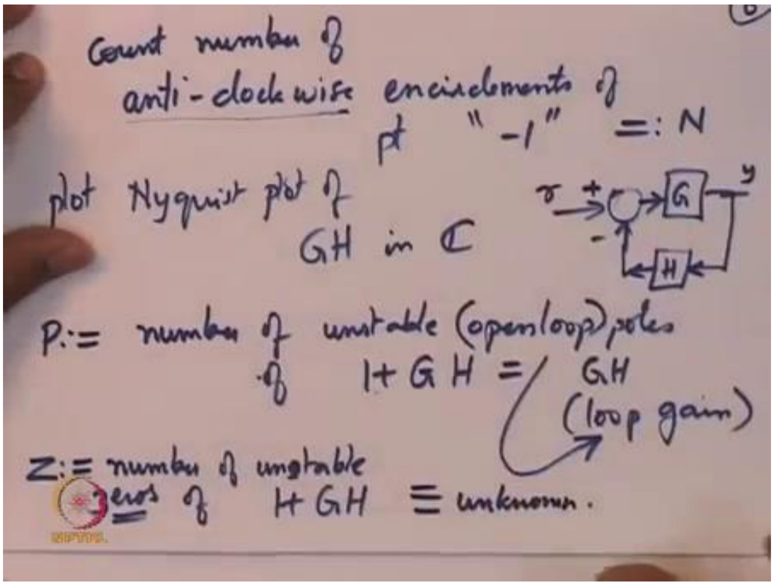

So, we will quickly review the Nyquist criteria for stability also we have the Nyquist criteria of stability as count number of anticlockwise encirclements of point minus 1. So, we are why minus 1 because we have negative unity negative unity feedback it is the H because that is more standard in various text books. So, consider this feedback we are going to plot the Nyquist plot of plot Nyquist plot of GH actually. So, first we will review the Nyquist criteria for stability for this case and then you will replace H by k.

So, around the Nyquist plot of GH on the complex plane and then count the number of anticlockwise encirclements of the point minus 1 and call this integer as N if the encirclement is clockwise then you count the number and you just put minus sign to it . So, this integer N can be positive or negative if it is positive then it has indeed the Nyquist plot has encircled upon minus 1 anticlockwise. If this N is negative that would be because we have clockwise encirclements of the point minus 1 and why is minus 1 important because we have negative sign here. And that is why minus 1 is important then we also have two more integers P is number of number of unstable open loop poles open loop poles of which transfer function of 1 plus G H actually.

Yes 1 plus G H is the important thing you see if you add a number 1 that poles of this is same as of G H which is the loop gainthat the poles do not change by just adding a fix number . If you add a fix number of course the zeroes might change that is indeed important. But, the number of unstable open loop poles of 1 plus G H and G H are the same that is because it is of G H this open loop should be replaced by loop gain if it is constant gain k in the feedback then this is nothing but G in that case .

So, it is number of unstable poles of 1 plus G H is how the Nyquist criteria Nyquist plot Nyquist criteria reads. But, it is exactly loop gain, because unstable open loop the number of unstable poles do not change by adding a constant and when the feedback path is a constant either k or 1. Then it is nothing but number of unstable poles of G and hence it is also called the number of unstable open loop poles open loop poles that is before you close the loop . If you close the loop with a dynamic transfer function called H then of course you should be counting the number of unstable poles of G H which is same as 1 plus G H and why is 1 plus G H important because 1 plus G H comes in the denominator in the transfer function from r to y consider this.

So, z is the eventual unknown we are using Nyquist criteria for stability to obtain this number Z this is number of number of unstable zeroes of 1 plus G H. This is the most important index integer in the three things it is to find z that we are trying to use Nyquist criteria for stability z is used because of it is zeroes. But, zeroes of which that is where many books in fact I have seen text books also that give this particular criteria wrongly it is number of unstable zeroes. But, not of open loop transfer function G not of loop gain G H, but of 1 plus G H.

This it is number of unstable zeroes of this which is actually unknown. It is unknown to whom now, it is unknown in this Nyquist criteria for stability. It is unknown that is what I will first clarify what is the if these are 3 integers.

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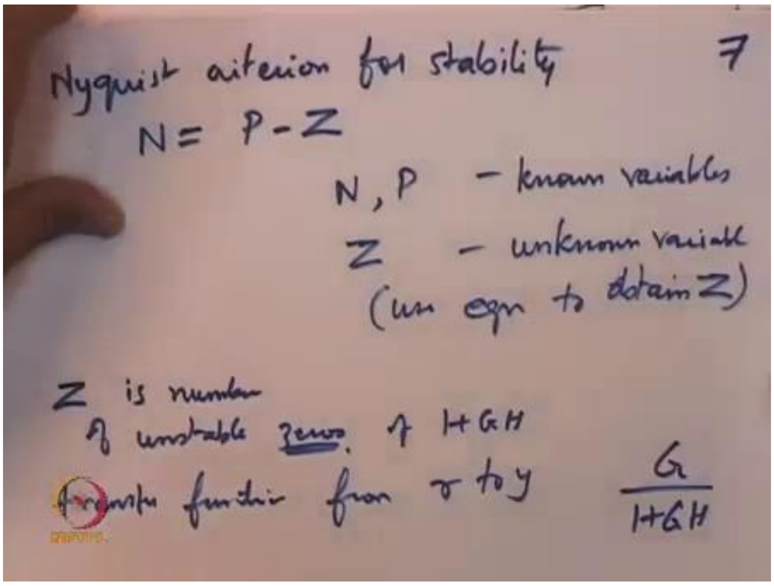

Then Nyquist criteria for stability says N is equal to P minus Z we are supposed to use the Nyquist criteria for stability this particular equation this equation there are three variables only 1 equation in 3 variables 1 equation we expect 2 to be knowns 2 to be known values. So, that we will use the equation to find out the third variable which can be unknown.

So, N comma P known and Z unknown means what use equation to obtain, so please note that this is extremely important this is often told wrongly in various books. So, Nyquist criteria for stability correctly is like this. It is N is equal to P minus Z 3 integers get related by the Nyquist plot what are these integers N is the number of anticlockwise encirclements of the point minus 1 anticlockwise refers to that. These encirclements should be counted positive, if it is anticlockwise it should be counted negative if it is clockwise. P is the number of unstable poles of 1 plus G H and Z is the number of unstable zeroes of 1 plus G H.

So, why Z is important? Z is number of unstable zeroes of 1 plus G H that is what I explained just now. But, why should we be concerned about the number of unstable zeroes of 1 plus G H that is because transfer function from r to y is nothing but G over 1 plus G H. So, the zeroes of the denominator are precisely the poles of the transfer function from r to y.

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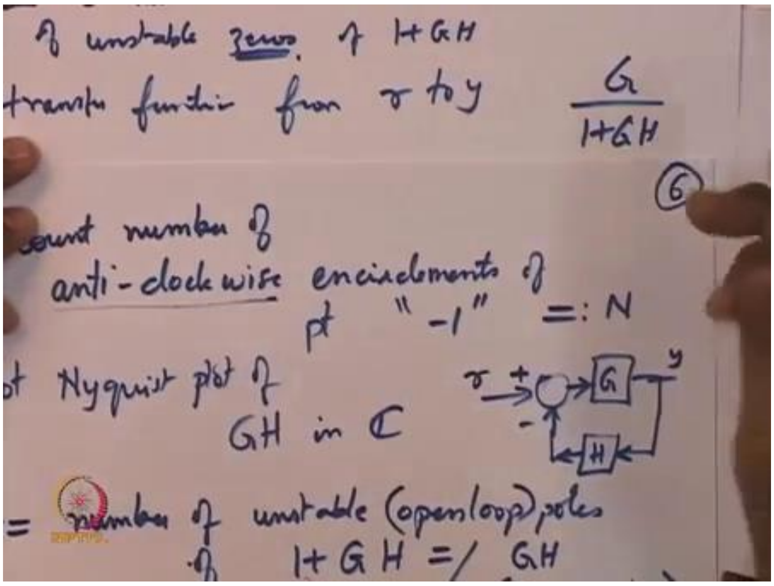

So, please come back to this particular figure in this figure, the transfer function from r to y the transfer function is nothing but G over 1 plus G H. One can obtain this by very straight forward block diagram reduction procedures. In this transfer function from r to y 1 plus G H comes in the denominator, and if you do not want unstable poles in the transfer function from r to y then this denominator 1 plus G H should not have unstable zeroes.

And it is to count unstable zeroes of 1 plus G H that were using the Nyquist plot Nyquist criteria for stability. So, we will conclude that the closed loop is stable if Z is equal to 0. Please do not confuse Z as the number of zeroes number of unstable zeroes of G nor of G H, Z is the number of unstable zeroes of 1 plus G H that is why Z. But, it is the number of unstable closed loop poles there when we are speaking on the closed loop that time Z indeed denotes the number of unstable closed loop poles.

So, it is to obtain Z that we are doing this exercise N and t are known we will use G knowledge of G and H to find out P we will plot the Nyquist plot. And obtain N refers to the number of anticlockwise encirclements of the point minus 1 minus 1 point significance comes. Because of this negative sign in the unity feed back because of this negative sign we have got 1 plus G H as the denominator that is how we will find using the Nyquist plot we will find P. And then we will use this equation plug in N and P and obtain Z.

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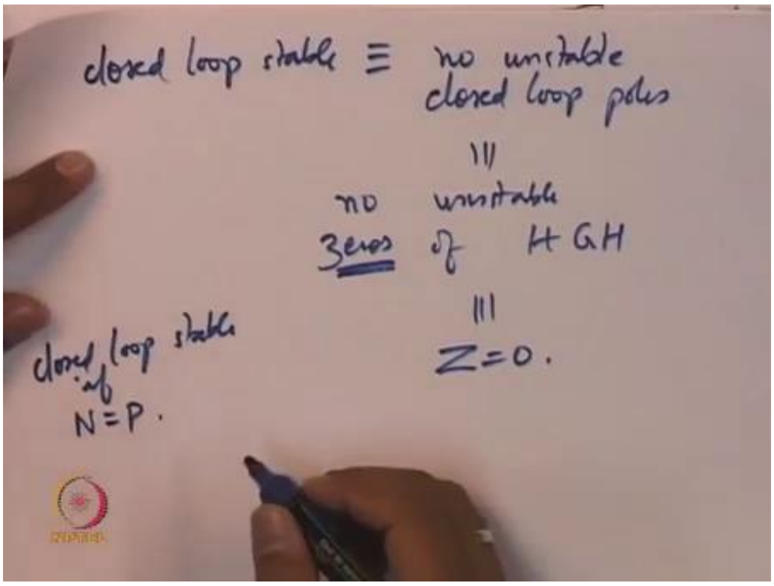

If Z is equal to 0 we will conclude that the closed loop is stable closed loop is stable, no unstable closed loop poles which is same as no unstable zeroes of 1 plus G H which is same as Z equal to 0 . We want to conclude closed loop is stable or not for that that is same as saying that the closed loop has 0 number of unstable poles which is same as the number of unstable. This no stands for number no stands for no unstable no unstable zeroes of 1 plus G H this 1 plus G H comes in the denominator hence we are interested in the zeroes of 1 plus G H and this Z stands for zeroes, but not of G H, but of 1 plus G H.

So, closed loop is stable in other words if N is equal to P . So, this is famous closed loop is stable if N is equal to P, if the open loop has unstable poles open loop G is all of concern if H is a constant. Otherwise you should be counting the number of unstable poles of the loop gain GH. So, if that P is not equal to 0 then you had better have that many anticlockwise encirclements of the point minus 1 in order to get the closed loop to be stable . That is how the Nyquist criteria of the stability should be read as if the open loop is already stable. And if you have only constant gain feedback in the feedback path then P is equal to 0. Then indeed you want N is equal to 0, if you want closed loop stability for that constant gain in the feedback path.

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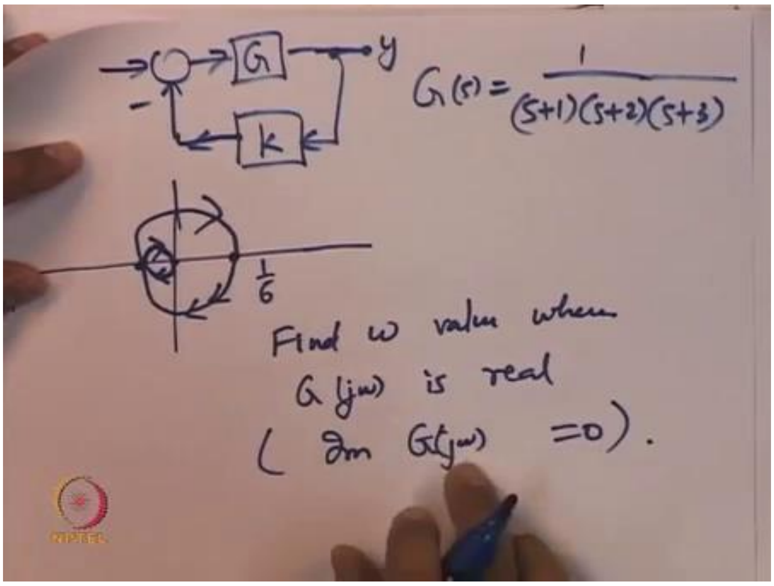

So, coming back to our example we have been working on the specific transfer function G of S equal to 1 over S plus 1, S plus 2, S plus 3 for this transfer function I already drew I already sketched that this is 1 by 6 it start from here . Since, we have only poles the body plot is expected to only have its phase plot decreasing. So, which means that the Nyquist plot is going to go on coming closer to the origin because the magnitude is also falling and also the phase is decreasing from 0o eventually to minus 270. That is when it reaches 0 amplitude and for the other part is just reversal point minus 1. We expect is very far why do we expect it very far because it just started at 1 by 6 here and the magnitude is only falling which means that it is coming closer and closer to the origin. So, it cannot cross minus 1, but nevertheless it is very important to know at which point it intersects the negative real axis that is what we will calculate quickly.

So, we will say find omega value where G of j omega is real which, means that imaginary part of G of G omega equal to 0. One might constantly ask why are we doing this for describing function method, but I should tell you that this same example we will use for describing function. We will use it for sector non linearity calculations or, so for circle criteria. So, for all these purposes same example will serve a very key role and of course we have reviewed a very important topic and linear systems theory namely the Nyquist criteria for stability.

So, we expect that for gain larger than some value even for saturation nonlinearity we expect periodic solutions we will quickly see that it cannot be unstable hence it has to be periodic . So, we are going to now calculate those values of omega where the imaginary part is equal to 0 and for that value omega when we substitute we will get the real part that will give us this real axis intersection one of course. We expected 1 by 6 the other values if any will be on the negative real axis this is what we will conclude very soon by equating imaginary part of G of G omega to be equal to 0.

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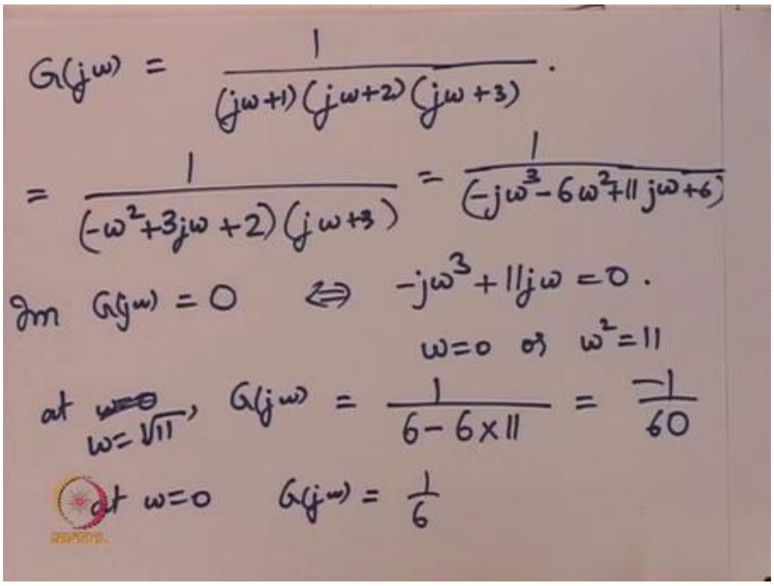

So, G of j omega is equal to 1 over j omega plus 1 j omega plus 2 j omega plus 3. So, which is nothing but equal to 1 over why do not we expand one these two brackets together multiplied inside this gives us minus omega square j omega. There is 2 j omega coming from here in this gives 3 j omega plus 2 this remains a j omega plus 3 this is equal to 1 over please do this bracket opening very slowly. One can easily make mistakes and that will affect all the conclusion later.

So, mistakes here are kind of unpardonable because we have to keep in mind that the consequences because of the small calculation mistakes are very bad later hence more care had better be given here, no excuses for not giving care here. So, this case minus j omega cube here then we have omega square term coming from minus 3 omega square here and again minus 3 omega square here. So, that gives minus 6 omega square then we have the j omega the j omega how many of them is what we have to calculate. So, j omega comes 2 j omega come from here and 3 and 9 j omega come from here that makes it to plus 11 j omega and finally plus 6 .

So, if we have calculated it correctly then this is equal to this, so imaginary part of G of j omega equal to 0 means you see that numerator is real denominator is all that is making this whole number a complex number. So, if the imaginary part of the whole G of j omega equal to 0 imaginary part of that denominator should be equal to 0. So, imaginary part this equal to 0 is if and only if minus j omega cube plus 11 j omega equal to 0. In other words omega equal to 0 or omega square equal to 11 these are the two situations that cause the imaginary part of the denominator equal to 0. Hence these are the only two situations which can cause the entire G of j omega to be a real number.

So, at omega equal to 0 G of j omega is equal to 1 over 1 over what in this part we expect that the imaginary part do not have to be evaluated because it will all cancel off that should indeed be checked once again. But, please do this part of the calculation yourself we are going to evaluate the real part. So, we are going to get 6 minus 6 into 11 which is equal to 1 minus 1 over sorry I such that omega equal to square root of 11. This is what we have evaluated at omega equal to 0 G of j omega equal to that is easier at omega equal to 0 we get this is equal to 1 by 6. So, these are the only two places omega is equal plus minus square root of 11 and omega equal to 0 are the only three places where we have real axis intersection.

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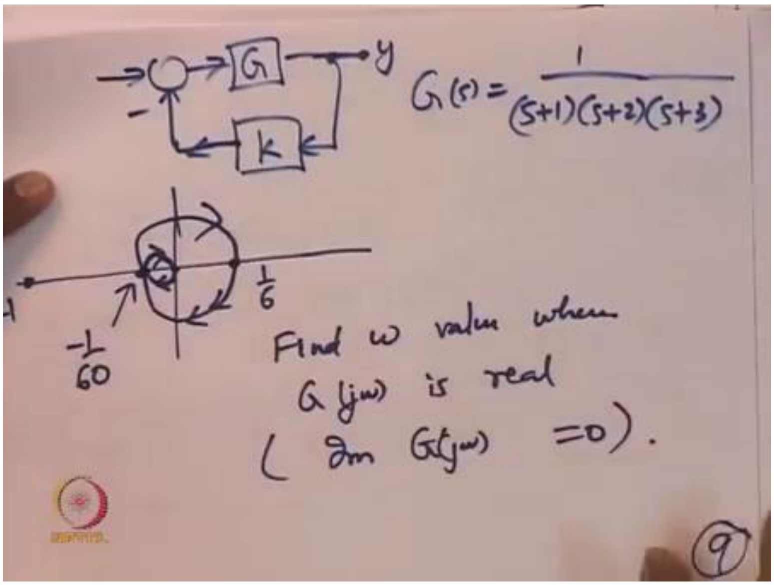

One of this cases is for the positive real axis the other two intersections happen to be at the same point that is expected because we expect that it is symmetric with respect to the with respect to the real axis. And hence if one part of the plot for omega positive if it intersects at some point for omega negative also it will intersect at same point . So, this particular point we have now inferred is equal to minus 1 by 60. We knew that this will closer to the origin than 1 by 6 because the bode magnitude plot magnitude plot is only decreasing because of only poles. So, to have only poles means it is going to be low pass filter which starts at 1 by 6 and goes on continuously decreasing to have any poles to have only real poles.

I should correct myself if one has only real poles and no pole at the origin that is the situation where the magnitude can only keep decreasing. So, the Nyquist plot from wherever it starts will keep coming closer and closer to the origin. So, these are the points where it is intersect at the negative real axis. So, what is this minus 1 by 60 what is the significance of this that is what we will see now.

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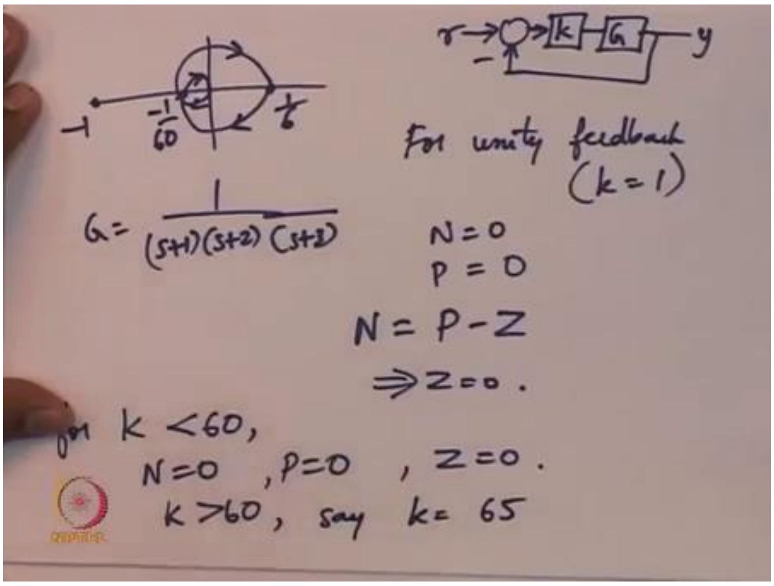

So, if we have now let us by a small manipulation put the k up there as I said it is only the loop gain that matters . If we have k here instead of 1 what you already concluded is that for unity feedback which is nothing but k equal to 1 k is equal to 1. In either this configuration where k is in the feedback path or k is here in feed forward path. In both these cases notice that the loop gain is the same and it is only the loop gain that matters as far stability is concerned. The transfer function from r to y is of course affected whether k is in the feed forward path or feedback path. But, the closed loop stability conclusions do not depend on where inside the loop these k and G are located as long as they are all in the loop they all give the same closed loop stability conclusion.

So, coming back to this figure for unity feedback for k equal to 1 we have already concluded that the Nyquist plot does not encircle the point minus 1 this is minus 1. This is 1 by 6 this we have said minus 1 by 60 even though I had already told that it has to be to right of the point minus 1. We have in fact calculated the point precisely it is that minus 1 by 60. So, we know that for unity feedback k equal to 1 N is equal to 0 P is also equal to 0 because G is stable open loop stable G is equal to 1 over S plus 1 plus S plus 2 plus S plus 3. So, all the poles are in the left of complex plane, so the loop gain G and hence k G also both have P equal to 0. Hence now we you plug this all into the formula N is equal to P minus Z this gives us Z equal to 0.

So, for unity feedback that is for k equal to 1 we have concluded that and the closed loop is also stable in addition to the open loop. But, now we go on multiplying this k for k equal to 2 3 4 5 notice that if we have done. So, much effort in plotting this Nyquist plot of G plotting the Nyquist plot of 5 times of G just means that we multiply this Nyquist plot by 5. And it blows up to multiply by any positive number means just scaling and making this Nyquist plot larger to multiply the Nyquist plot by minus 1 does not mean the reverse this like this. But, what it means is we rotate this by 180 degrees to multiply this by minus 1 does not mean that we swap this like this we mirror image this with respect to the imaginary axis.

But, to multiply by minus 1 in fact means to rotate this whole plot by 180 degrees either plus 180 or minus 180. That is immaterial that indeed makes a difference the you see the whether it is rotating a point by in a clockwise or anticlockwise orientation that is going to get reverse if we mirror image this plot. But, the whether it is rotating clockwise or anticlockwise that does not change orientation of the plot does not change by rotating this rotating the entire plot by 180 degrees. So, please note that these are entirely different things when you multiply by minus 1 then keep note to rotate this plot by 180 degrees rather than to mirror image the plot about the imaginary axis.

So, now what we are able to say is for k larger than one all for k positive for larger than 1 this plot only becomes larger and larger and it will not encircle the point minus until you multiply by k equal to up to 60 for k less than 60 N is still equal to 0. And by whatever k you multiply of course the number of unstable poles of k G do not change by multiplying by a constant k . So, P continues to be 0 and hence Z equal to 0 , but what happens when you put k larger than 60 for k larger than 60 if you multiply this Nyquist plot by k larger than 60. Then this Nyquist plot would have become large enough, so that this point of intersection on the negative real axis comes to the left of the point minus 1.

So, let me take an example let me take say k is equal to 65, let us take an example and for the ease of understanding let us explicitly plot it for this for this situation Nyquist plot of G of S equal to 65 over S plus 1 S plus 2 S plus 3 .

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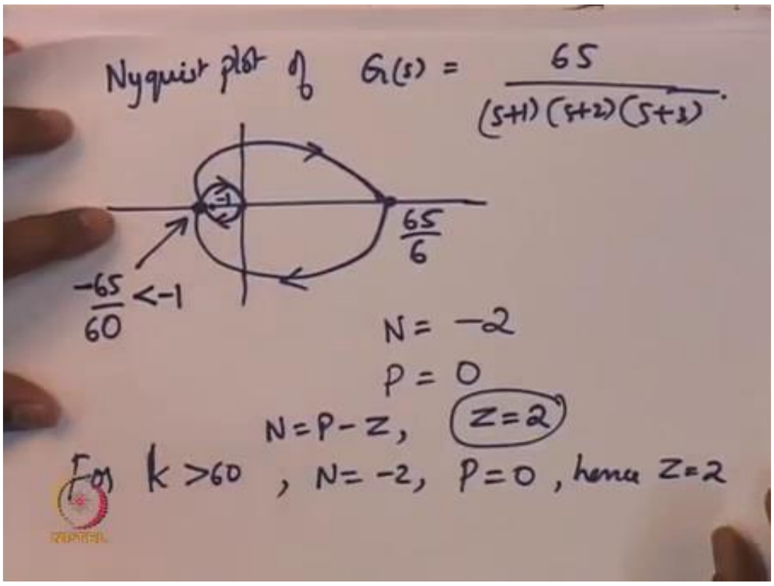

Are we going to do all the calculations again of course not we have already plotted for 1 over S plus 1 S plus 2 S plus 3 all we are going to do is multiply it by 65. So, we have got that it starts from 65 by 6 and slowly comes inside we do not know where the point minus 1 is. But, notice that this point is equal to now minus 65 by 60.

Nyquist plot is useless without the orientations especially, now that the minus 1 point is getting encircled how did we conclude that the minus 1 point is getting encircled this negative real axis intersection has been calculated as 65 times minus 1 by 60 which is nothing but minus 65 by 60 which is slightly less than minus 1 which concludes that the point minus 1 is inside here. Now, we can count how many times it is encircling this point minus 1, where is the point minus 1 it is here.

So, it has encircled clockwise , so N is equal to 2 it has encircled the point minus 1 twice, but clockwise that if it is clockwise then we are supposed to take this number 2 as negative minus 2 P continues to be 0. And hence now we are going to put N is equal to P minus z, so we have got Z is equal to 2 . So, the one of the best things of Nyquist criteria is that everything fits well like magic by no for no Nyquist plot will you ever get Z negative. It is now possible to say that the number of unstable right half the number of unstable closed loop poles is minus 2 we can have negative number for N for P and Z we have to have positive numbers only.

So, if you have if you have P is equal to 0 and have instability then the encirclement of the point minus 1 has to be clockwise all these things are. So, well rigid they are all, so interlinked that one part ensures the other part. So, well, so I am trying to say that you will never encounter a situation where Z has become negative. So, what it means is there are 2 unstable poles for k larger than 60. This is the conclusion for any larger value of k also for k greater than 60 N equal to minus 2 P continues to be 0 of course. Hence Z equal to 2 we have two unstable poles of the closed loop for any value of k greater than 60. This is how one can use Nyquist plot for also finding the precise range of k for which we have closed loop stability just like one can use Ruth Harvards criteria also. So, now let us come across saturation nonlinearity and see what this says for the saturation nonlinearity.

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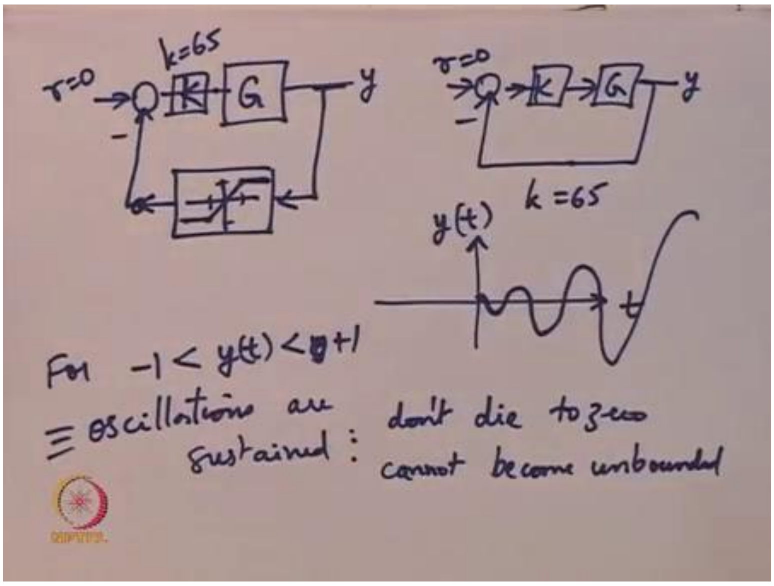

We have a constant gain k here and k is equal to 65. This is why we have we have 2 we have k here we are going to analyze two systems closely here also r equal to 0. Now here we know that for k is equal to 65 we have closed loop stability even when r equal to 0 to have instability means that if the initial conditions are not equal to 0. For very small perturbations we will have oscillations building into the system oscillations that slowly grow slowly grow until things become unstable. So, this is how y locks as a function of time y of t if it starts from nonzero initial conditions then we have the instability oscillations that slowly grow. In other words they cannot be sustained these oscillations are slowly going to grow and become very large why because 2 poles are in the open right half complex plane for k is equal to 65.

But when these oscillations are very small when y is very small then we expect that it is inside the range plus minus 1 . So, they will continue to grow inside this range as long as the output is inside this range it will grow. But, the output cannot after all y is limited what comes out here cannot be greater than plus minus 1 and it is getting amplified by only 65. Do we really expect that oscillations will really become unbounded. In this example here because of the saturation in the feedback path the oscillations cannot become unbounded does that mean that they will all die down to 0 no they cannot die down to as long as the oscillations are smaller than this range as long as y is less than plus minus 1. There will always be a tendency to grow because if y is for minus 1 less than y of t less than y less than less than plus 1 as long as y is inside this range.

The input here the y input to this saturation nonlinearity block looks at this block as a linear system and hence what comes out will only want to grow as it goes like this why would it want to grow of course. This is just gain one for y inside this range this is just gain 1 why would it want to grow because in that case it is like this system where you have 1 in the feedback path and for k is equal to 65 we know that this is unstable. So, it will want to grow because of the dynamics of the k and G together and the feedback aspect of this configuration it will want to grow. But, once it grows beyond plus or minus 1 it is getting chopped it is getting saturated. So, we cannot have a situation like this here .

So, we have to have oscillations we have concluded that oscillations are sustained what is what is the meaning of sustained do not die to 0 why they do not die to 0 that is just ((Refer Time: 44.09)) it does not die to 0. Because as soon as the range of y the oscillation here come smaller than plus minus 1 this block behaves like a linear system with unity gain in the feedback path that time it is like this system and we know that this is unstable. So, the oscillations cannot come smaller than plus minus 1 the amplitude will be greater than one cannot become unbounded why it cannot become unbounded.

Because we know that there is saturation in the feedback path at this point the gain can never exceed the value of the signal. Here cannot exceed 1 in magnitude and this is at most 65 this is at most 65 magnification r equal to 0 in both the cases external input is 0.

We are looking at oscillations from this autonomous system. So, that is why we know that this system cannot have unbounded trajectories either it cannot become oscillations cannot become unbounded. In other words it is sustained that is the this is an example of an oscillator where we have sustained oscillations. But, of course this is not a linear system we need more techniques to find out the amplitude and frequency of the signal of the oscillation here of course. At least the frequencies not hard to find at least approximately, but exact values are very complicated task we will use describing function method to find out the amplitude.

In particular for this example for k equal to 65 70 for different values we would like to find out the values of the amplitude. This is an example where describing function method will help. So, what exactly is the describing function we saw some motivation for describing function for finding out some technique is required for finding out the gain and the frequency when we have non-linear element in the loop in the feedback loop.

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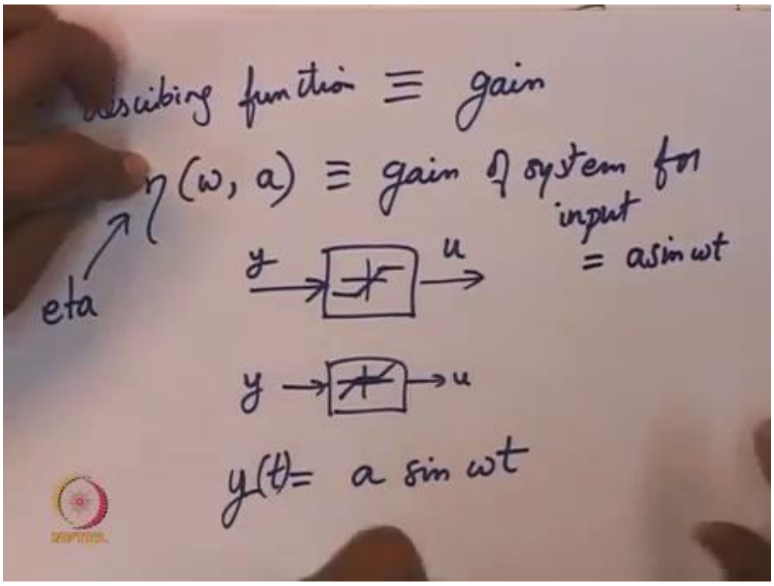

So, one should think of the describing function as a gain it is a function of it is a gain . We are going to denote it by eta again in the depends on the frequency and amplitude, so for every block linear or non-linear we will we prefer calling the input as y output as u only because in our application. We will have this non-linear element in the feedback path for example it could be a saturation nonlinearity which we have seen or it could be a dead zone or it could be a dead zone like this.

So, corresponding to every block we want to associate notion of gain of this operator. And we are going to keep in mind that this gain can depend on the frequency omega it can also depend on the amplitude what are this amplitude and frequency we this is when the input y is equal to the signal y of t is equal to a sin omega t . We need a little more development before we reach the stage where we define the describing function as a gain it is a complex gain. It has a real part and imaginary part, but the motivation of this is completely linear time invariant systems why because for linear time invariant systems we the transfer function of the system is nothing but a gain nothing but a complex gain.

So, hence for non-linear systems also we are going to associate a gain of this system a gain of this operator and this gain of the operator we are going to call as eta. This is called eta Greek alphabet for eta and this gain this gain of system for input for input equal to a sin omega t for this particular input. We are going to associate a gain of the system and this gain we are going to allow it to various function of omega. The frequency of the sinusoidal input we can also allow it to depend on the amplitude of the input signal. So, what is the motivation for this because the motivation for linear time invariant systems behave like this.

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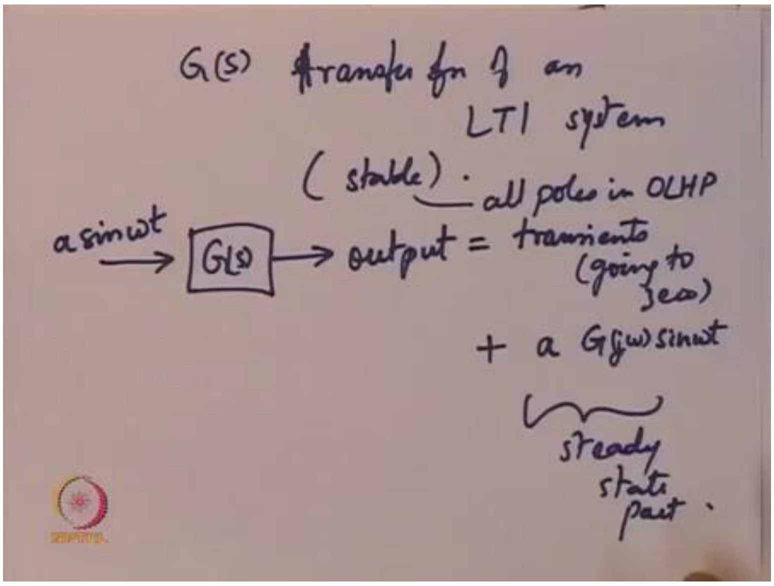

One can check that if G of is a transfer function of a system of transfer function of an LTI system. Let us also assume that it is stable then when you give input a sin omega T to the system G of S what comes out is output equal to some transients what is the meaning of transients? It goes to 0 stable all poles in open left half complex plane no poles in the on the imaginary axis also transients going to 0 plus steady state part. There is a steady state part that we are going to write now this steady state part turns out to be nothing but a G of j omega sin omega t. This is the part we will focus on for the rest of this lecture and also the next lecture.

So, when you give an input a sin omega t this a sin omega t might have suddenly been switched on that is why we might have some transients, but the poles of because all the poles of the transfer function are in the open left half complex plane. There is nothing on the imaginary axis there is nothing in the right half complex plane hence all this transience because of this sudden switching on go to 0. But, eventually there is a steady state part this steady state part is of course it has sin omega t. This is what we like to think of as a forcing function the forced part and the transience natural response and the forced response we are speaking about the forced response.

The forced response has exactly the same frequency sin omega t if a was the amplitude there is a here also. But, there is a complex number associated to complex number that is equal to multiplied by sin omega t of course. This is primarily notation you give a signal that is real a sin omega t how can the output become complex. I mean how can it have an imaginary part we measure signals that are functions of time. They are all real part at any time instant if value of function of t on oscilloscopic measure.

We do not measure complex signals this is the notation G of j omega is indeed a complex number. And this complex number have some meaning the real part is imaginary part have some important meaning, but at the way it is written here it is just notation. So, what is the meaning of this notation then if it is just notation.

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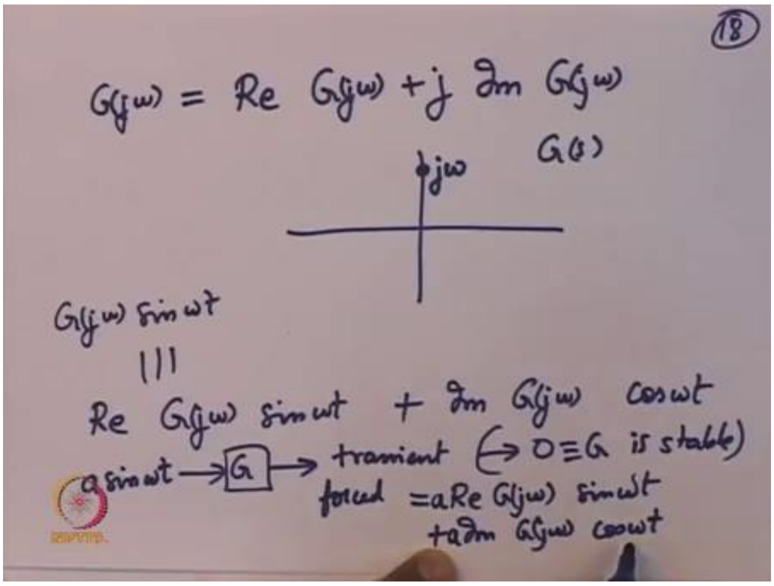

So, if G of j omega is equal to some real part plus j times imaginary part. We can break its real part and imaginary part there is some transfer function G of S we evaluate it at a particular value of omega. Then evaluate G at some complex number there is a very rare chance that it has become a real number it could become a real number. For example, if the Nyquist plot for this omega intersects the real axis except for that rare case this G of j omega has the real part. And the imaginary part both imaginary part by itself is some real number it is got multiplied to j. Now, it has become purely imaginary and now you added another real number now together it has become a complex number.

So, when we say G of j omega times sin omega t what we mean by this is real part of G of j omega times sin omega t plus imaginary part of G of j omega times cos omega t to associate the imaginary part with sin omega t. And this j this j when applies to sin omega t we will think of it as cos omega t . This is a very important fact what is that important fact you take a linear time invariant system with transfer function G you give it a sin omega t what comes out has a transient part that goes to 0. Because g is stable by here we mean no poles on the imaginary axis no poles in the right half plane. And there is a forced part this is also steady state part.

At steady state this a sin omega t will have an effect only on the forced part the forced part is equal to real part of G of j omega times sin omega t plus imaginary part of G of j omega times cos omega t. So, one can check this by first principles by substituting this into the differential equation every transfer function after all is nothing but differential equation between this.

And this in the differential equation for output also you substitute sin omega t and cos omega t the coefficient that come out sorry I forgot a factor a here . If the amplitude is larger the output also will have the amplitude exactly larger, but how much part comes with sin omega t how much part with cos omega t. That is determined by the real part and imaginary part of G of j omega. Now, you see there is no j here, now the output is a real signal what happened to the imaginary part of G of j omega as far as our earlier notation is concerned that is got translated to cos omega t amplitude. So, how do you how does one verify this we can take a simple example and also verify this.

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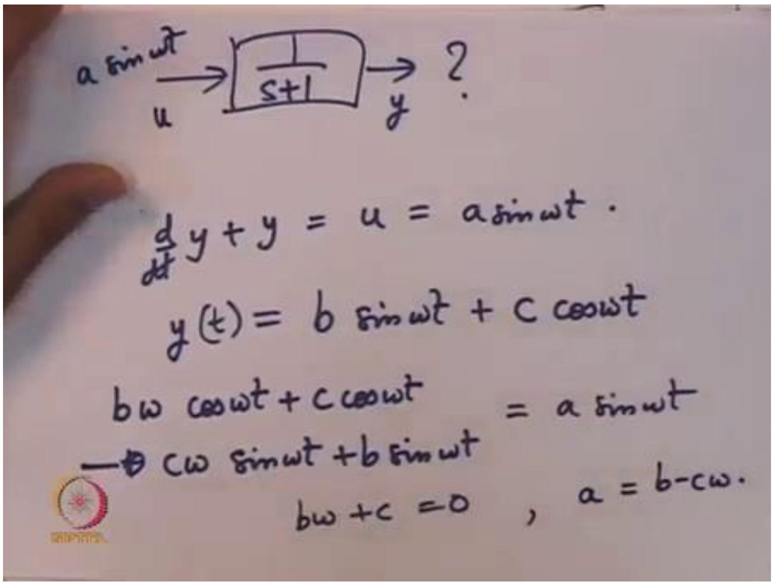

Suppose the transfer function of some system is one over S plus 1 input is a sin omega t output is question mark. So, this particular let us say input is let us come back to more traditional notation of using input u output y. So, the meaning of this transfer function is nothing but d by dt of y plus y equal to u. If this is the differential equation of a system then we associate this transfer function to this system, now u is equal to a sin omega t. So, will write u is equal to a sin omega t here, now for y of course there is a e to the power minus G term which is homogenous. For every initial condition there is a part, but that is going to die to 0. So, we are interested in the forced response part, so we will write that this is some b time sin omega t plus c times cos omega t we are going to find out values of b and c.

So, we will substitute this into the differential equation here and when we substitute we get derivative of sin omega t is nothing but cos omega t. So, we get b omega cos omega t plus c cos omega t plus c omega sin omega t negative sin here plus b sin omega t. So, this part here is the derivative of y and this part here is nothing but y. So, this is equal to a sin omega t, now sin omega t will equate to the coefficients will equate and cos omega t there is nothing cos omega t here, so sin omega t.

In other words we say that sin omega t and cos omega t are independent functions of time if this has to be satisfied for all values of time. Then we can equate b omega plus c equal to 0 and a is equal to b minus c omega. So, we are going to solve these two equations simultaneously for obtaining the values of b and c and by solving this we will get the values of b and c. We have two unknowns b and c and two equations in b and c we will use this to find the values of b and c.

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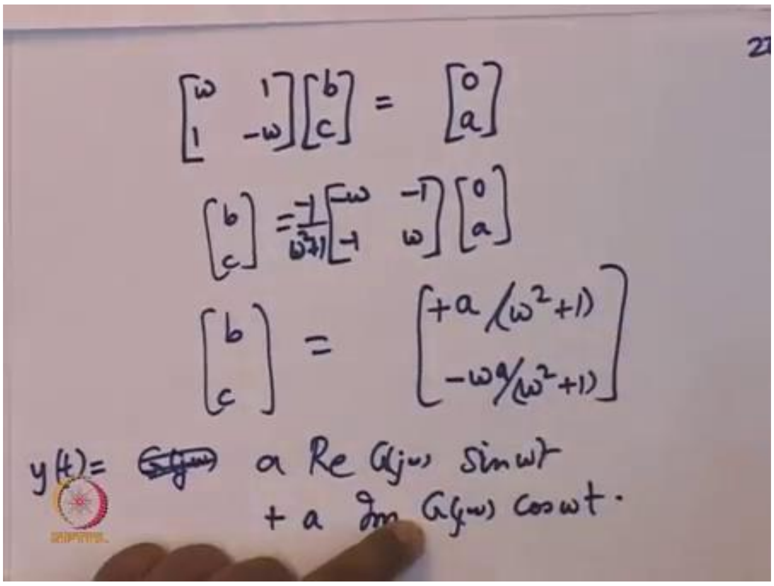

So, what do this translate to let us write this into a matrix b c equal to 1 of them is of course a the other one is 0. The coefficients of b and c is 1 minus omega and the upper equation is omega 1. So, in order to get b c all we to do is b c equal to inverse of this matrix applied to 0 a. The inverse of this matrix is obtained by just minus 1 minus 1 applied to 0 a divided by the determinant of this matrix. Determinant of this matrix is nothing but minus omega square minus 1. So, it is 1 over minus 1 over omega square plus 1 , so this is how we can get b and c.

So, b and c is equal to this is nothing but a second column of this. So, this is minus plus a over omega square plus 1 and here we have minus omega over omega square plus 1. This is the value of b and c, what we can do is we can check whether. So, there is a missing distantly we can check whether these are indeed the real imaginary parts of G of j omega. So, G of j omega from the theory that I said we have we have got output y of t equal to a times a times real part of G of j omega is the coefficient of sin omega t plus a times imaginary part of G of j omega times cos omega t .

So, we will quickly check whether real part of G of j omega is nothing but one over omega square plus 1 and imaginary of G of j omega is nothing but minus omega over omega square plus 1. After checking that we would have rectified this formula that we wrote for linear time invariant systems which is what we will give this motivation for defining the describing function of nonlinear system.

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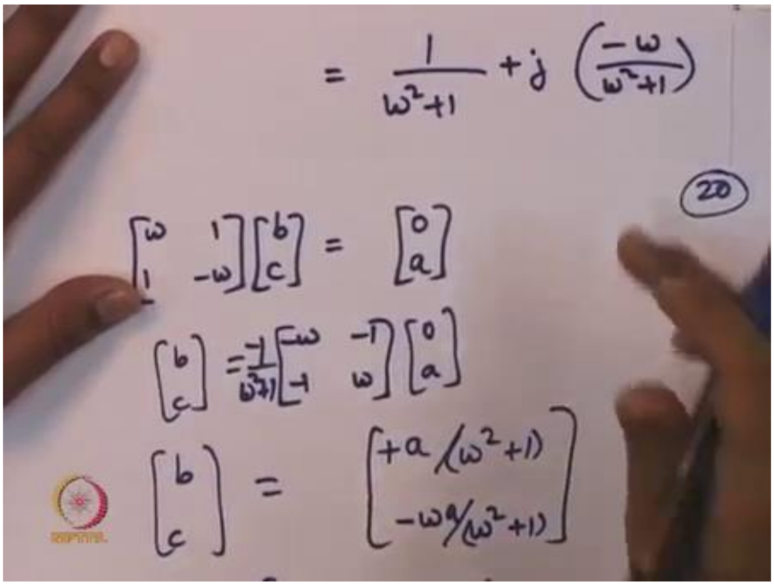

So, G of j omega is equal to 1 over j omega plus 1 this is equal to 1 minus j omega divided by omega square plus 1 this is exactly equal to 1 over omega square plus 1 plus j times, so this is what we have. So, obtained here we have obtained that the real part is nothing but b over a notice that there is there is a factor the part that comes is sin omega t.

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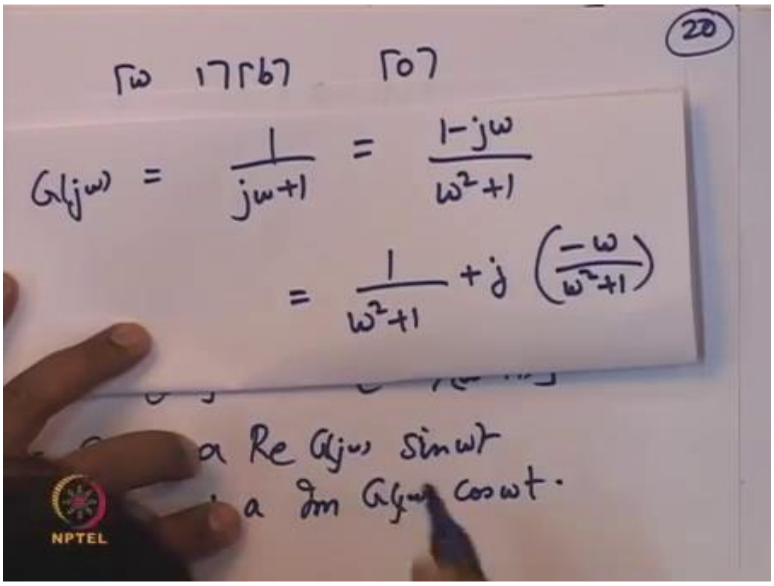

Of course has a factor a that comes because the input has a sin omega t if you scale the input by a constant of course outputs will get scaled by the same constant both for sin omega t and cos omega t. But, how much portion has gone to sin omega t how much portion has gone to cos omega t for linear time invariant systems. That fraction does not depend on a very important point to note that the linear time invariant systems. It is only the relative partitioning relative means I do not mean to say that they add up to 1 that is not the case. But, a does not play a role in these constants real part of G of j omega and imaginary part of G of j omega in other words the magnification also has some contributions coming into cos omega t. But, scaling by a does not change these relative contributions.

So, real part of G of j omega indeed is equal to 1 over omega square plus 1 which is what we got as d acts as for. This a factor and the part coming with cos omega t was exactly minus omega or omega square plus 1 which is nothing but the imaginary part this is what we will like to generalize to describing functions also to for non-linear systems. Also using this notion of describing function where we allow dependence on omega and also for non-linear systems that is very much possible quite expectedly we will see that further case of saturation nonlinearity.