Digital Signal processing & Its Applications
Professor Vikram M. Gadre
Department of Electrical Engineering,
Indian Institute of Technology, Bombay
Lecture – 05 b
Linearity- Homogeneity and Additivity

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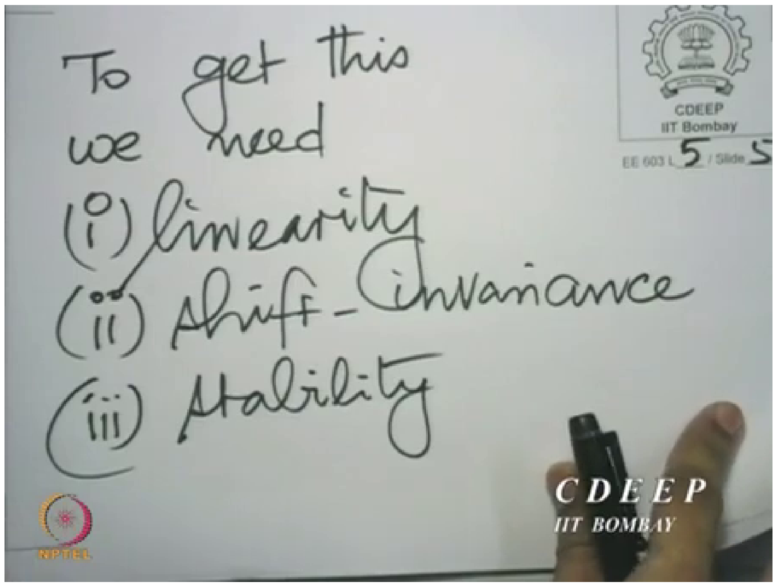

So, lets you see lets first take up property number one linearity. What does linearity mean? Linear in fact linearity is an off quoted property in the context of systems.

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And in one word linearity is summarized by what is called the principle of superposition. Literally the word superposition means putting one on top of the other. Superpose; when I superpose one upon another I am putting one on top the other. And the mathematical meaning of superposition is the following. Here again, I conceive of three experiments.

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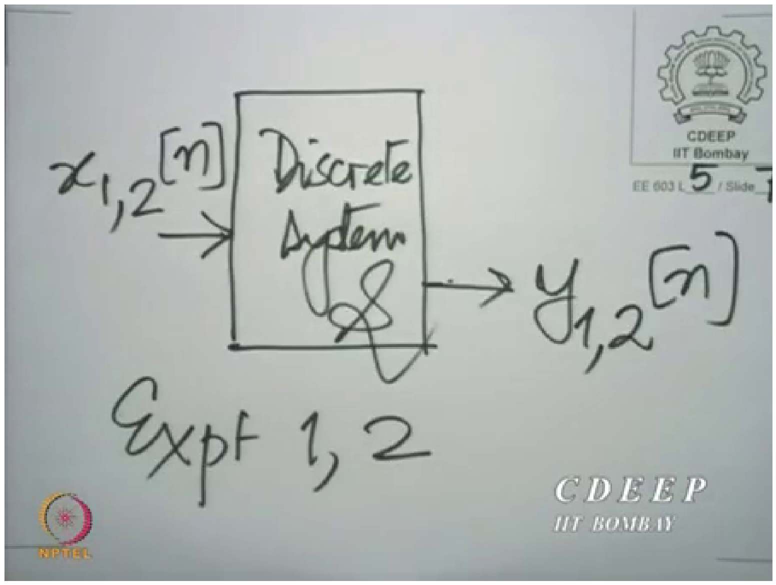

And I am more abstract now. I have this discrete system. I shall denote that discrete system by script S. As I have done here. I apply to it in two different experiments. I shall now number them experiment one and two, the inputs $x_{1,2}[n]$. And observe the corresponding outputs $y_{1,2}[n]$. Again, I am using what I call multiple concept notation here. I am using this notation to condense my drawings.

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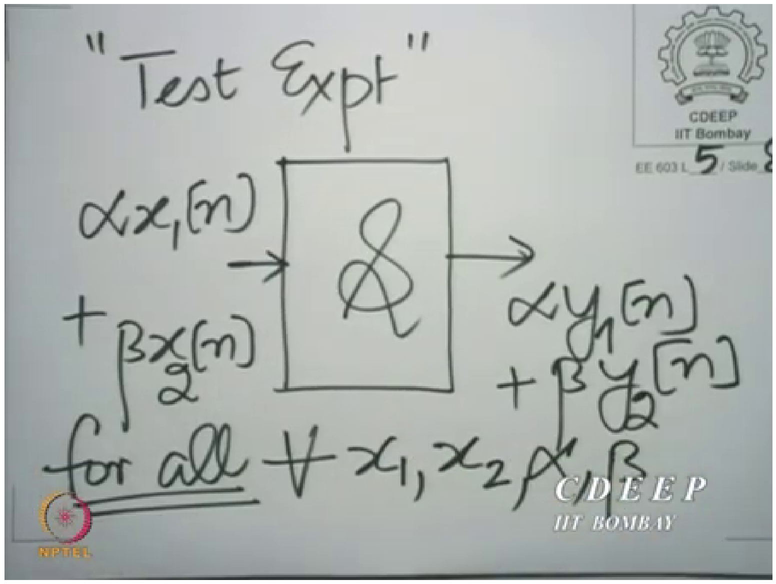

The test experiment is to take the same system. Give it $\alpha x_1[n] + \beta x_2[n]$ and ask what the output is. If the output is also $\alpha y_1[n] + \beta y_2[n]$ and this wholes, now I bring in one more notation. I bring in this notation to denote for all x_1, x_2, α, β. α and β are in general complex constants $x_1[n]$ and $x_2[n]$ are in general complex sequence. We know now why we have to allow for complex sequences.

We have agreed to use rotating complex numbers in place of sine wave. So, I must allow for complex sequences. And if I am allowing complex sequences I must also allow complex constants. Needless to say, real sequences or real constants are a special case. Is that right? Now, what we are saying in the principle of superposition is when I superpose by virtue of linear combination two inputs.

So, I take any linear combination of two inputs and give it as the input to the same system I get the same linear combination of the two outputs. Not only this this is true no matter which two inputs I took and which α and β are used. Now, important thing in superposition is the for all part. I must stress this right in the beginning.

It might very well be true that the system exhibits this property for some specific x_1 and x_2. But, that does not satisfy us. It is only when the system unconditionally exhibits this no matter what x_1, x_2 are and no matter what α and β are that we are convinced the system is linear. If there is a violation even for one particular case of x_1, x_2, α or β we must classify the system as non-linear.

Now, we can reflect a minute on why this should be required. One thing is very clear the condition that we had put on the system a couple of minutes ago is very similar to what we are saying right now. In fact, what we are saying right now is a kind of more general statement on that condition. Is not it?

There we were saying this for the context of rotating complex numbers of phases. Now, we are saying it for any inputs x_1 and x_2. You see it is clear that if the system is linear in the way that we now understand. Quite a bit of what we asked for a few minutes ago in connection with rotating complex numbers seems to be satisfied, intuitively. Quite a bit of it seems to be satisfied. However, this is only intuitive this is not really rigorous.

We are first going to build ideas by imagination or intuition and then we are going to put down rigorous proofs. Intuitively it is clear. You know we are asking for some kind of a linear combination thereof two rotating complex numbers rotating with different angular velocities and we got the corresponding outputs there as we expected here. I want to emphasize another point here. You see when you talk about $x_1[n]$, $x_2[n]$, $y_1[n]$ and $y_2[n]$ you must remember they are all sequences, they are not points, they are not numbers.

$x_1[n]$ is a whole collection of numbers indexed by the sampling index n. And so, 2 is $x_2[n]$ so 2 is $y_1[n]$ so 2 is $y_2[n]$. So, it is not enough to look at a few points n and come to conclusions. You are talking about the whole integer set denoted by n obeying what we are saying here, superposition point by point on the index n. So, not only is it a superposition principle for all possible x1, x2, α and β that super position holds point wise at every end this holds.

So now, we understand that it is a serious demand that we are making on the system. This is the property of linearity. Now, let us break this property into two smaller properties.

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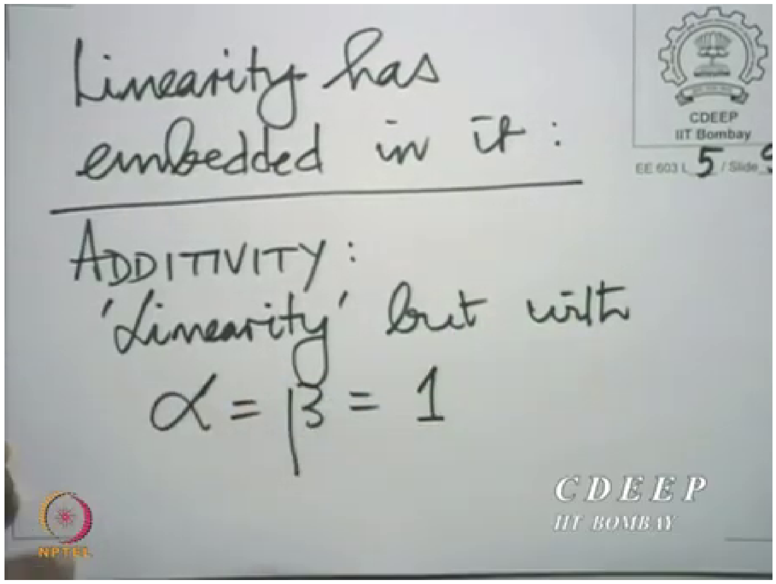

So, linearity has two properties embedded. One property is what is called additivity. And additivity is the restricted version of linearity where we are only asking for linearity but with α equal to β equal to 1. So essentially, we are saying if I take two inputs x_1 and x_2 if I were to add them the outputs corresponding to x_1 and x_2 also get added. That is what additivity means.

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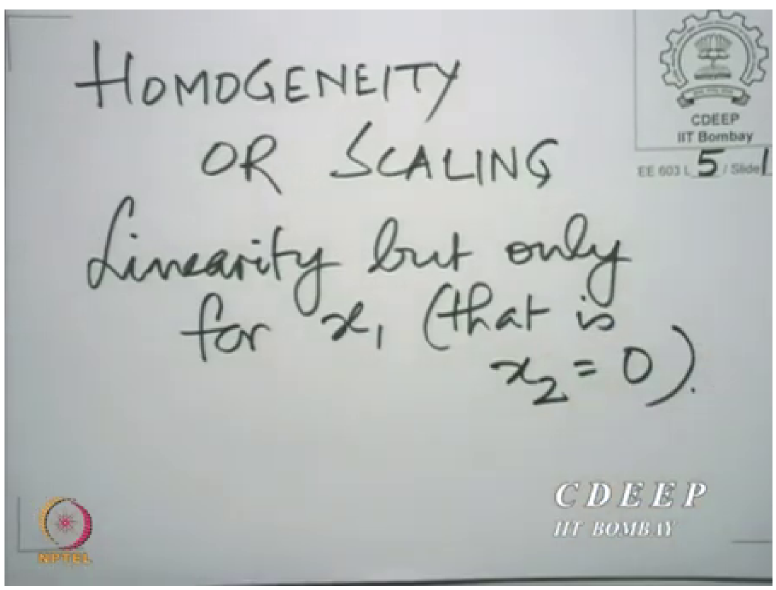

And the other restricted form of linearity is what is called homogeneity or scaling and that is essentially linearity but only for well this is tricky. Only for x_1 that is x_2 equal to 0. So, it is a 0 sequence that we apply for the second. What it means is I essentially give an x_1 I study the output x_2 is anyway 0. I take α times x_1 and give it to the system the output should be α times y_1.

So, essentially homogeneity of scaling says when I multiply an input by a constant the output is multiplied by the same constant. Now, I break linearity into these two properties because there are examples of systems which can obey one but not the other. I am now going to illustrate examples of all the three.

Systems that are linear, systems that obey additivity but not homogeneity and well, I leave the third one to you as an exercise. Is that right? So, I take the first two.

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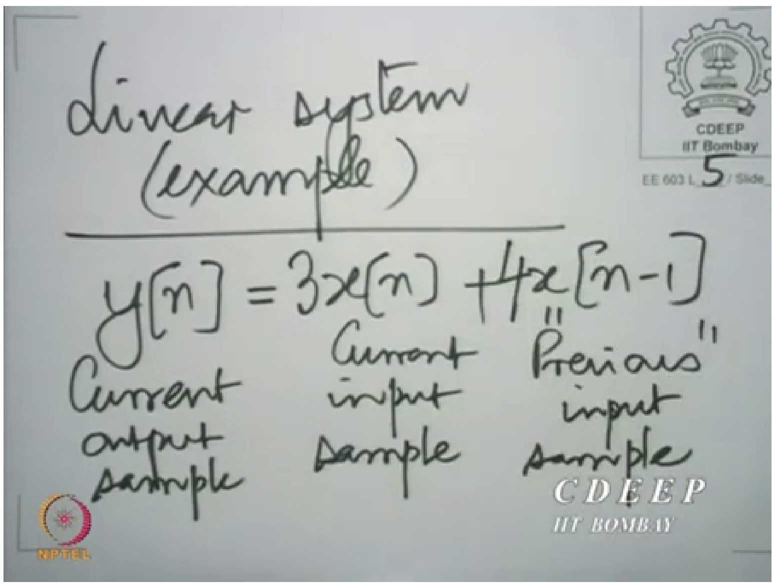

Linear systems, example: We describe a system by the following equation. Well, how do you specify a system? A system is specified by the way it relates the input and the output samples. More generally a system is specified by a mutual relationship between all the output and all the input samples. But it is more convenient to use what is called a calculable description.

A calculable description means I can explicitly determine sample by sample the output from the input. And we will most of the time prefer to use what are called calculable descriptions. Otherwise, if you have an implicit relationship so I can say $y[n]^2 + x[n]^2$ is a constant that is an implicit relationship. But it does not help me calculate y[n] from x[n].

So, we will most the time try to use explicit relationships. Anyway that is a technical point. So, let us take the example of a system where the current output sample is x of n is given by the current input sample plus the previous input sample. Since, we are just beginning on our discussion of systems I am trying to be very clear what we mean.

But after a while we would not do this. In fact let me go a step further let me take any linear combination let me take 3 times the current input sample and 4 times the past one. And let us show the system is linear. Well, the argument is very simple indeed and I recommend that you pay careful attention to the steps of the argument because this would be useful in other examples as well.

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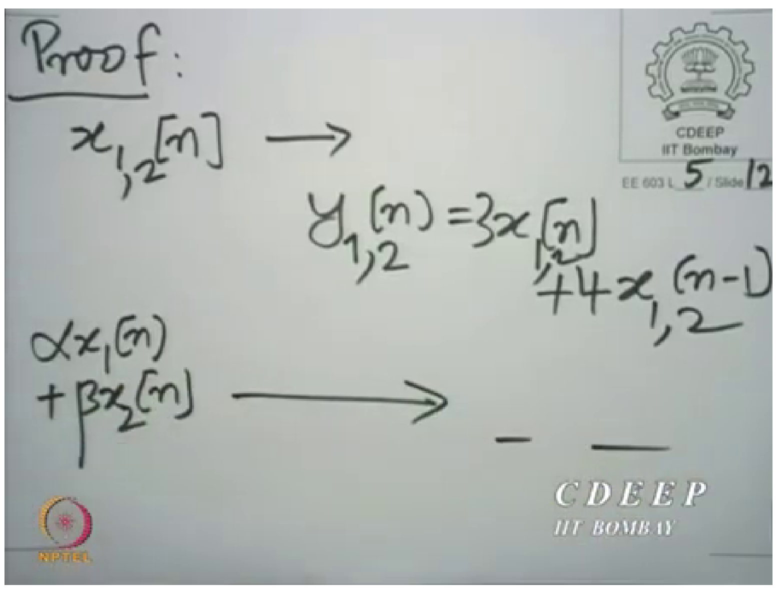

We note of course that $x_{1,2}[n]$ would result in the output $y_{1,2}[n]$ is equal to $3x_{1,2}[n] + 4x_{1,2}[n-1]$ and we use the same multiple notation because the same thing happens to x_2. Now, let me keep it α times x_1 plus β times x_2. And let us write the output down on the next page. Now, you must be careful. Here it is it is all right because you can almost transparent you know. The output is going to be 3 times the input here plus 4 times the past sample of the input and let us write that down clearly.

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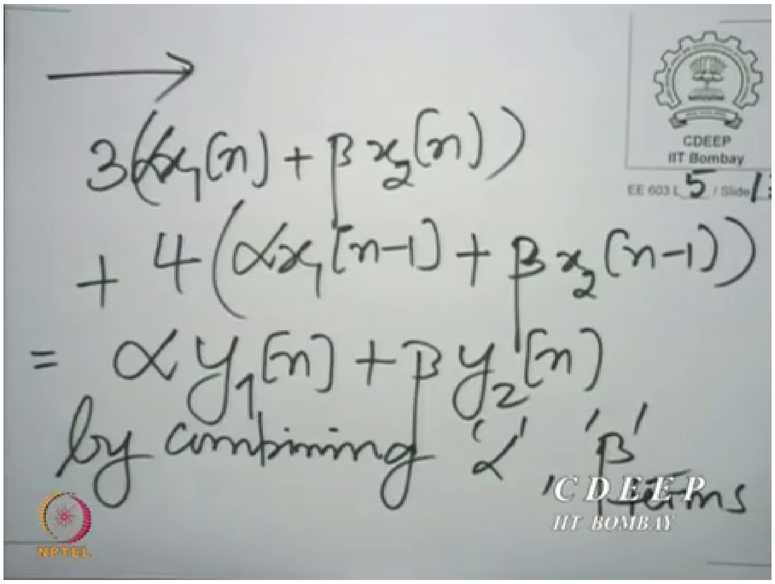

The output is $3(\alpha x_1[n] + \beta x_2[n)] + 4(\alpha x_1[n-1] + \beta x_2[n-1])$. But, the past sample is also $\alpha x_1[n-1] + \beta x_2[n-1]$. And of course, it is easy to see this can be written down as $\alpha y_1[n] + \beta y_2[n]$ by combining terms with α and β, simple enough.

You just expand the brackets you get terms combining α and β there. You know $3(\alpha x_1[n] + ..)$ and so on so I leave that. So, it is very clear this system is linear because this happens for any x_1, x_2 any α β. Is that right? Now, let us take an example of a system which is not linear at all.

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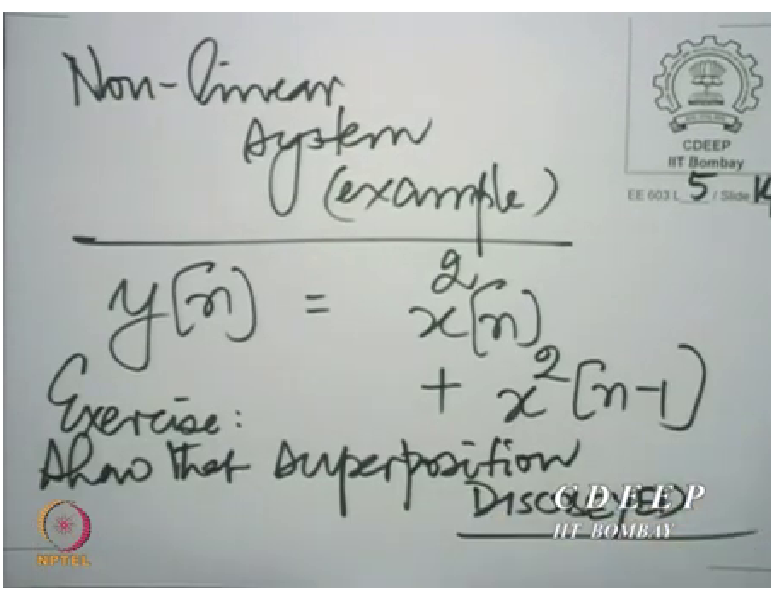

The output is 3 times the current sample $\alpha x_1[n] + \beta x_2[n]$ plus 4 times the past sample. But, the past sample is also $\alpha x_1[n - 1] + \beta x_2[n - 1]$. And of course, it is easy to see this can be written down as $\alpha y_1[n] + \beta y_2[n]$ by combining terms with α and β, simple enough.

You just expand the brackets you get terms combining α and β there. You know α 3 times x1 n plus and so on so I leave that. So, it is very clear this system is linear because this happens for any x_1 $x_2$anyαand β. Is that right? Now, let us take an example of a system which is not linear at all.

Non-linear system - Well, very simple, take any nonlinear operation, you could have x square n plus x square of n minus 1. So, I square the current input sample and add to it the square of the past sample. And I leave it to you as an exercise. Show that superposition is disobeyed. I leave it as an exercise to show that it is disobeyed. Well, what I mean by disobeyed is?

Now, you must understand this point here. By disobeyed I mean that I must produce one counter example.

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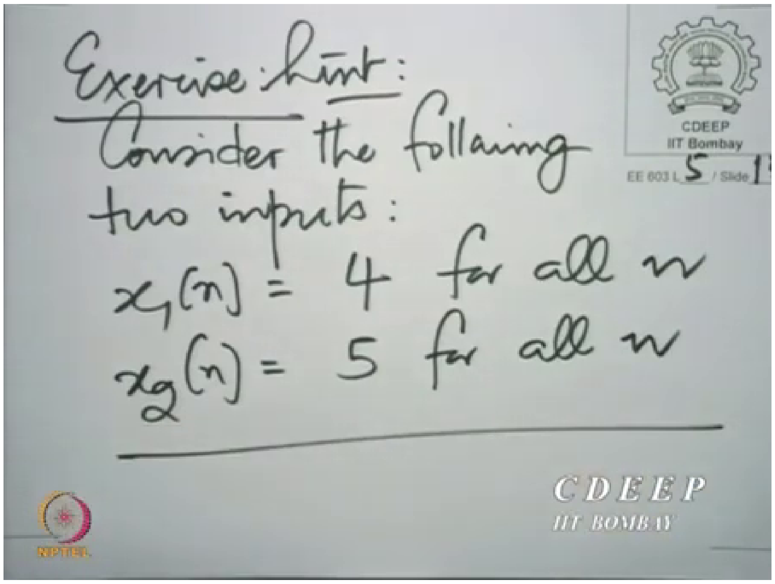

So well, in fact, I leave it to you as an exercise to consider the following two inputs. Exercise hint consider the following two inputs $x_1[n] = 4 \, \forall \, n$. And $x_2[n] = 5 \, \forall \, n$ and show that in this case it will be violated. So, I am giving you so you see it is very important when you in general deal with proofs that when you want to make a positive statement. So, you are trying to prove a system is linear notice what we did.

We were trying to prove a system was linear. The way we proceeded is independent of example. We said take a general input x_1 or x_2 and proceed and then come to some conclusions. And the conclusions did not depend on the particular α and β that we chose. Then we start to disprove or to prove against a particular statement. So, we wanted to prove the system is not linear.

Now, there one counter example is enough. So, once I take this troublesome input and I prove that the system does not satisfy linearity or superposition for this, it is enough for me to conclude the system is non-linear. But, the reverse is not true. If I take some examples and I notice superposition is obeyed in those examples it is not adequate to conclude the system is linear. And in a minute you will understand why.

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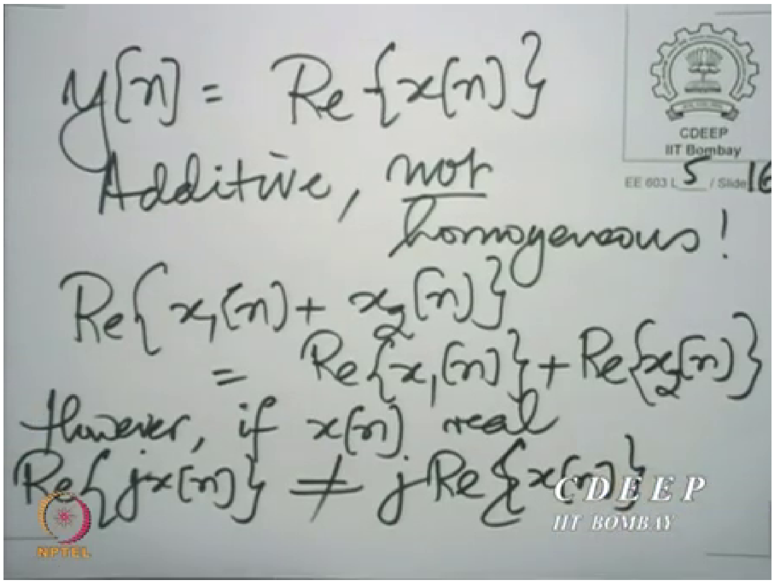

Let us take the following system; y of n is the real part of x[n]. Now remember, x[n] can in general be complex. We will show the system obeys additivity but not homogeneity. In fact, it is very clear that real part of $Re\{x_1[n] + x_2[n]\} = Re\{x_1[n]\} + Re\{x_2[n]\}$. So, additivity is of course true.

However, if x[n] is real, real part of j x[n] is not equal to j Re{x[n]}. In fact, if x[n] is real the real part of j x[n] is 0, identically 0 and therefore the system is not homogeneous. Now, here you could have run into trouble had you taken 2 real inputs $x_1$and $x_2$and tested homogeneity on them only with real constants.

So, you may have you could have in principle taken an infinite number of examples where you have real x_1 x_2 and real αand β and you observe superposition to hold. But, the system is not and it is not homogeneous in spite of all those being true. It is not linear therefore the system is not linear.

It obeys additivity but not homogeneity. Now, I leave it to you as an exercise to construct a system, which is homogeneous but not additive. And I give you a hint in the direction of this exercise. The hint is typically when you take you know what is called an ordered choice of numbers. You know so if I have a list or if I have a, if I have a collection of a few samples of the past and if I take I order that collection according to its amplitude or magnitude.

And then I choose one of those maybe the greatest or the least or someone is, one of them in the middle. And I do this consistently as I proceed then that process tends to be homogeneous. But I am giving it to you only as a hint that is one possible case. But there are other ways to do it too, there are examples of systems that are homogeneous but not additive. And I leave it to you as an exercise to construct them.

And there too one needs to be careful in the same way, one cannot simply come to conclusions based on specific example. So, what I am trying to emphasize is proof positive statement means prove independent of example. Proof negative statement means take one counter example. Now, when you want to negate a statement, when you want to disprove a statement, it is not always possible to come out with a whole class of situations in which that will be violated.

It is not always convenient to do that. If you can do that that is of course wonderful. But, it might not always be possible. It might often be easier to construct a specific or a specific set of counter examples which would disprove a statement. Anyway that is so much so for principles of proof and disprove. And now we make one more observation.

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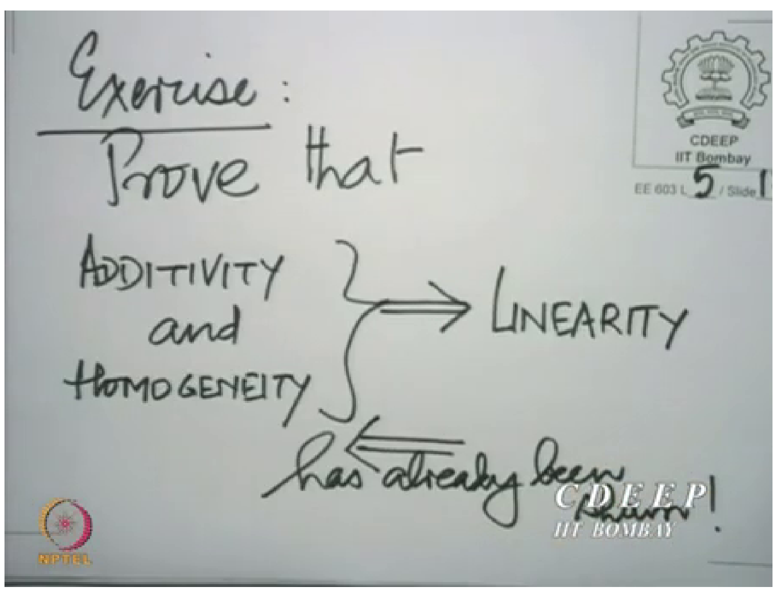

The observation and this again I leave to you as an exercise is - Prove now, here it is a positive statements you need to prove it independent of example. Prove that additivity plus homogeneity or scaling. Together imply linearity. In fact it is both ways, this way has already been shown. It is already shown because the very definition of additivity and homogeneity includes specific cases of linearity.

So, if a system is linear of course you know that it obeys additivity and homogeneity because of the very definition. But the other way is what you need to prove and I leave it to you as an exercise. So, if you are guaranteed a system is both additive and homogeneous it is also linear. And I leave the proof to you. Yes, so what so then for the first of the properties that we wanted to discuss. The second property is equally interesting and that is the property of shift invariance.

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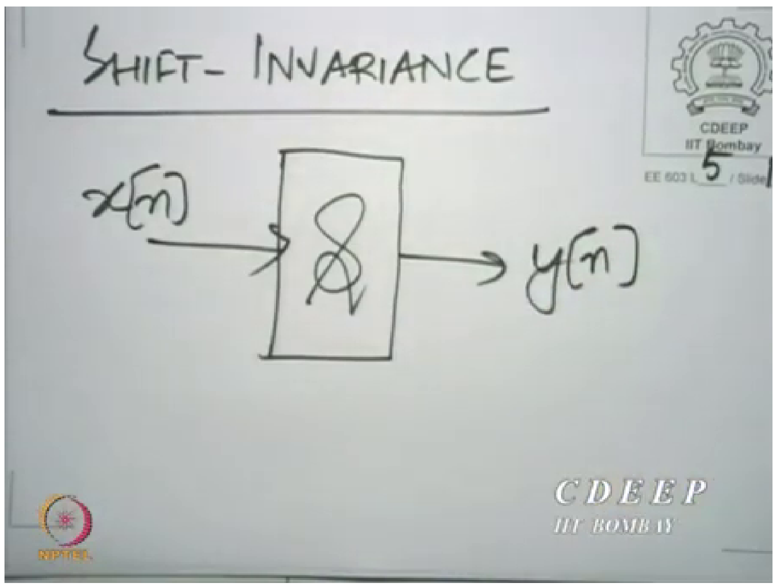

To test the system for shift invariance we need to perform only two experiments. We first apply to the system and input x n and study the output y n. And then we take the very same system S apply to it the input $x[n - n_0]$ where n_0 is an integer and a constant.