Non Linear Dynamical Systems
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Lecture - 34
Dynamical Systems on Manifolds-1

So, welcome everyone to the next lecture, so today and over the next two lectures, we will cover a topic called dynamical systems on manifolds. So, what is a dynamical system and what is a manifold dynamical system, of course you have seen, it is nothing but a differential equation, but today we will see in little more detail what is a manifold that arises in non linear dynamical systems and hence it is relevant. I would say this is one of the ways that people work in non linear dynamical systems to be more precise.

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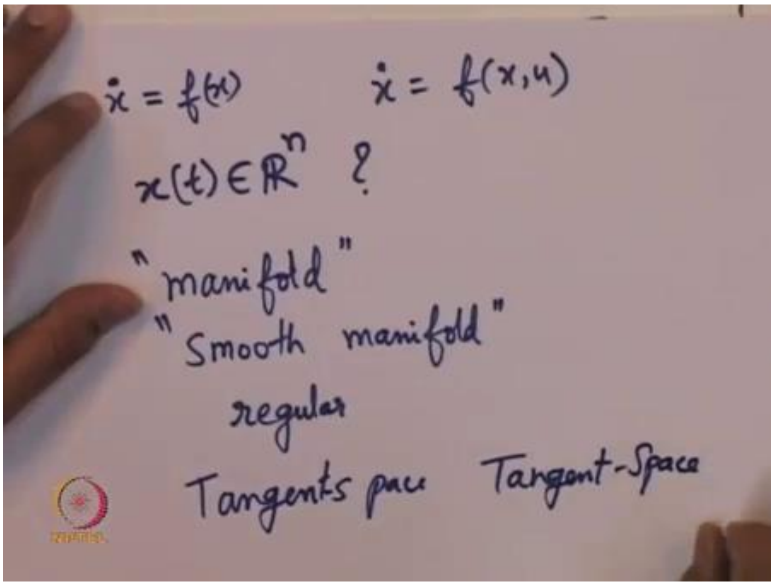

So, when we take x dot equal to f of x may be x dot equal to f of x comma u if there an input. So, of course this f and this f are different clearly this f has only one argument x. On the other hand, this f has two arguments x and u in which it is implicit that u is an input to the system, but then the values of x itself at any time instant R.

Here is a question, so can x t take any value in r n or is it that x of t is required to be in a in a subset of r n is this subset a sub space or is it a more general set, does it have some notion of dimension when we speak of R n? When we speak of vector space R n, we speak of it being n dimensional, but when x of t takes it values in not necessarily the whole of R n, but in the subset. Then, what is the meaning of dimension, for those purposes, we will speak today in more detail about something called manifold. Manifold, generally speaking are smooth manifolds, what is smooth, that we will see soon. In other words, it is also called regular, what is regular about it at every point its dimension is fixed as you change the point the dimension of the manifold.

Once you give the notion of manifold property called dimension, once we give that, we can speak about locally is a dimension constant as you change this point. So, that is what we will use to define a regular manifold, that is also is a smooth manifold. Then on a manifold, we will speak about tangent space, why is tangent space relevant, because it is in the tangent space that the vector field lives, sorry for this bad handwriting. So, tangent space, we will speak about the notion of tangent space, if time permits we will see some examples of manifold some equations today only.

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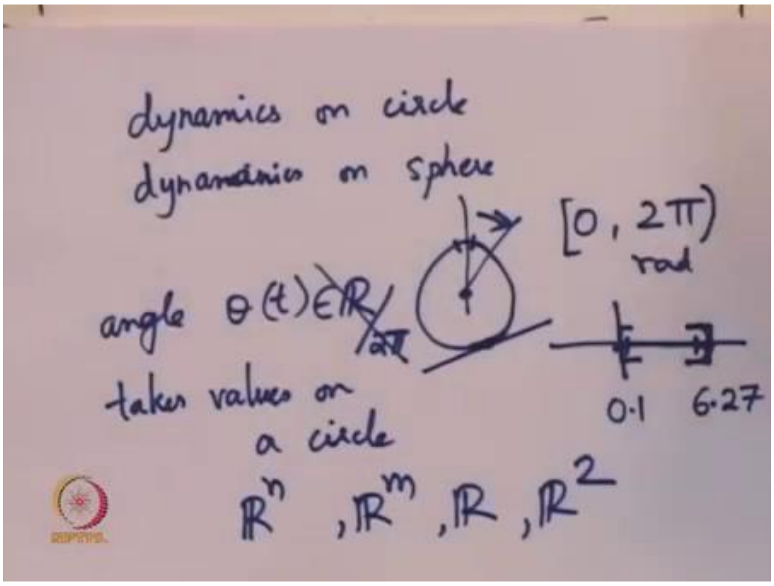

So, why is this relevant for dynamical systems? So suppose we say dynamics on circle or dynamics on dynamics on sphere. So, it turns out that when we are talking about the rate of change of angle for example, we know that angle varies from 0 to 2 pi, but not just that the angle equal to 2 pi is same as 0. So, in that sense it is incorrect to view this set this angle as a interval like this, why am not making it closed at both sides because the angle equal to 2 pi same as 0. So, we let angle equal to be either 0 or 2 pi, we cannot let it be equal to both 0 and 2 pi because they are actually the same. Then, it appears if we write it as interval like this, it appears some point some angle value here and some angle value here, let us say 0.1, this is radians.

When we say 0 to 2 pi, clearly the angle is being measured in radians, but suppose angle equal to 0.1 and angle equal to 2 pi corresponds to 2 into 3.14, which is let us say 6.28. Suppose, 6.27 yeah is slightly less than 2 pi, these two values of angles are not actually very far. So, one should note that this particular point is actually the same as this given that these two angles are same. Hence, it open interval, it is not a not a good way to pictorize this particular set of all values, where angle takes the values.

On the other hand, if we let that particular angle be denoted like this on a circle and we say that this is theta equals theta is being measured like this, so we know that as theta increases and 0.1 angle and 6.27 angle indeed are close. So, here it is also explicit that angle 0 and angle are the same, so when angle for example, when dealing with the angle it is resemble to think about the angle as angle theta yeah takes values on a circle. At any time instant, we can say that theta of t is some particular point on the circle, so that is one example where we will like to think that our theta of t.

It does not take arbitrary real values even though it is mod 2 pi, we know that the angle is the same when if it is referring by a integral multiple of 2 pi in spite of that this is not a good set because it does not suggest that 0 and 2 pi angles are the same. For example, all the 0.1 and 6.1 angles are actually very close, on the other hand, instead of this, if we let that the theta of t takes its values on a circle, then this is a correct representation of this set where theta takes its values. Now, we can ask that this particular set the circle is it one dimensional or two dimensional. So, the question the next question that arises is that what is the notion of dimension for such a set which is not R, if it is a vector space if it is R n or R m or R 1 or R 2, the plane clearly the dimension is here.

It is n m here, it is 1, 2, but then for such more general sets, so this what we were like to call a manifold, the circle is an example of a manifold, but for such sets what is the notion of dimension. So, very soon we will make this more precise we will like to say that if we were sitting at a particular point on the circle. Then, locally this just looks like a line, for example when we are on this earth, when we are on this planet earth at that time we know that actually the earth is a very big globe. It is a big circle, sphere at the particular place, where we are it looks like R 2, we like to think that we are on this particular sphere, which is R 2.

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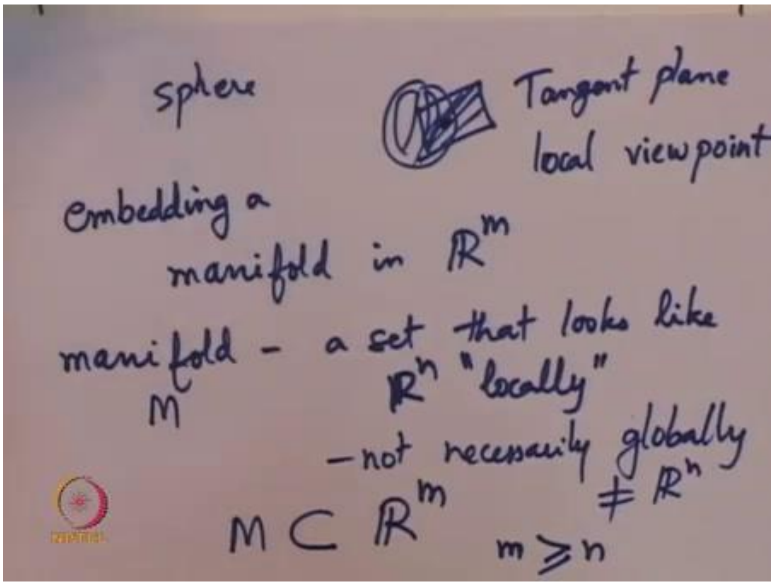

So, let us draw sphere here, so when we are at a particular point on this sphere, then when we draw a tangent to this at that particular point tangent plane. We are not at a very particular point, we are not very concerned that far away this particular plane indeed gets rotated and becomes a sphere locally. At a particular point, it looks as good as a plane, so that that is as far as local view point. So, we then like to say that the dimension of a set is what it is when viewed very locally, so how do we make this more precise, so for that purpose we will speak about embedding a manifold in R m.

So, manifold just loosely speaking a set that looks like R n locally meaning where ever we are at a particular point view around it. Then, it will looks just like R n, but not necessarily globally, not necessarily globally when we view the entire set together. Then, we need not view R n its only locally that we think it is R n more generally such a manifold m might have to be embedded in a dimension R m vector space and clearly in that case m will be greater than n.

So, as a extreme case it might be in fact equal to m in which case R m was equal to R n that time R n was itself equal to R m except for that or it can also be special case R m equal to R n is also manifold. That time R m is also manifold open subsets is also a manifold, so clearly manifolds can be bounded, it need not be unbounded like R m. So, one can have smaller subsets also, now so how does one characterize it too typically.

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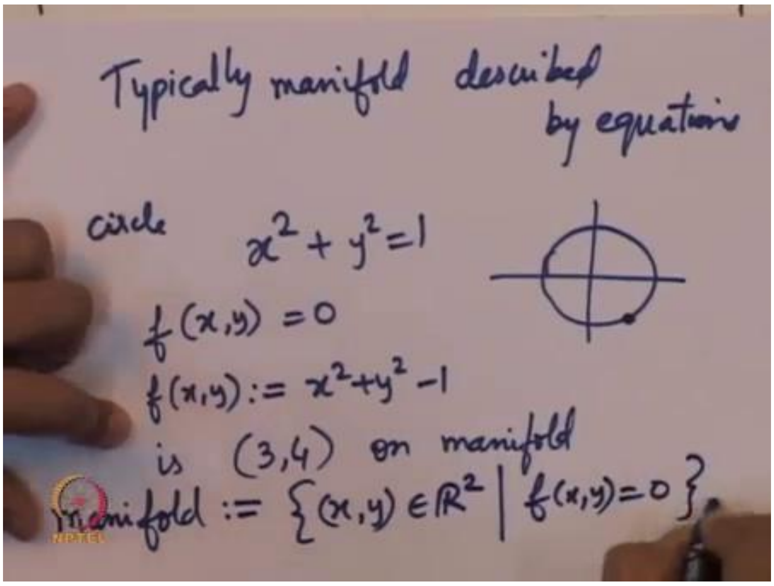

Typically, manifold described by equations for example circle, if we take unit circle as far as the angle, it does not matter what the radius of that particular circle is, but let us consider the radius is equal to 1. So, x square plus y square equal to 1 is the radius is that particular circle that we already drew a circle centered at origin and the radius equal to 1. So, we can view this as f of x comma y equal to 0, where f of x comma y is defined as x square plus y square minus 1.

So, it turns out that manifold can be written as solution to a system of equations in this case there is only one equation and two variables, one can write f of x comma y equal to 0, where f of x comma y is defined like this. So, now what is the particular point a particular point will be in the manifold? It will be on the circle if it satisfies f of x comma y equals to 0. So, let us take a point is 3 comma 4 on manifold, what was the manifold defined as this particular manifold. The circle was defined as the set of all x comma y in R 2, such that f of x comma y equal to 0. So, the definition of our set, so far this definition is concerned 3 comma 4, we can check.

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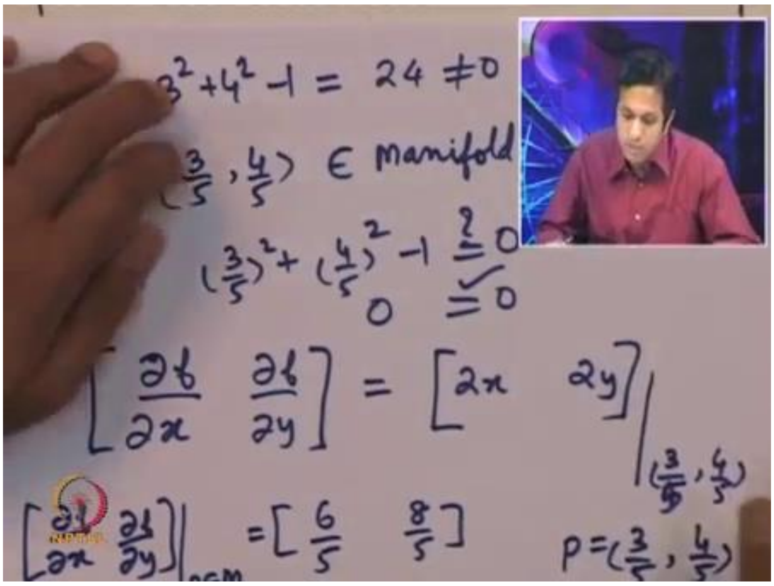

Here, 3 square plus 4 square minus 1 is equal to 9 plus 16, 25 minus 1 that is 24 that is not equal to 0, hence it is not a its not on the circle. So, let us consider 3 by 5 comma 4 by 5 is, this an element of that manifold is this in that set of all points which satisfies the equation. So, we can check this it turns out that this will indeed be equal to 0, 3 comma 5 square plus 4, 3 by square plus 4 by 5 square minus 1 is that equal to 0, it is equal to 0, this will turn out to be equal to 0. So, we see that this particular point is on the manifold, now what we can do is we can take this so called Jacobian del f by del del x del f by del y.

We can consider this particular matrix that particular matrix turns out to be 2 x 2 y and this matrix in general will have x and y because f was dependent on x and y f was a function of x and y. So, this one, we will evaluate at a particular point, for example 3 by 5 comma 4 by 5 at this particular point manifold, when we evaluate it, we get 8 by 5, we get 6 by 5. And we get 8 by 5, this is what we get as del f by del x del f by del y evaluated at a point yeah at a point p in the manifold for the point p equal to for p equal to 3 by 5 comma 4 by 5. We get it equal to this, now we are able to speak about the dimension of the manifold more concretely using this.

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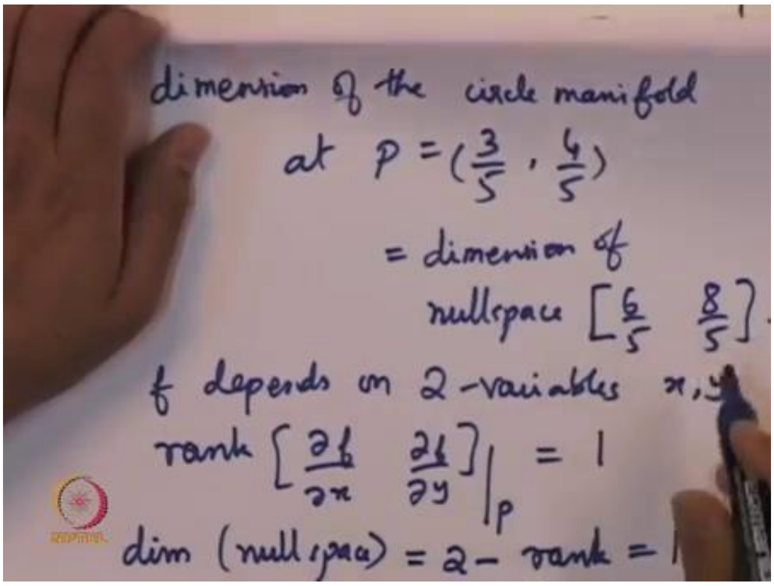

So, let us come dimension of the circle manifold at p is equal to 3 by 5 comma 4 by 5 equal to dimension of null space of which matrix, of that particular matrix that we obtained 6 by 5, 8 by 5. So, this is this is a particular matrix that we get by evaluating del f by del x del f by del y and we can speak about its null space, why we have to speak about its null space we will see in detail soon. So, f depends on two variables x comma y, rank of this particular matrix del f by del x del f by del y evaluated at a point p at which point p, p equal to 3 by 5, 4 by 5 turn out to be equal to 1.

Hence, dimension of null space null space of a matrix is set of all vectors that equal to 0, where that matrix acts on it, we will give a formal definition in the next slide this is equal to 2 minus rank of this matrix, which was equal to 1. This is equal to 1, this is the dimension of null space at that particular point, so notice that the matrix del f by del x del f by del y depends on x and y. When you substitute different points, you get different matrices and in general ranks might change even though the column is the same.

Hence, the dimension of the null space might change in general, but one can verify that at every point p on the manifold the dimension will indeed be one the rank of the matrix will be 1. Hence, the dimension of the null space will indeed be equal to 1 and hence the circle manifold is what we will like to call a regular manifold. This is what we will see in more detail now, so before we see in more detail we will just give a formal definition of null space of a matrix.

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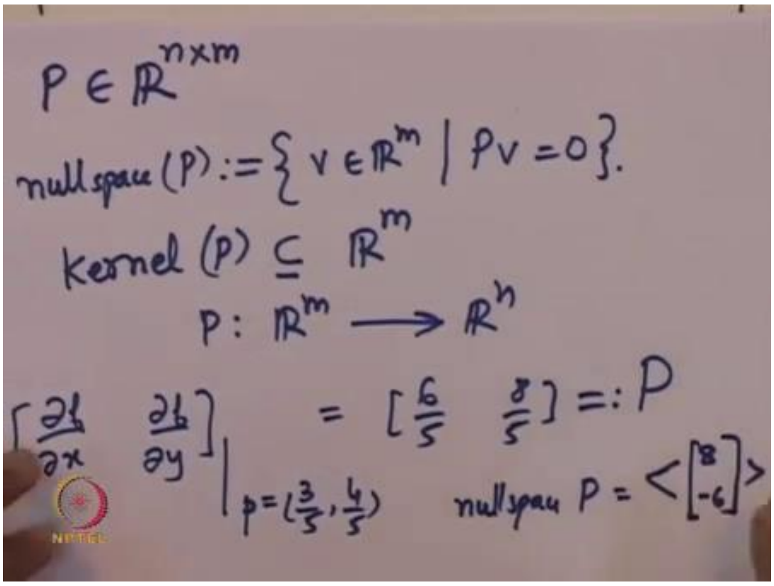

Suppose, we are giving a matrix P, capital P that is different from the point p that we just now saw with n rows and M columns then its null space null space p is defined as set of all vectors v sitting in R m such that p v equal to 0. So, take a matrix with all real entries r for real entries with n rows m columns, and then its null space is defined as set of all vectors v such that p times v equal to 0 matrix p times vector v equal to 0. So, some other words for this purpose is called kernel of p also means the same thing. So, this is set of all vectors that go to 0 null space and kernel, both mean the same they are both in general a subset of R m.

If p is a map from R m to R n, then null space and kernel both mean the same that is what is defined here are subsets of R m, why do we say p, which has m columns maps R m into R n. It has m columns when it acts on a vector the way it has written here, it will require the vector v to have n components and hence the vector v is an element in R n, so null space is an element in this to come back to our particular problem. So, del f by del x del f by del y that particular matrix when we evaluate at particular point small p equal to 3 by 5 comma 4 by 5.

Then, we had got that this one is equal to 6 by 5 and 8 by 5, this particular constant matrix we can look at the dimension, the set of all vectors that go to 0 and that turns out nothing turns out to be nothing but null space of p. Let us call this particular matrix as capital p its null space is nothing but the span of eight minus 6. So, span means you take linear combinations of this particular vector and that particular that set which you get by linear combinations of this is precisely equal to the null space of this. They are precisely the vectors v, which get sent to 0, this you can verify by just plain multiplication.

So, what is the dimension of the span of this exactly one we have only one independent vector and any linear combinations will all generate a one dimensional sub space. So, this is dimension one, so that is how we conclude that the circle locally at every point gives you a null space of dimension one and hence it is a manifold of local dimension one. The next question that arises is at the point p we verified what about the other points, at the other points also will it be indeed null space dimension equal to 1. That is indeed the case that you verify yourself, but we will define the dimension of manifold a little more generally.

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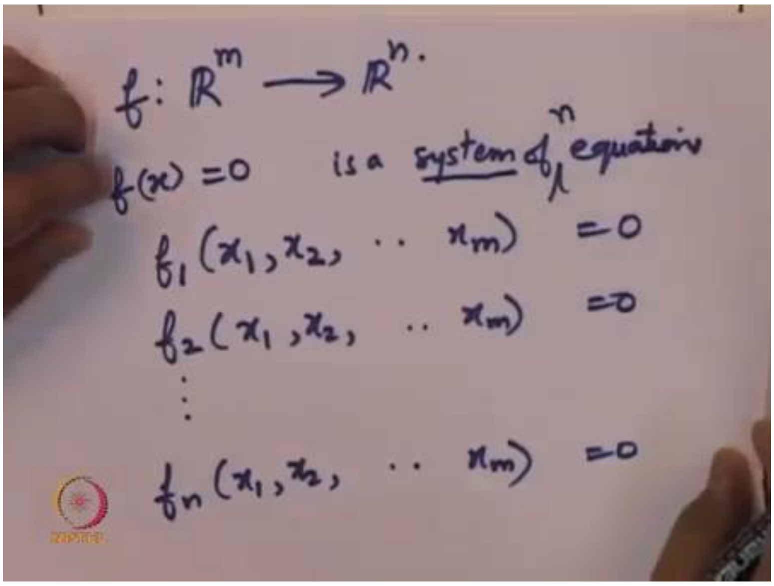

So, suppose f is a map from R m to R n and we say f x equal to 0 is a system of equations, please note this x here is different from the x we wrote in the previous example, why because f acts on R m and gives you R n. If f acts on x, x has to have m components already and f of x itself has n components in more precisely. We can say f 1 of x 1 x 2 up to x m equal to 0, f 2, x 1 x up to x m equal to 0 like this up to f n equal to 0.

So, there are actually n equations that are why wrote system of equations the system of n equations to be precise and each equation involves m variables. So, this system of equations may or may not have a solution in general, so suppose you take a particular point x 1 up to x n that satisfies all these n equations that particular x point will include into the manifold. So, more generally manifold are defined like this a large class of manifolds are all defined as solution to a system of equations solution to a system of n equations and this already makes that manifold subset of R m.

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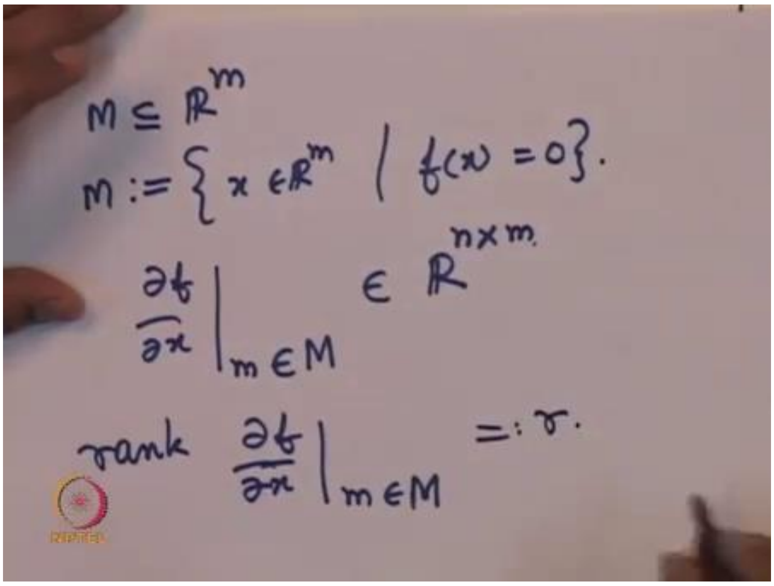

So, what was our manifold was a subset of R m more precisely, it was set of all x in R m such that f of x equal to 0, so n equations are satisfying. Now, suppose we define del f by del x, this we can evaluate at a particular point small m in M, when we evaluate it at a particular point, then this matrix that we get after evaluating becomes a constant matrix with how many rows? It has exactly n rows because f 1 up to f n, n functions are getting differentiated and how many columns it will have, it gets differentiated with respect to m components. Hence, it will have m columns, so this particular matrix this matrix we have has n rows and m columns.

One can speak of rank of this particular matrix del f by del x, after evaluating we speak of rank of constant matrices as far as this course is concerned. So, we will find out the rank of the matrix only after evaluating it at the particular point m on the manifold. Of course, in principle this matrix is defined for any point in R m. We can evaluate it at any point R m, but then we are interested in what happens to this matrix at the point on the manifold. Hence, we are going to evaluate it at the particular point m inside the manifold

M, capital M. So, this particular rank yeah that decides what is, that will help in finding out the dimension of null space, suppose this rank is equal to R, suppose r is a particular number.

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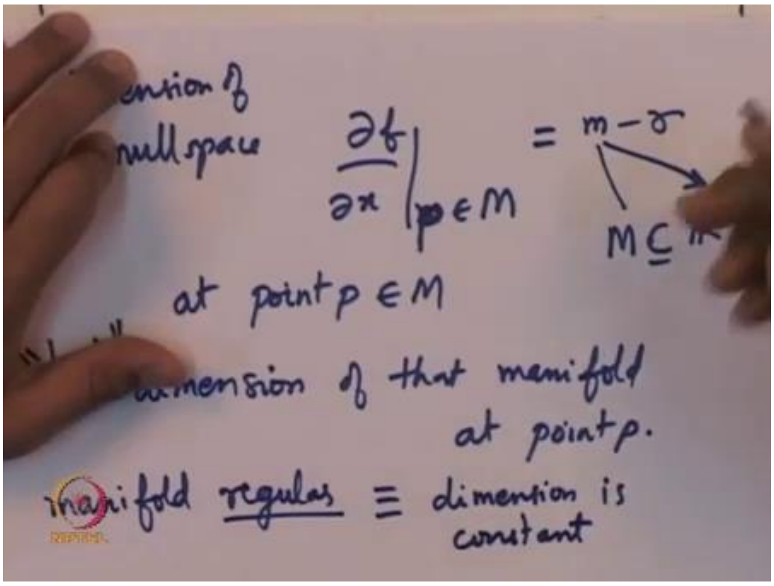

Then what is the dimension of the null space in that case, now this is equal to this dimension of null space of del f by del x after evaluating it at a particular point in the manifold will be equal to m minus R. So, I should point out a few things about the notations, here this m was because capital m is a subset of r m it is it is a integer. So, this m is because of this particular m, it has this particular matrix has m columns while this m is a particular point p on the manifold. So, it is better that I change this to a particular point p on the manifold where dimension of the null space at point p in the manifold.

At point p in the manifold, what is the dimension of the null space m minus rank of this constant matrix, which constant matrix the rank, the matrix that you get by evaluating the matrix at this particular point p on the manifold. So, this is the dimension of local what is local about it because we have evaluated this matrix at this particular point local dimension of that manifold at point p. So, now we can ask when you quote for different points p does the dimension change does the dimension what is the dimension m minus r does the rank change m itself does not change, because its entire manifold is a subset of R m.

So, m itself will not change the number of columns of this matrix will not change, but the number of the rank itself might change depending on the point that you substitute. So, does the R change with the point p, where you evaluated this matrix of functions you can find once in for all, but depending on where you will evaluate it its rank might change? If the rank does not change depending on the point p of the manifold, then the dimension of the null space will also not change because m minus r is the dimension of the null space at that particular point p.

So, we will call this manifold is called regular if dimension is constant what dimension dimensionally dimension of null space will be constant if rank of this particular matrix is constant. So, such manifolds are called regular manifolds and they are the ones easiest to study and we will study only them. So, what are examples of manifolds circle sphere all the ones we can think of.

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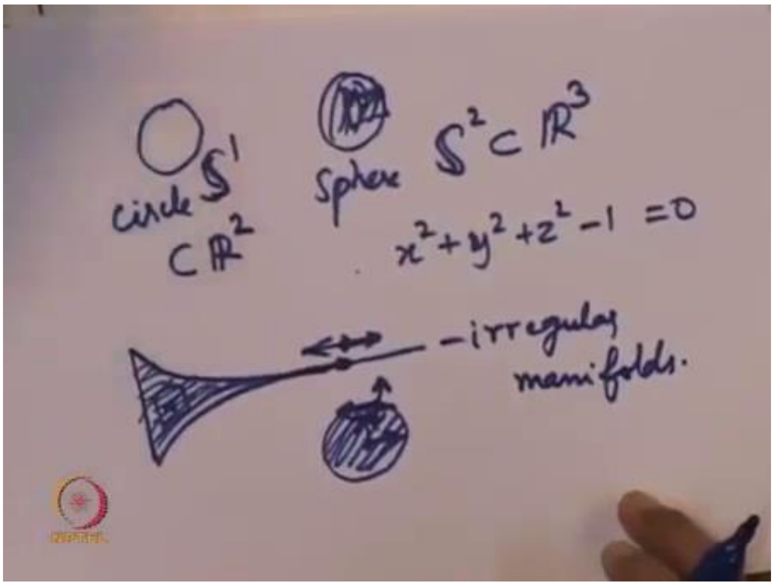

So, circle is a one dimensional manifold embedded in R 2 because it is embedded in two dimensional plane sphere, this circle, this sphere, sphere is also called s 2, this is called s 1. So, s 2 is the sphere s 2 is a subset of R 3, while s one is a subset of r 2 yeah, so this is also a regular manifold in the sense that at any point you can evaluate the particular function how is s 2 sphere defined.

It is defined using the formula x square plus y square plus z square minus 1 equal to 0 in r 3 in three dimension why x, y, z 3 components. If you take one equation, then that equation unless that equation is real unless it does not set any constraints, we expect that a two dimensional degree of freedom is there and this degree of freedom is exactly the dimension of null space we were talking about. So, at any point there are two nook and directions, one can move and those two directions are indeed the null space of this particular del f by del x y z that we get by using this equation, so hence this sphere is of dimension 2.

So, what is an example of an irregular manifold, so look at this particular set the interior of the set. Here, it looks like R 3, here it looks like R 2 that this particular point you can go anywhere in these two directions, but as this becomes like this the same set when we are here there are only one independent direction. Either we go here or the negative of that gives the opposite direction, so there is only one independent direction at this particular point. On the other hand in the interior here one can go in two directions, similarly if we have a circle and its interior on the interior we have two dimensional.

Here, we can go in two directions independently, but on the boundary we have a problem, we cannot go here; we can go like this and the inside that is in there are some constraints where all we can go on the boundary. So, these are the situations where we can say the dimension is not constant this is an irregular, these both are irregular manifolds. So, with that we will not look into more detail about how manifolds are defined. What is the meaning of its dimension at a particular point because our entire example will have manifolds with constant dimension at every point and they are the regular manifolds?

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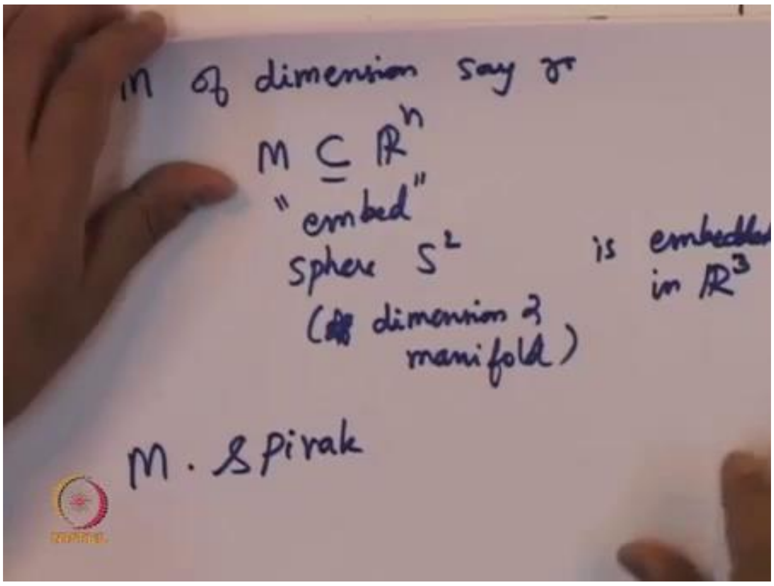

So, what is a vector field defined for a manifold, so take a manifold m of dimension, say r, so this r is not to be confused with the rank r we had in the previous pages. Suppose, it is dimension is r and this m we will like to embed it in r n embed, what is embed about it? So, what is embed about it even though as we said the sphere itself is dimension two manifold the sphere is physically being placed in R 3. So, we embed it in a larger dimension vector space if required sphere s 2 manifold, it is a manifold of dimension 2 manifold is embedded in R 3, R 3 meaning x y z our space has dimension 3.

So, one can think of the sphere s 2 can even though it is a manifold of dimension two, it cannot be placed in R 2, one has to embed it in a larger dimension vector space R 3. So, it is an important question about manifolds about what dimension vector space you have to minimum go larger and embed, so such theories explained in more detail in books by Spivak, M Spivak. He has one book on calculus on manifolds one thin book, but he has many more volumes, which speak about such questions in much detail and also more complex questions about manifolds.

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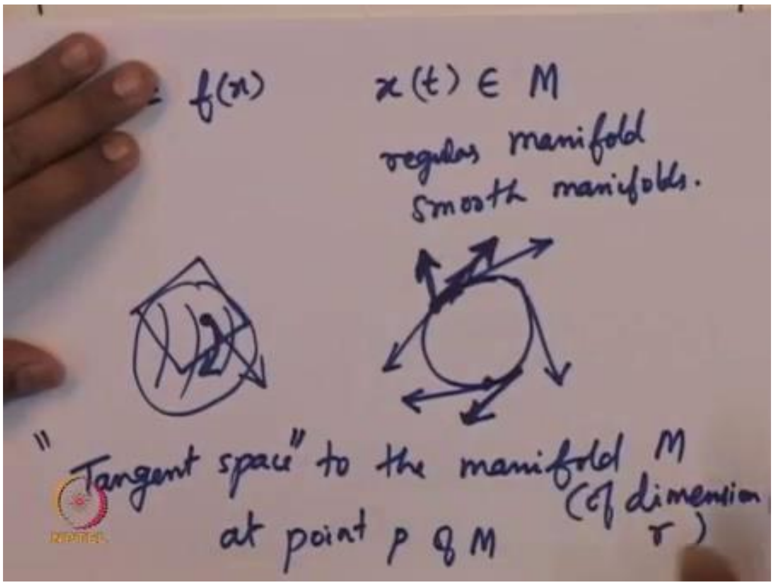

So, as far as we are concerned we are dealing with dynamical systems where the variable x evolves on a manifold. So, consider x dot equal to f of x, for the time being x this is a time invariant system and x of t takes its values in manifold m and we do not want this manifold m to have a dimension that is varying. So, we will call it a regular manifold, regular manifolds are the ones which are also called smooth manifolds, one can speak of infinity functions defined on manifolds tangent spaces defined on such manifolds in a more general setting. So, let us take an example we will like to say that while x was on the manifold the vector field itself x dot.

This function x this f is different from the f that we had used for defining the manifold there f was such that its solution set of all solution was defining the manifold, but right now the manifold is already defined if required. It is also been embedded in a larger dimension vector space, now this f is defining the vector field it defining the dynamics. So, take a sphere and this x is evolving on the manifold only, now we will like to say that x dot is a vector in which in which set we will like to say that it lives in the tangent space to the manifold at that point. So, take another circle, let us start with a circle, so x of t takes its values from the manifold and suppose the manifold is a circle.

Suppose, at some time instant it is here, then x of t the fact that x of t has to remain on the manifold x dot itself takes its values in a tangent space in a tangent line to this manifold either positive or negative, that is where the rate of change can be. Why it is important to know the vector x dot itself cannot be out going out why because you see if you if one is to required to be on this manifold the circle, then the rate of change cannot suggest that we go here. It will clearly come out of the manifold immediately, but if we say it has to go in this direction it will go little in that direction and one gets a different point on the manifold and one evolves like this.

At this point, we might say we have to go like this here like this of course, we are not going to move here in the next time instant infinitesimally after little amount of time we will reach here and there. The tangent is at a different point, so x dot is equal to f of x it is a differential equation at a particular x on the manifold f x is a vector in the tangent space to the manifold at that point. So, more precisely tangent space to the manifold of dimension, we said r of dimension r at point p of suppose the manifold was manifold m manifold m. So, it is a tangent space to the manifold it is tangential to the manifold, but tangential at which point at the point p of that manifold, hence space is if the manifold itself was dimension r locally it looked like R.

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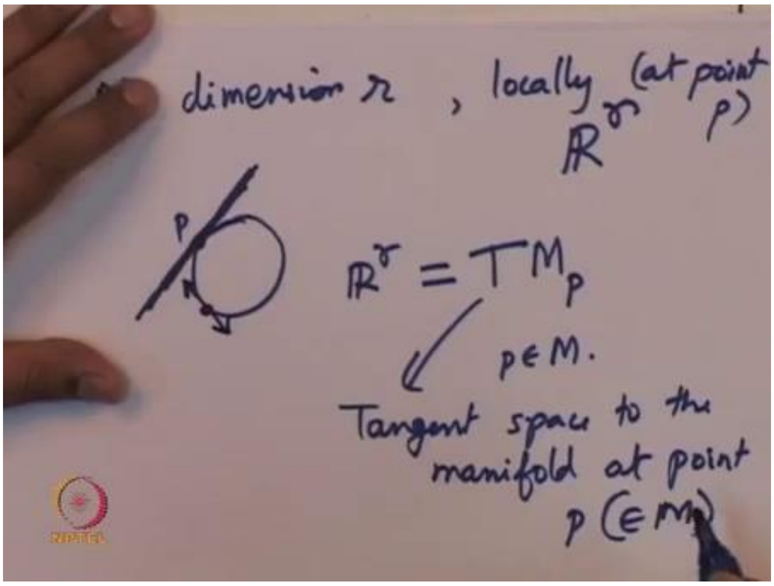

So, dimension R m of dimension r means locally it looks like r, dimension locally at point p it is like r. For example, a sphere as I said let us take a circle, for example, at particular point, we said that it looks like a line, so when we draw the tangent line to this particular circle at point p. Then, the tangent this tangent space is certainly r is equal to tangent space to the manifold at the point p, where p is the manifold. So, what is this tangent space to the manifold at point p, point p, of course p has to be also in the manifold. We do not consider tangent spaces to the manifold and the tangent space itself is also tangential at some point p not on the manifold.

That is not going to happen because x of t lives in the manifold at any time instant at this point suppose this was x of t. Then, it can move either her or here, so it has it is forced to be tangential to that particular manifold at every time instant that is the rate of change. Hence, that particular set of all vectors where the rate of change can belong to that is called as a tangent space to the manifold at that particular point p.

That tangent space not just looks locally like r manifold was looking like manifold was locally like r at every point p because it was a r dimensional manifold this tangent space. On the other hand in fact is equal to r, it is equal to r, where what about the origin of this particular vector space that origin of this particular vector space that origin of this vector space exactly the point p that is the important thing.

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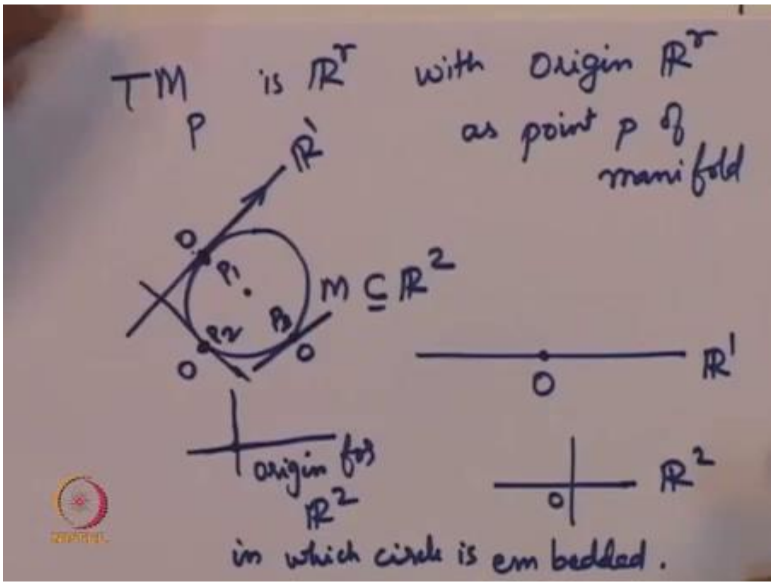

There are different copies of this tangent space, tangent space to the curve at the point p is r with origin of this vector space as point p of manifold, what do I mean by this, let us take the circle. Let us take this particular point, so this is one dimensional manifold the circle and this is a point p this is a origin of this particular line. So, we speak of line as r 1, this is a origin we speak this as r 2, and this is a origin, now when this r 1 happens to be the tangent space to this particular manifold, this manifold m itself is embedded in r 2.

The circle itself is embedded in r 2 where this perhaps is origin for r 2 in which circle is embedded.

So, this our coordinates and our particular circle is here, so notice that this particular circle does not have its centre is not as origin like the earlier circle this is some other circle whose centre is somewhere else, radius is not necessarily 1, this is a manifold. Example for manifold this also can be written by such system of equations like we had written before, but now on this manifold point p is here and its tangent space is here the origin. So, tangent space itself is r 1 to that particular manifold at point p and the origin of that particular vector space r 1 its origin is 0, which is exactly the point p. So, if we have another tangent space at this particular point this is origin for this for this called p one this p 2.

Similarly, if we have this as another point p 3 and this origin of that in that sense, we have plenty of tangent spaces. We do not have one two or three tangent spaces this circle has been embedded in r 2 there is a origin of r 2 in which this manifold has been embedded that origin is a origin of r 2, but we are speaking of this manifold, which is one dimensional manifold. For this manifold this point p 1 at which that tangent space is tangential to the manifold this point p 1 at which that tangent space is tangential to the manifold at point p 1.

That p 1 is itself is origin of that tangent space as I said the tangent space is not locally like r 1. It is in fact r 1. So, where is the origin to this particular of this vector space its origin is exactly p 1, what is the significance of this point p 1 being origin that we will see when we actually consider the differential equation.

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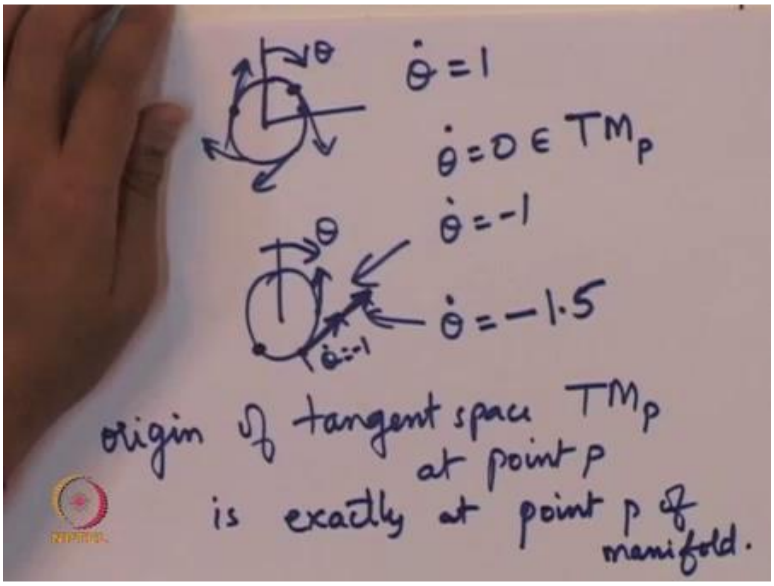

So, let us consider a circle in which theta dot equal to 1, so at every point we will like to say that this how theta increases. So, we may say this is why this not anticlockwise, now we only take anticlockwise as positive, this is not a manifold and one can have any convention as far as this manifold is concerned. So, now at every point rate of increase of theta is equal to 1, so the rate of change is equal to a vector one in that direction, what about theta dot equal to 0, where would that vector be at every point, it is a vector of length 0. So, it would just be there neither left side or right side where would theta dot equal to minus would be theta, we had said is increasing like this increasing clockwise theta dot equal minus 1 would be that theta is decreasing.

So, it would be in the opposite direction, at every point this would correspond to this was our theta was increasing like this at every point the vector would be pointing like this. It is decreasing like this at a particular rate what about the length of the vector length of the vector indeed denotes the rate of change and length of the vector has actually unit theta by time rate of change of angle with respect to time. Hence, the length of vector itself cannot be directly related to the coordinates r 2 in which the circle has been embedded.

So, except for the length of the vector the direction itself has lot of significance, but it also has some relative significance in the sense that we know that if this theta dot equal to minus 1. Then, this is this vector corresponds to theta dot equal to minus 1.5, because it is longer than this vector. So, the vectors within the tangent space can be compared with respect each other, but a length of a vector in the tangent space is not directly comparable to the length in the manifold. Elements in the tangent space have dimension value divided by time in that sense units are different.

So, notice that we can see that theta dot equal to 0 means that at that particular point arrow has length 0. It neither increases nor decreases and that corresponds to the origin the 0 is in that tangent space to that particular curve at that particular point, hence it is exactly the origin. So, it is at point p increasing or decreasing, so important conclusion is origin of tangent space origin of the tangent space t m at point p is exactly at point p point p of manifold.

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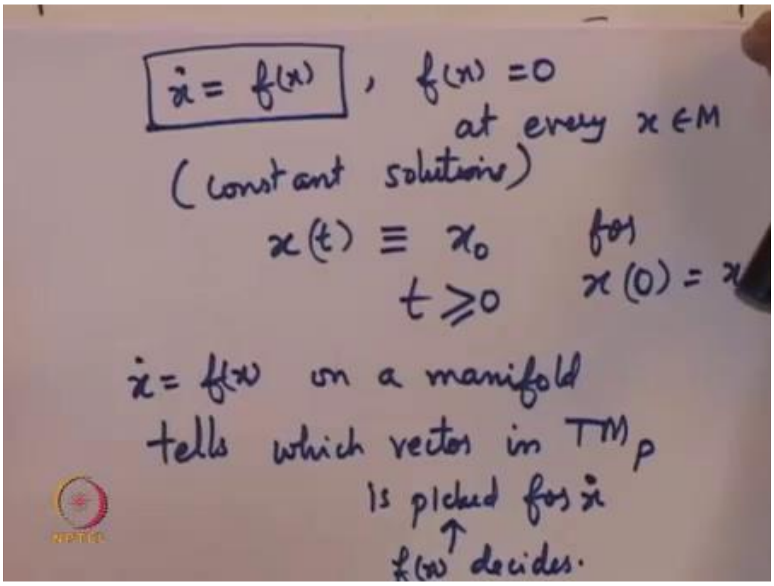

So, as an example x dot is equal to f of x f of x is equal to 0 at every at every x in the manifold this is like constant solutions, all solutions are constant x of t is identically equal to x naught for x at time t yeah at any time.

We have at t equal to 0, it is equal to x naught it will identically be equal to x naught for all time t for all t greater than or less than equal to 0. Also, why because rate of change is 0, so tangent space at every point which vector has been picked, so this differential equation from all the vectors that are possible in the tangent space at that particular point. At an initial condition, f of x picks a particular value in that tangent space, it tells us which direction it will evolve. If you say a constant vector field, it means it will pick the

0 vector, so it will remain there that there is no arrow there is no arrow because arrow has length 0.

So, that is an example where the origin is in fact the vector field has been has picked, hence x dot is equal to f of x on a manifold tells which vector in t m at a point p is picked for x dot, this picking is what f of x is doing f of x decides. So, this a way of understanding a vector field that a vector field of particular differential equation tells that at a particular point x on the manifold f of x tells you which particular vector in the tangent space. The manifolds at that point p has been picked and has been defined as x dot when you integrate you go to particular future time.

There, f is evaluated at a different of the manifold, but the fact that at each time instant the vector belongs to the tangent space ensures that the vector does not the rate of change does not make the x go out of the manifold. The fact that the dynamics are constrained to be on the manifold that is guaranteed by the fact that f of x is an element of the tangent space f of x is not suggesting a vector outside the tangent space. That ensures that the dynamics remain on the manifold to the manifold dynamics remain on the manifold. So, this set of three lectures this and the next two are not intended to be into lot of depth about this way of seeing non linear dynamical systems it just suffices that we take the union.

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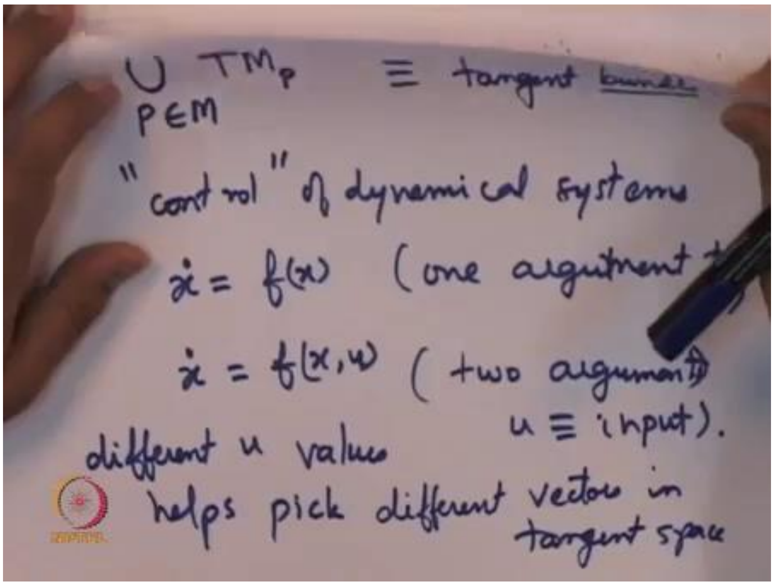

We just introduce some words so if it is a union over all points m in the manifold of the tangent space to the manifold of the point p the union all this is called tangent bundle. What is a tangent bundle, it is a bundle it is a collection of tangent spaces. How is the collection being defined we said T M P is the tangent space to the manifold at the point p, but you can vary this point p for all p in the manifold that union that collection together is what defines the tangent bundle. One can speak what is the structure of the tangent bundle itself this questions that is asked from a research view point since many years.

What about control of dynamical systems, so how is control viewed here till now we had been viewing x dot equal to f of x at every point p. We pick only one vector from the tangent space, but if you have x dot is equal to f of x comma u, so as I said as I warned in the beginning of today's lecture, these are different there is only one argument this is a different system of equations. Only one argument to f, while here we have two, so when you see ways papers one should note that this f has two arguments.

We would like to say that u is an input u is an input to the system, now here different us different u values helps pick different vectors in vector in the tangent space different vectors in tangent space more precisely. It helps you pick a whole family of vectors in the vector field what family, let us take an example and see.

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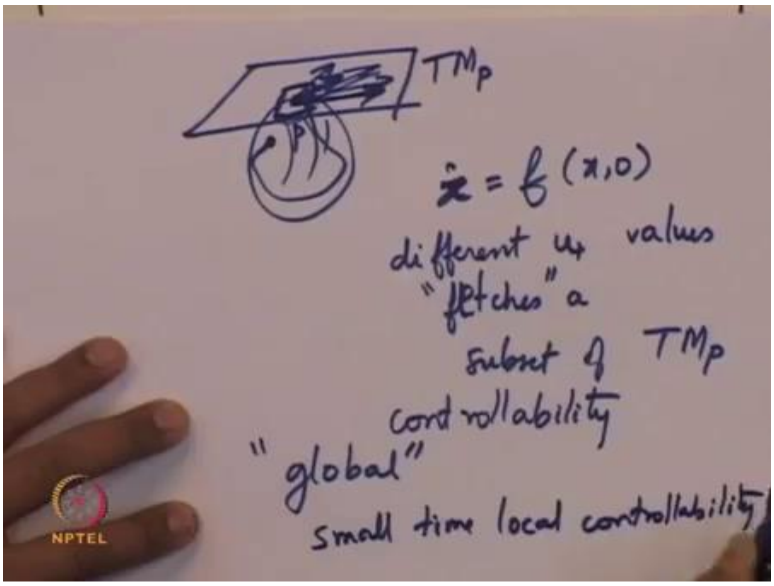

Suppose, we are on a sphere and at this particular point this Particular point p, this is a tangent space to the vector field tangent space to the manifold at the point p. Suppose, this is how f of x yeah x dot is equal to f of x comma 0, when 0 input is given, this is how it is, but when some particular value u 1 is given it helps to say here. So, it is possible that this is a whole class of inputs that you get u 1, so different u 1 values different u values fetches all this vectors in this. Whatever has been shown fetches a subset of t m p, one can speak whether this it fetches this subset defined in different values this f of x comma u will be different values you see. So, it will help in picking not just one vector in t m p, but a whole collection of vectors. In that sense, this whole family of vectors from the tangent space can be picked by taking the different values of u. Now, we can ask what about controllability, so control itself means that you pick a whole collection of vectors that whole collection is defined.

Now, inside that collection which vector you pick is about choosing particular value u controllability is about whether that collection is enough. You can go anywhere in the manifold, so global manifold global controllability is that you can find some trajectory to go to every point. Perhaps, you need lot of time to go there on the other hand you can ask that by just very small quick manipulations, can we go to every point nearby. So, this is what we will say small time local controllability, so given the fact that the different u values might give you more than just one vector in the tangent space to the vector to the manifold at point p.

By different choices of u, can you go around the open interval neighborhood of the manifold at that point p, can you go if that open interval neighborhood is made smaller and smaller, that is what is called small time local controllability. On the other hand, global controllability ask the easier question can you go from any point to any point by some choice. When you go to different point you get different family of vector fields which is the collection of the tangent space at that point. So, by such careful choice can we go from any point to any point that is what global controllability about? So, this is as far as the different questions that are asked using this language using this notion of tangent spaces. So, we will see some more properties of tangent spaces in a few minutes in the next lecture.

Thank you.