Nonlinear Dynamical Systems
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Lecture - 35
Dynamical System on Manifolds-2

Welcome everyone to this next lecture on tangent spaces and manifolds. So, we had just began with the definition of tangent space and a manifold. We also said what is a tangent bundle, and it helps to know little about some more examples of manifolds.

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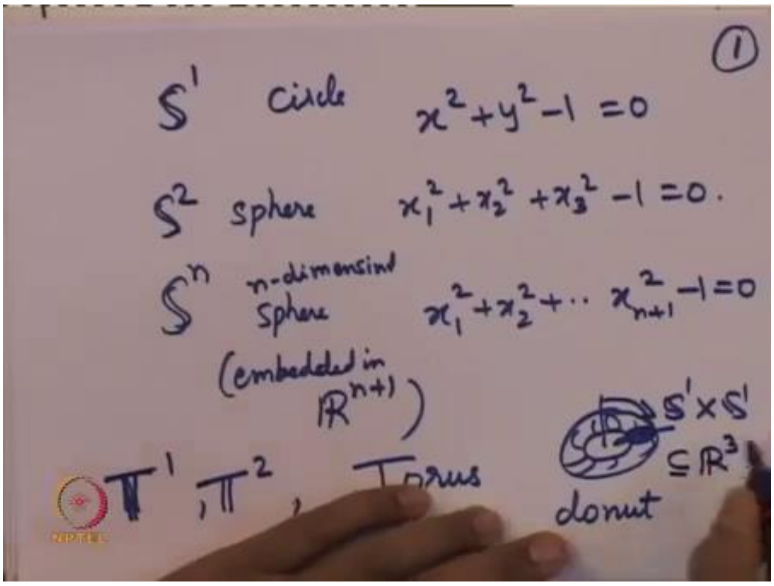

So, we saw some standard terms S 1, we said is a circle, circle meaning we just mean x square plus y square minus 1 equal to 0. What is defined by this equation? S 2 was our standard S 2 sphere here, which we prefer calling x square x 1 square plus x 2 square plus x 3 square minus 1 equal to 0. Similarly, we can have S n the sphere n dimensional m n dimensional sphere embedded in R n plus 1, R stands for the set of real numbers super script n plus 1 means n plus 1 components, which we will say x 1 up to x n plus 1. So this we will say S n. Another importance object is so called torus T 1, T 2, etcetera. So, this are called torus.

So what is a torus? This torus is also what can be thought as donut T 1, T 2, T n and more generalizations, but this turns out, here in this case this turns out to be S 1 cross S 1. Any point is denoted as what angle around this torus and, then if you cut the torus at any particular angle, then we get another circle. In that sense, the torus with one hole is embeddable in R 3.

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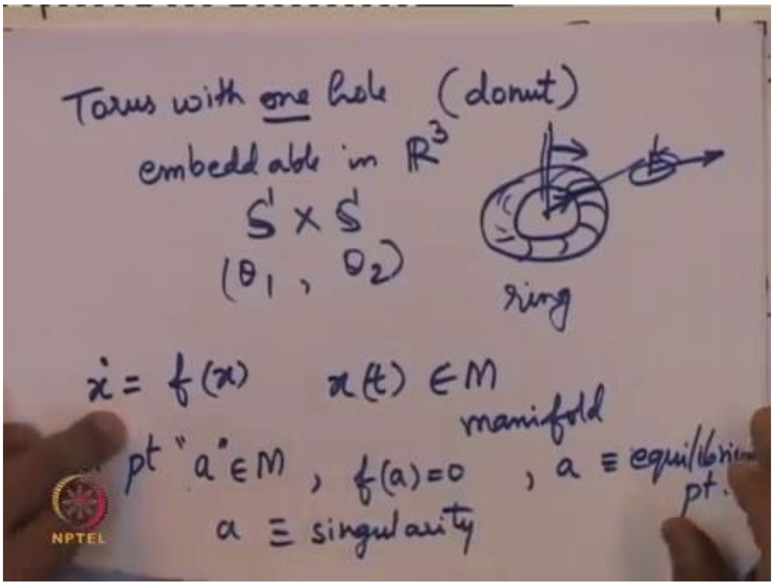

So, this is subset of R 3, let me write this in little more detail. Torus with one hole, which we should think of as like our donut. Donut embeddable in R 3 and this turns out to be S 1 cross S 1 and why is that look at it like this. There is, this is like a ring, which we can hold with by putting our hand inside this. So, this can also be thought of like a ring, a ring whose cross section itself is a circle. So, one can tell at any point, any point on this ring can be told at what is the angle with respect to such a frame. Let us say, and once it is in that cross section, once we have identified the cross section using this angle, one can say which point on this cross section by another angle. That cross section suppose it is like this then with respect to radically outward of the outer of the bigger of this frame, one can say what is the angle with respect to this circle. In that sense one can see that it is indeed S 1 cross S 1.

So, there are different ways this particular topic requires a good amount of imagination and also a good amount of rigor to be able to do this systematically, and not just keep imagining various objects. So, it is easy to see for this example that, this is indeed one to one correspondence with the set S 1 cross S 1. In other words, one angle theta 1, another angle theta 2, these two together describe the precise point on the torus. We are not including the interior of this particular donut, we are just speaking about the torus, but this is the torus with one hole.

So, this is also, some standard manifold that people deal with. This particular thing is very relevant for example, in robotics where we have two angles, two angles, one at one joint another at another joint and one can ask that we are speaking of the evolution on a torus and what kind of dynamical system, what kind of equilibria do we have. This brings us to the next topic that, once we are given with a manifold, one can ask, what can we say about the equilibrium, what can we say about the singular points?

So, given a dynamical system f of, x dot is equal to f of x in which x of t evolves on a manifold. So, unless we specify otherwise, in this course we are going to be dealing with only smooth manifolds, what we defined as regular manifolds. So, on this manifold, we have this dynamics and if it turns out that at point a on the manifold m, if f of a equal to 0, then we will say this a is an equilibrium point, equilibrium point. It is an equilibrium point, why because if f of a is equal to 0, rate of change of x with respect to time 0.

So, x dot is 0 hence, x remains at that point, in that sense it is equilibrium, but if being on, if being in equilibrium means. So that particular point a is also called a singularity, the singular, there is some singularity, it does not mean that some particular matrix is singular. Singularity just means that, something is different here and in this particular case all the components of this function f are all 0, it is equal to 0 vector hence, the rate of change of x at that particular point is 0, and in that sense it is in equilibrium point. One can ask, what happens about a neighborhood. So, this is something that we have seen in detail, so one would have to linearise at that particular point.

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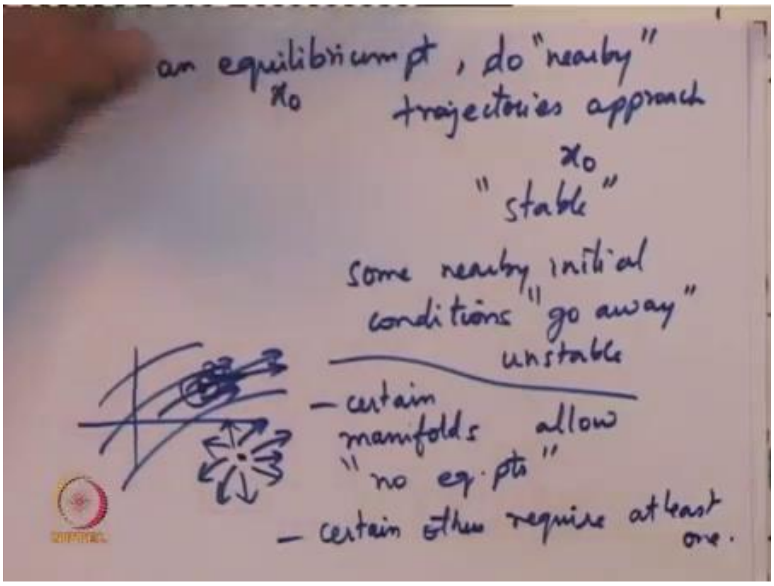

So, at an equilibrium point at an equilibrium point do nearby trajectories approach, which point, suppose that equilibrium point was called x naught, do nearby trajectories approach x naught. If all nearby trajectories approach x naught, then we have decided to use the word stable, if all nearby trajectories approach x naught asymptotically, if they converge to x 1, we will call it asymptotically stable. If they do not blow up, if they remain in that small neighborhood, then we will call it just stable.

On the other hand, even if some, if some nearby trajectories go far, if some nearby initial conditions, this are things that we already saw in more detail when we were studying lyapunov stability. But if some nearby initial conditions go away, then we already called it unstable.

So, these all studies are relevant only at an equilibrium point, why is it relevant only at an equilibrium point. Because let us consider r 2, so if this is a point which is not a singularity, then that point itself is not going to remain there, as a function of time it is going to evolve go further. Because the vector field at that point is not 0, it is non zero hence, it will move in that direction. So, nearby points also it is likely to be non zero, if this function f is continuous and if it is non zero at a particular point near by it cannot suddenly become 0. So, nearby also it will be non zero.

So, all those points are going to, also any way move. So, it does not ask do nearby trajectories approach this point, this point itself is not even an equilibrium point. So, the question about stability or instability automatically applies to only equilibrium points, so but consider this particular point where the trajectories are, let us say changing in a way because so the neighborhood is indeed worth studying in detail because that point happens to be in equilibrium point.

And one can ask that, can we stabilize this? If there were an input one can say that, we will like to stabilize this, so stability is a question that we also apply only to equilibrium points, and it depends on the manifold. It is possible that certain manifolds, certain manifolds allow, certain manifolds allow no equilibrium points, certain others, certain others require at least one. And now, speaking about a global property of this manifold, what is global about it?

Not just locally it is, it turns out that certain manifolds force you to have some equilibrium point at least. If you want the, if you want the dynamical system f x dot is equal to f of x, if the function f should be continuous, then certain others require at least one. So, this is what we will see in little more detail in this lecture.

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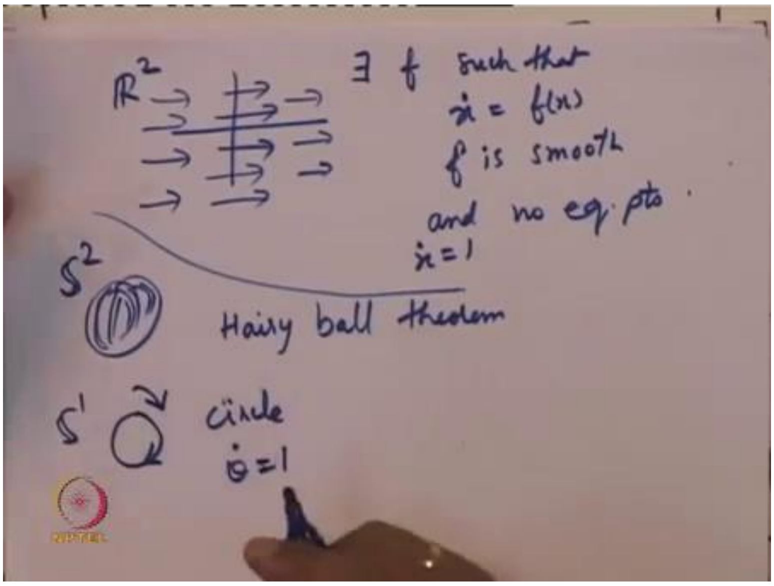

So, let us take an example, so suppose we speak about R 2, the question is if somebody tells us, can you draw a vector figure R 2, in which there is no equilibrium point? Does there exists f such that, x dot is equal to f of x, f is smooth, smooth meaning in this case it is just continuous and differentiable, so more generally smooth word could also mean infinitely often differential, any number of times the derivative exists. f is smooth and no equilibrium point at all.

So, answer is yes, one can just make all, one can say x dot is equal to 1 so that, so that no point is an equilibrium point. On the other hand, the same question answer is no, if you want this f to be smooth, and x evolves on a sphere. On a sphere, it turns out on S 2, S 2 turns out the answer does not exist. So, we cannot have vector field that is continuous and there is no point, where the vector field is 0. In other words, there is no, there is no equilibrium point at all, such a situation cannot happen as this sphere S 2 is concerned unless you let this f to be discontinuous.

So, this turns out to be a very important result, which we will see in more detail today, that result is called hairy ball theorem. So, this requires us to develop a little more concept, but why I am trying to draw your attention to this is that, it is a property of manifold, even the stability of the equilibrium point, the equilibrium point itself appears to be of a very local nature. The fact that f has to be continuous and it has to eventually cover the entire manifold, forces some properties on the manifolds itself. It requires something on the manifold for existence of an f whether or not equilibrium points should exist for that f. Let us ask about a circle.

On the other hand, S 1 circle, is it possible to think of a equilibrium point? is it possible to think of a dynamical system in which there is no equilibrium point? Yes, we can just have theta dot equal to 1, that continuously it is rotating like this, and its going on rotating so at no point there is an equilibrium point. So, hairy ball theorem speaks about S 2, and it says that one is forced to have an equilibrium point.

In fact if one requires only simple similarities, then one will in fact have at least two singular points and this turns out to be a to a very celebrated and familiar result to all of you which is that the number of phases edges. And vertices of any polyhedral satisfy a relation.

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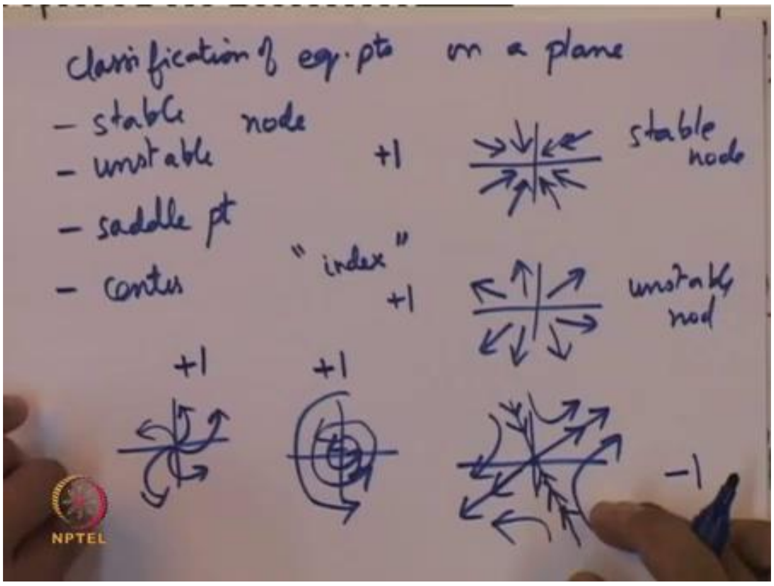

So, let us comeback to classification of equilibrium points on a plane. On a plane we already saw some examples, stable unstable node. We saw a node, we saw saddle point, we also saw center, what where these?

All trajectories are coming inwards, this turn out to be the case, which turned out to be the case, if the matrix a had Eigen values which were both real and negative this was what we called stable node. We saw another case where when both Eigen values were real and positive, that time they this one was an unstable node. On the other hand, a saddle point where was, were Eigen values are real, but one is positive one is negative. For example, this is a Eigen vector corresponding to positive Eigen value, let us say.

So, everything goes away, and if this is an Eigen value corresponding to negative Eigen value, we are speaking of a plane when we have linearised above the equilibrium point, and we are considering the Eigen values of the matrix the linearization at that point. And at all other points, it is like this.

So, we also saw centre, we saw the situation where Eigen values of a are complex, but depending on whether the real part is positive or negative, we can have oscillation that are coming inwards or going out. This was the situation where the Eigen values are complex, but on the imaginary axis.

So, the oscillations are neither coming in nor going out, that is as well as the linear system is concerned. And in this case it turned out that the linearised system the real part is 0, but the second order non-linearity might cause that the oscillations come inward or go outward. And that is why one cannot use the linearised systems conclusion about it being a center for the origin non-linear system also for the equilibrium point on the non-linear system.

So, as I said for the case that Eigen values of a are complex if the real part is non zero. If the real part is positive, then these oscillations are going out. On the other hand, if the real part is negative it comes inwards. So, these all we will like to classify as something called the index of the vector field, so it will turn out that this stable and unstable node the index is 1. So, we are going to very soon define a notion of index of an equilibrium point, which will turn out to be plus 1, plus 1 for stable node unstable node.

For the saddle point it will turn out to be minus 1, we will verify it for a few examples. For centers it will be plus 1 and for focus stable and unstable focus also it will turn out to be plus 1, then only the saddle turns out to be a little special for which it will be minus 1. So, we are going to see in more detail what is the meaning of index of a vector field is. So, let us take a equilibrium point.

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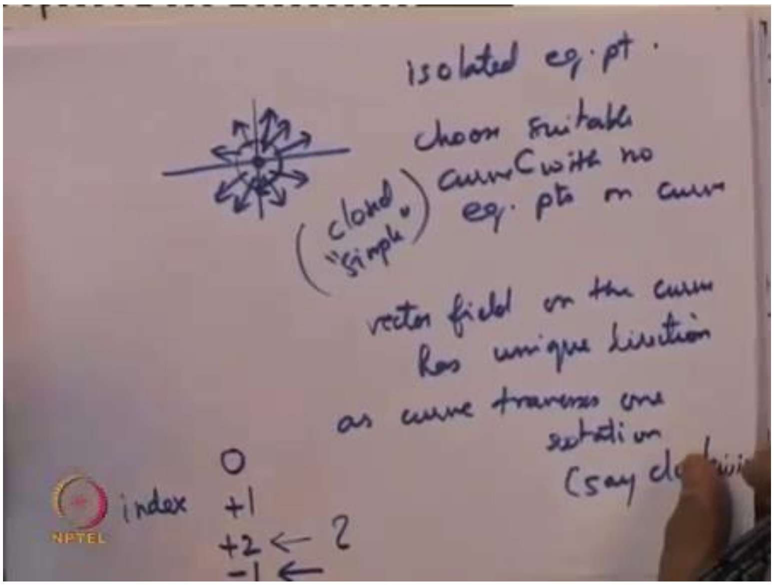

Suppose this is an equilibrium point and we have vectors all around, the vectors themselves. First we will consider an isolated equilibrium point, what is isolated about it? There is some neighborhood within which this is the only equilibrium point, we are able to find some circle, some curve such that that curve contains this equilibrium point and that curve contains this equilibrium point in the interior and this is the only equilibrium point inside it. If such a curve can be found small enough, then we will call that that equilibrium point is isolated, in the sense that it is not sticking to any other equilibrium point. In that sense it is isolated at least some sufficiently small neighborhood contains only this one.

Now, we will like that there is such a curve, which does not contain any equilibrium points on it. Choose suitable curve with no equilibrium points on curve, let us call this curve, so this curve is a closed curve, it is also a simple curve. In the sense that, we are not going to allow this curve to have self intersections. Starting point and end point are the two only points that are common, no other intermediate points are repeated.

So, that is indeed the case and we can also give it in the orientation, we can give it in either clock wise or anti clockwise, that is not the issue. Now, we will ask that, look as we go along this curve, the we can the vector field at any point on this curve has a unique direction because there are no equilibrium points on the curve. If there are equilibrium points on the curve, then the vector field has length 0, and hence it would have no direction and that would cause the problem to us.

So, we chose a suitable curve which has no equilibrium points on the curve, and we choose that this curve has this equilibrium point only, and that is possible because this equilibrium point is isolated. So, now this vector go, also goes through a rotation, notice that this vector is pointed like this, it is like this, and as we go along that curve, this vector is also turning and if this vector, if this curved traverses in clock wise, it turns out that this vector also has turned clock wise.

So, vector field, vector field on the curve has unique direction, has unique direction because the curve has no equilibrium points on it, has unique direction as curve traverses one rotation, say clockwise. As I said, the direction clock wise anti clockwise will not matter. Let us choose this clock wise, then we can ask whether that particular vector also has rotated, how many number of times and whether it is a whether the orientation on that vector also has remained the same or not.

So, if we have chosen the curve to be clockwise, has a vector rotated clockwise? is the first question and has it rotated say how many number of times once or more? So, index is defined as plus 1, if rotation once in a same direction, plus 2 if it is rotation twice in the same direction, minus 1 if it is rotation one, but in the opposite direction than the curve. Then, the fact whether it is same or opposite is what decides whether sign and the number of rotations of course, is decided by how many times it has rotated. It remains the question, do there exist equilibrium points where you have two rotations?

It seems unreasonable that the vector rotates in a opposite direction as we rotate in the along the curve in a particular direction, let us say clockwise. So, most easy to think of is plus 1, where the vector field rotates in the same direction as a curve itself. So, notice that if there is no equilibrium point inside, if there is no equilibrium point inside, then the vector field does not rotate any net rotation, it might just change sings like this, but it may not complete a rotation.

So, 0s also possible. In fact if you have multiple equilibrium points isolated equilibrium points inside. Then, the curve will add all these indices and it will be an algebraic sum, that turns out to be a extremely neat concept that all these indices of equilibrium isolated equilibrium points, add up for a curve that contains all of these. So, if the curve contains none, if it contains no equilibrium point, then the index of that curve will also automatically be 0 that means, the vector field undergoes no net rotation. So, let us see an example now let us take for unstable equilibrium point, for unstable node we have already verified.

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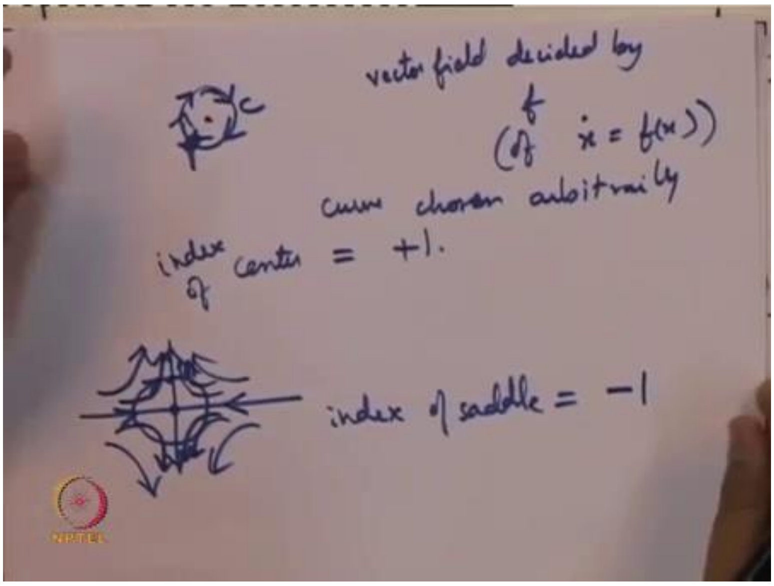

Let us take a centre and let us take that particular periodic orbit itself as a curve c. So, we see that at every point the curve c, the vector field is tangential to the curve itself. And hence, when the curve undergoes one rotation the vector also undergoes exactly one rotation in same direction.

So, now notice that even if the, even if we are considering sorry, this particular arrow should have been like this. Even if the curve was chosen anti clock wise positive that, does not change the directions of the vector field. So, vector field decided by f of x dot is equal to f of x, is decided only by f. Curve chosen pretty arbitrarily only important constrain is that in order to decide the index of a equilibrium point one should chose a curve that contains only this particular equilibrium point. And that curve should not have any equilibrium point on it.

This curve should be, should contain precisely one equilibrium point inside it, precisely the equilibrium point for which we are trying to find out the index. Moreover, this curve should be a simple and closed curve. Except for this constrains it is to be chosen arbitrarily, one can ask it need not be the periodic orbit that is decided by because this being a centre, could be like the periodic orbit between the opposite orientation also. And still it turns out that the index of that equilibrium point is independent of which curve has been chosen as long as it satisfies these conditions.

So, centre one can verify that it will, it is indeed index of a centre equal to also plus1. Now, let us verify a saddle point. Now, let us take a curve like this, let us orient it positive, so this particular point is equilibrium point. So, we see that when we start at this point on the curve, it is pointing upwards. So, this is how we should fill. So, now you can check that, as you go along this curve, in fact at this point it is tangential hence, opposite to the curve and this point it is inwards like this.

So, we see that as you go along that curve, this particular pen that this particular pen that I was showing has rotated by one number, but in opposite direction. So, index of saddle equal to minus 1, why because if you choose the curve clockwise, then you took the vector field along this curve. As we are traverse on a curve in clockwise direction, that particular vector turnout to rotate by one number of times, but in the anti clockwise direction, that is why the index of the saddle point is equal to minus 1. So, now that brings us to the question that, do there exist equilibrium points with index 2? So, this particular question sure would have haunted the dynamical systems community for many years and indeed there is this was told to me by my teacher.

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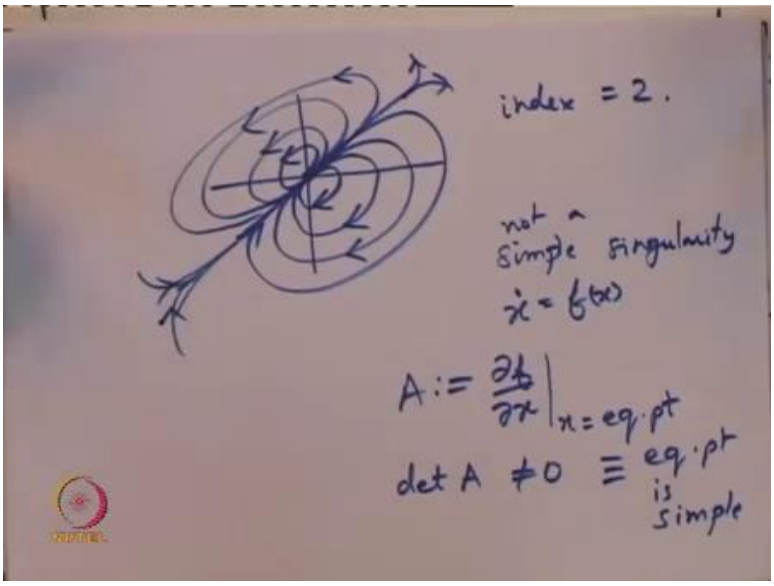

So, it requires some effort to construct. One can check that this particular equilibrium point. Of course, trajectories is for smooth vector field unless we have non lipschitz properties. Trajectories do not intersect, it is just that they are very close by and they eventually separate like this. So, for such a vector field, we can draw curve and check that the vector rotates by 2 times, when you go around this particular equilibrium point once, so such a vector field has index equal to 2.

So, it turns out that this is a little a special side, this is a special vector field for it to have such a index, and we will yet to see what is special about it, that it is not a simple singularity. It is not single simple, it is a singularity because it is a equilibrium point, what is not simple about it, that we will see that the linearization. If x dot is equal to f of x of this, then you take the derivative of f with respect to x, then it will be square matrix then you evaluate it at the equilibrium point x equal to this particular equilibrium point. And this matrix, Eigen values is what decided everything.

So, you look at the, suppose you call this matrix square matrix, it is a 2 by 2 matrix determinant of a. Non 0 will decide that that equilibrium point is simple. What is the meaning that the determinant of a is 0 for the linearised system? If the determinant of a is 0, it means that there is, the equilibrium point is not isolated as well as the linear system is concerned. The non-linear system equilibrium point might be isolated, but the linearization is suggesting that there is a continuum of equilibrium points and that is indeed what happens for linear systems if the matrix a is singular.

So, if the determinant of a is non zero, then the matrix a is what we will like to see as a non singular matrix and for such a, for such a situation that isolated equilibrium point, we will say is a simple similarity. If the determinant of a is non zero. If the determinant of a is, then that equilibrium point even if it is isolated, we will say is not a simple singularity. So, only with non simple similarities one can have index more than 2, more than 1.

So, index equal to 2 or more is possible only for non simple similarities. One can check that the saddle point stable unstable focus they all have, there are simple similarities and hence, the indexes are plus or minus 1 only. So, this brings us to the one of the last topics of this one of the last sub topic of this topic that is about the hairy ball theorem so one can ask now that suppose they are given with a sphere.

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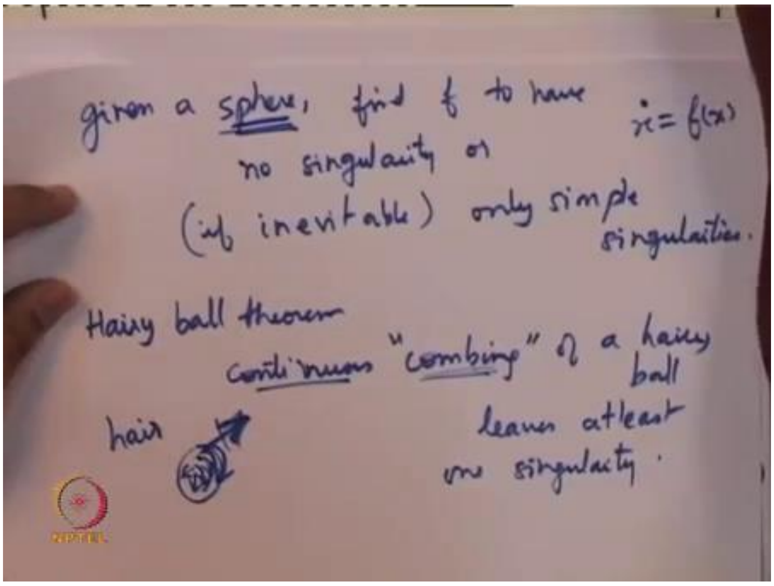

Given a sphere, find f to have no singularity or if inevitable only simple singularities. So, this is the question in a sphere, find a function f, what is this function f? Because we are trying to construct dynamics like this o, we see, if there are no similarities it means, that you can never stabilize the system at any point, there cannot be any equilibrium point itself, that is the consequence if there is no singularity. If there are singularities, then we will like that there are simple singularities because there linearised, because we will ideally like linear system linearization and linear control to operate there.

So, if it is not a simple singularity, then we need more complex systems because the Eigen value at the origin suggested that at steady state there is a non zero value, it is not converging, but it is staying close by only. Asymptotic stability requires that all equilibrium points are in the left half, linearization at every equilibrium point is in the, has Eigen values and left half complex plane.

So, simple singularities are good in that sense. So, it turns out that there is this person, not there is hairy ball theorem that tells that no singularity is not possible requires. At least two simple singularities, if you allow the singularities to be non simple, then one would suffice. Hairy ball theorem says that, continuous combing of a hairy ball leaves at least one singularity. What is combing of a hairy ball?

Suppose we are given with the ball, and suppose we are, this ball has lot of hair along it, this hair each hair is like our regular hair h a i r. If this hair denotes a vector field at that point, at each point there is some hair that starts at the origin of the of the tangent space at that point and it is, it defines the unique direction and we are now asked that we are required to comb it in a continuous ways.

So, this tells that the f has to also be continuous in x, so what is combing about it because we want that hairs are all in the tangent space, in that sense they are tangential to the ball, they cannot be standing out, the hair cannot be sticking out like this, they better get combed on that ball. So, combing means that particular vector is in that tangent space continuous, the function f is continuous, hairy ball meaning, the ball has hairs and each of this hairs are nothing but vectors in that tangent space. Then, tangents space is one that is getting forced because of these combing forces.

Now, we will like that there is at every point there is one hair, that is non zero length, that is the meaning that there is no singularity. So, is it possible that we can continuously comb without a singularity? So, it turns out that the answer is no. The hairy ball theorem says that, there will at least be one singularity. This is the meaning that when we comb there is a at least a point on the head, where all the hair are going away or going round and round, this are what is well known when we comb the hair, comb our own heads hair for example.

So, what is this one singularity? This one singularity is inevitable because this sphere, the fact that we are given with a sphere, this sphere has a manifold it is forcing that the manifold, all the isolated similarities when we add the indices we end up getting number 2. That is the sum of all the indices of every equilibrium point, isolated equilibrium point and we have defined index only for isolated equilibrium point.

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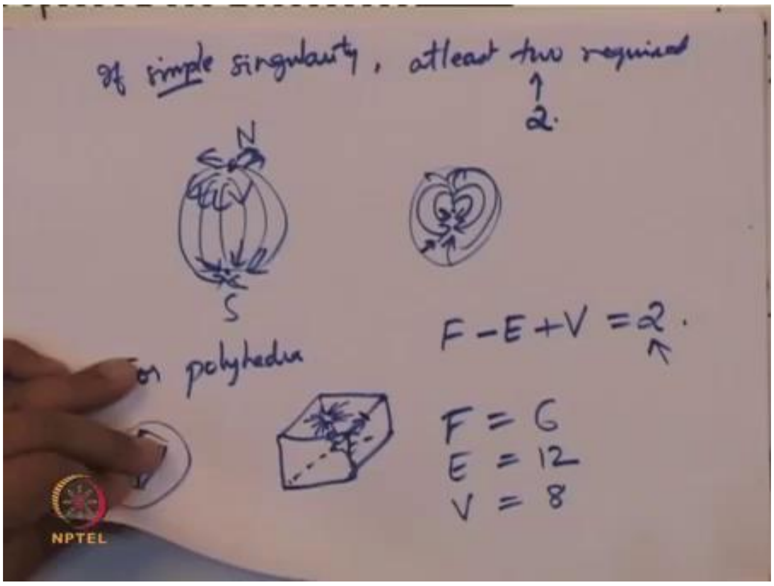

So, let us now say if simple, if simple similarities at least two are inevitable. At least two required to this number 2, this extremely special. What is special about it? For example, we can now think of a ball like this, let us say we have a north pole and a south pole. So, we can think of vectors leaving form north pole and all converging towards the south pole. This is an example of a vector field, which is continuous leaves a north pole or comes towards the south pole.

So, two simple singularity, two singularities only, both are simple. Why because this one, the north one is a unstable node while the south one south pole is a stable node, so they both are simple singularities, they both have index 2, index 1 each. So, the total sum of all indices for all equilibrium points is exactly 2. So, we can see that we can construct such a case. Using that particular non simple singularity, we can also think of this, on this sphere and this goes all along like this.

So, one can think of a non simple singularity or singularity itself of index 2 defined like this on this sphere. So, now what is special about 2? So, that brings me to one of the last very good relation with, for a polyhedral. So, for polyhedral, the Euler's theorem for polyhedra says that the number of phases minus number of edges plus number of vertices is equal to 2. For any close polyhedra with no holes importantly with no holes.

So, for example, for a cube, so the phases the number of phases is equal to 6, the number of edges is equal to 4 on the top, 4 on the bottom and 4 vertical, that is 12, and the number of vertices we have 4 on the top, 4 on the bottom that is 8. So, we get that 6 minus 12, that is minus 4, 6 minus 12 minus 6 plus 8, that is plus 2. So, we get 2. So, what is the relation between this and singularities on a sphere.

So, one can now think of a sphere and we can try to sort of look at this having phases, edges, in which we put this points all the edges are marked and the phases are like different regions on the sphere. So, polyhedra we can think of is actually very similar to a sphere, in some topological sense it is nothing but a sphere, with all this lines marked on the sphere. Now, inside each phase one can have let us say for example, a stable node and at each node, at each vertex we can have an unstable node everything going away and between two vertices it will turn out that there will have to be a saddle point. Let me draw this figure little larger.

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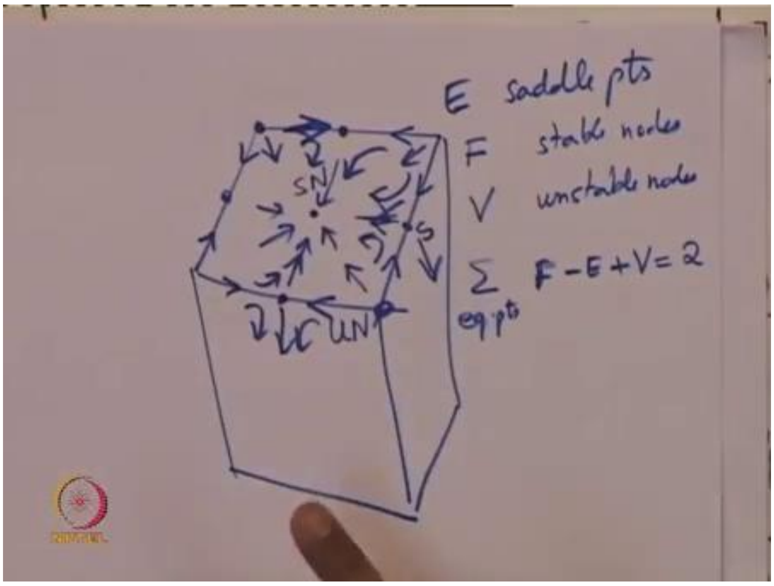

So, let us take this cube, so this is at the centre of each phase, we decided to have stable node stable node, at each vertex we will have a unstable node, at the centre of each edge we will have this saddle point. So, that sorry… So, we see that at the centre of every edge we need to have a saddle point, if we have to be able to do this systematically continuously more precisely.

Sorry, this arrow it should be inwards because all our vertices have been decided as unstable nodes, so it is all going away from vertices at it goes away form every vertex, then between two vertices there is exactly one edge, that centre of that edge is where both seem to be coming in. So, why do we make that stable as well as this edge is concerned, but if we have the centre of every phase to also be stable node, then it will be going away from this particular point, so fine that that way lets allow this to be a saddle point.

So, notice that this is a systematic continuous consistent way of placing the saddle points. How many saddle point would we have placed? E number, the number of edges. How many stable nodes would we have put? F number of stable nodes. And, how many unstable nodes would we have put number of vertices? V number of unstable nodes. So, now we know that this, we can add the indices for all of them. So, we know that indices of saddle point is minus 1.

So, the sum of overall equilibrium points will in fact give us, F minus E plus V because both stable and unstable nodes have plus 1 as their indices. So, F and V both come with plus sign and E on the other hand because it corresponds to the saddle point which has index minus 1. It corresponds to minus sign here. So, this one is what we saw for polyhedra is equal to 2 and that is exactly what we also saw for that north pole and south pole, there also it turn out to be 2.

So, what it say is that this sum being 2 is a property of the sphere, it is not a property of whether you take a cube or a pyramid. The fact that every phase you associate a stable node and every vertex you can have a unstable node, and then at the edge you are forced to have a saddle point, and then sum of all vertices will, sum of all indices of equilibrium points will exactly turn out to be 2 by this particular formula. And the 2 is like a invariant of this particular topological object called the sphere and all this polyhedra are in that sense homo topic to a sphere.

So, this brings us to the end of seeing how the heavy ball theorem says that it is inevitable that, for a sphere we cannot have a situation where for a continuous function F we do not have any equilibrium points at all. We do not have any singularity, such a situation is not possible.

One last question is somebody can ask, can you inter change the role of stable node and unstable nodes? In other words can you have a stable node at every vertex and unstable node at every phase? Yes, that is still possible, still at the edge you will require a saddle point where as we just some arrows reversed, but then the formula will still turn out to be

2, that is because sum of all the indices of equilibrium points will still turn out to be 2 because we know that both stable and unstable node both have index plus 1.

So, this is one of the extremely important, extremely enchanting topic within non-linear dynamical systems about how it is related to Euler's polyhedra formula. And through that, to something called algebraic topology, but then I do not work in this nor do I know enough. I want just you to know about this and tangent spaces on the other hand finds wide applications in control, when dealing with non-linear systems especially on manifolds with that we will end this lecture on tangent spaces and manifolds.

Thank you.