Digital Signal Processing & Its Applications
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Lecture – 06 a
Characterization of an LTI System using its Impulse Response

A warm welcome to the sixth lecture on the subject of Digital Signal Processing and Its Applications. We take up today further discussion on the properties of systems that we are embarked upon in the previous lecture. All right, so let us once again put down a few thoughts that we had come up with, in the previous lecture.

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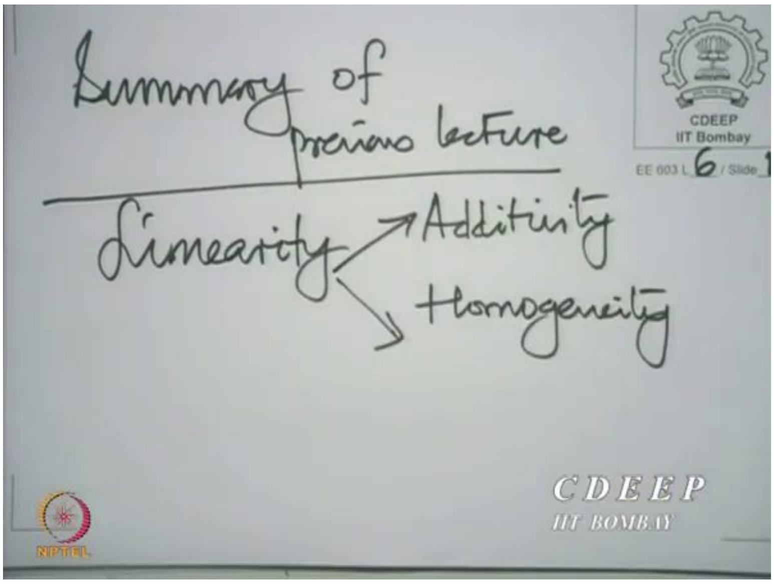

So, in the previous lecture, we have essentially looked at linearity and shift invariance. And we had seen that linearity comprises of two properties, additivity and homogeneity. In fact, we had defined additivity and homogeneity by using special cases of linearity, so we had defined them as properties, which were subsets of the properties of linearity.

However, we had remarked and I had left it to you as an exercise to prove that relationship is two way. In other words, additivity and homogeneity together come to make linearity once again, that means, together, they are sufficient for linearity to hold. So it is both ways.

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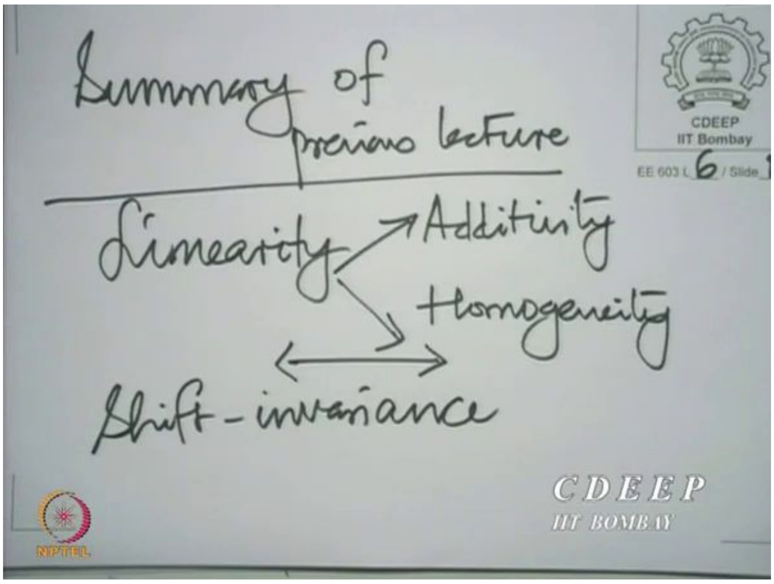

We have also looked at shift-invariance. And I might remark, that when we are very clear that the independent variable is time, then shift-invariance is sometimes called time-invariance. So, in many texts or in many references, we encounter the term time-invariance and the meaning is shift-invariance with the independent variable being time.

Now, we also explained the context from which these properties arose. The context was that we were trying ultimately to insist or realize the property of behaviour with respect to rotating complex numbers of phasors. And ultimately, we were trying to insist on certain properties pertaining to the response to sinusoidal excitation.

And we explained why that was the case. That was because sinusoids are not only found in nature, at least in an electrical context, but they are also very smooth functions. Now, what we intend to do today, first, is to continue what we had just embarked upon in the previous lecture, namely, to look at a Linear Shift-Invariant system and to characterize it completely.

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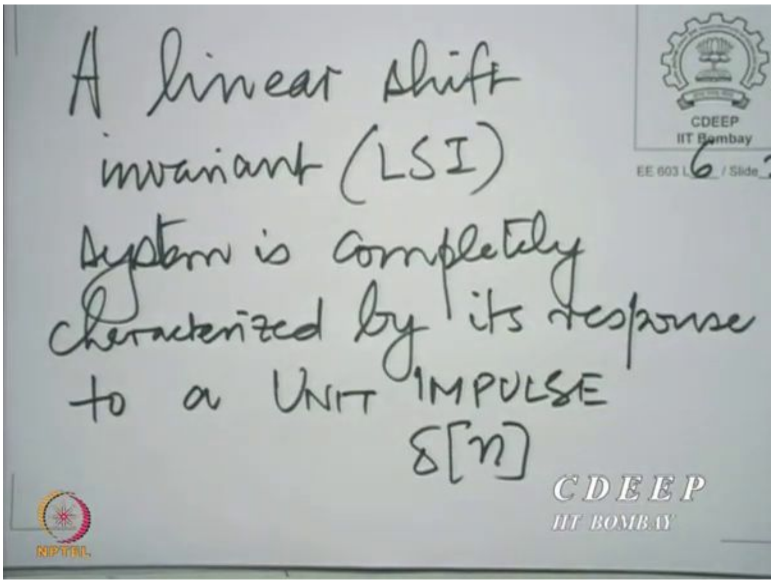

We have remarked in the previous lecture that a Linear Shift-Invariant system and now we will use an abbreviation for this for Linear Shift Invariant system is completely characterized by its response to a unit impulse $\delta[n]$. You see, we had agreed that we would use essentially three steps in this process.

And we also agreed on what we mean by characterization, we had also cautioned ourselves, that we should not bring in more than we have put in the definition right now. Right now, in the definition we have not brought in any sine wave, we have not brought in any rotating complex numbers, he have brought in no system functions or transfer functions.

We only have the knowledge that there is a relation between the input and the output and that relationship obeys the properties of linearity and shift invariance. With this, we must now prove. But if I know the response to one particular input, namely the unit impulse, I know everything about the system. In other words, I know the output that would appear for any input. And that is what we are now going to prove.

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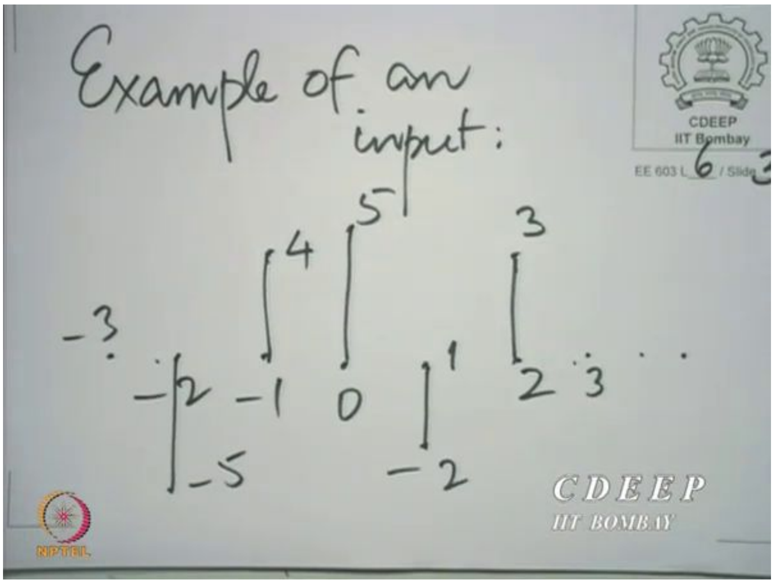

Well, the first step in the proof is to observe that one can express any input in terms of the unit inputs. And that is very easy to see. So let us take an arbitrary example of an input. So at *0*, let the input take the value *5*; at *1*, let it take the value *(-2)*; at *2*, let it take the value *3.* And for a moment, let us not confuse matters; we will not worry about what is present at *2, 3, 4* and so on, after beyond *2*.

At *(-1)*, let it take the value *4*; at (*-2)*, let it take the value which say (*-5)*; and then let it do whatever it wants to after that. And we will show that if we focus our attention on this part between (*-2)* and *2*, we can express this as a combination of appropriately shifted, unit impulses, shifted and scaled. So you see, it is very easy to see that this sample at zero can be obtained by putting a unit impulse here and multiplying it by *5*. So this can be contributed by what is called $5\delta[n]$. So let me write that down.

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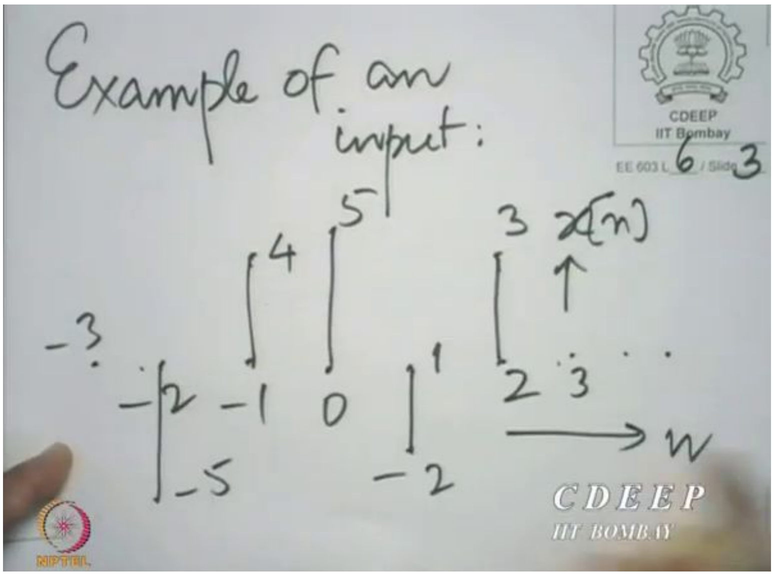

In fact, let me take any other specific points. So let us take the point $n = 2$. The point at $n = 2$ can be contributed by shifting the impulse to 2, in other words, writing $\delta[n-2]$ and multiplying it by *3*. So this is contributed by *3* $\delta[n-2]$.

And similarly, this is contributed by *-2* $\delta[n-1]$, essentially, the unit impulse shifted forward by *1*. If the unit impulse is shifted backward by *1*, you would put an impulse here, and if you multiplied it by *4*, then you would get this sample and similarly for this, and similarly for any other.

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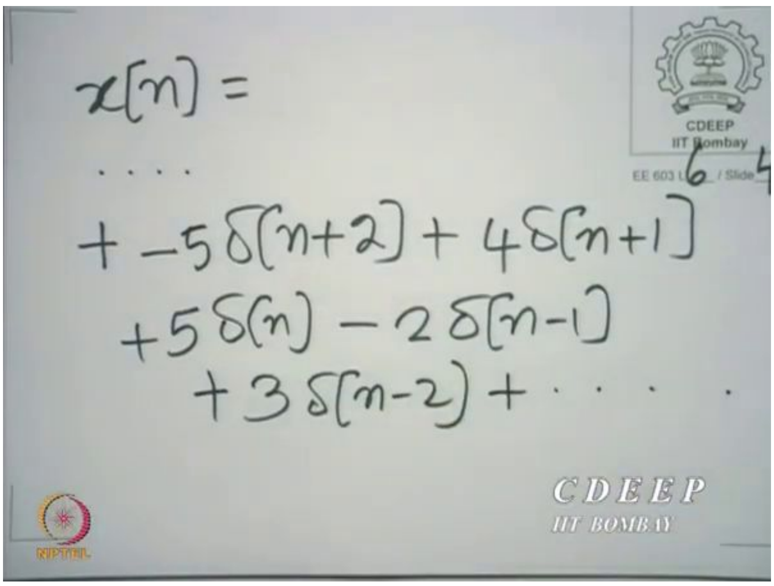

So we make the remark that this *x[n]* can be written as well, many other things, together with $-5\delta[n+2]+4\delta[n+1]+5\delta[n]-2\delta[n-1]+3\delta[n-2]$ and of course, many other things. In fact, we could now come up with a more general expression here.

If we notice, a typical term in this summation is really the sample value at a point *k*, multiplied by an impulse, unit impulse shifted by that *k*. So for example, this is a term for *k = 0*. This is the term for *k = 1*, this is the term for *k = -1*, and so on, so forth.

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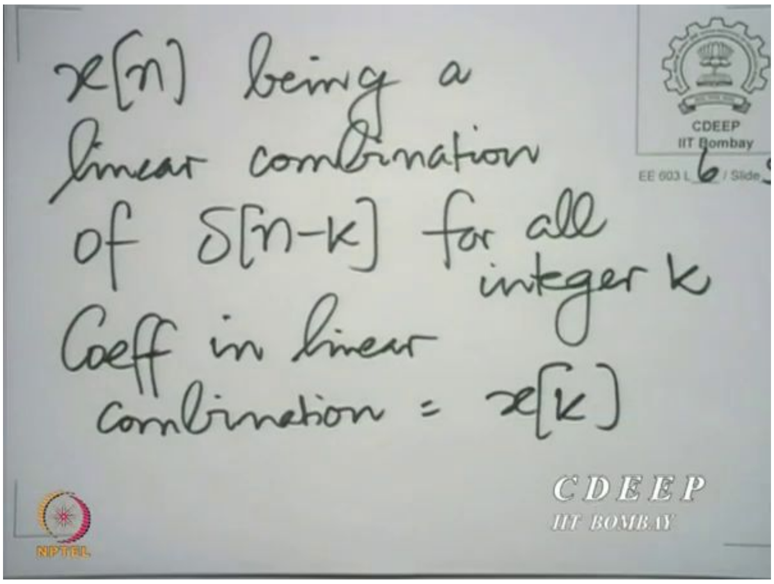

So we have in general, *x[n]* being a combination of $\delta[n-k]\ \forall\ integer\ k$. And the coefficient in the linear combination is *x[k]*. Now, although it is very easy to understand this, particularly because we have just taken an example, one must also appreciate some finer points in this discussion. The finer point is that *x[k]* in this discussion is now a constant, it is not a sequence for any particular *k*.

Secondly, $\delta[n-k]$ is a sequence not a constant, although it has only one sample, it is a sequence right? So, what we are saying is that a given input, any given input can be thought of as a combination of several sequence, each of which have only one sample. These finer points must be understood, because they are important in the proof.

I must repeat that if there are any questions on the way, you must ask them, then and there, do not wait until the discussion is done to ask questions. So, if there are any questions, or any point in time, you must raise your hand right there and ask the questions.

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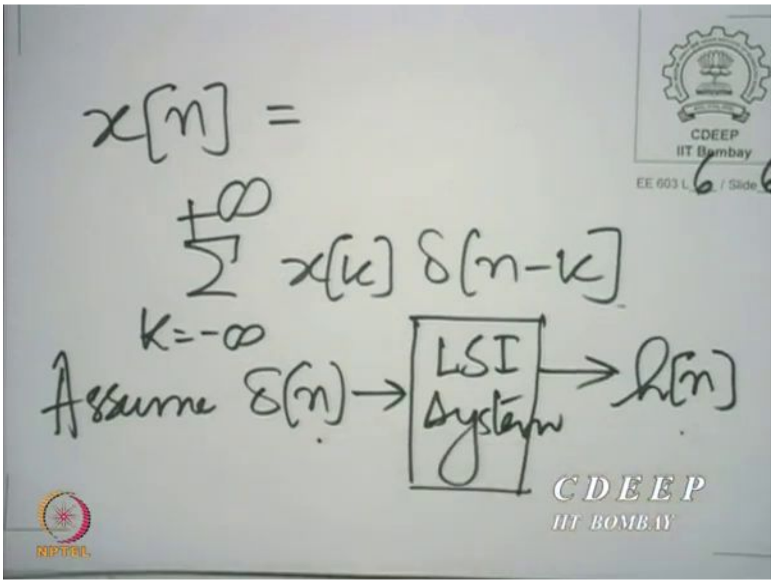

So, what you are saying in effect is that *x[n]*, the whole sequence *x[n]* is a summation over all *integer k*, $x[k]\delta[n-k]$. Note, this is a constant, this is a sequence. And we assume that when we put *h*, when we put $\delta[n]$ into this LSI system, we get a sequence *h[n]* emerging as the output. So, here is a sequence that goes in and a sequence that comes out. Please note. Naturally this is the unit impulse response, the response of the system to a unit impulse.

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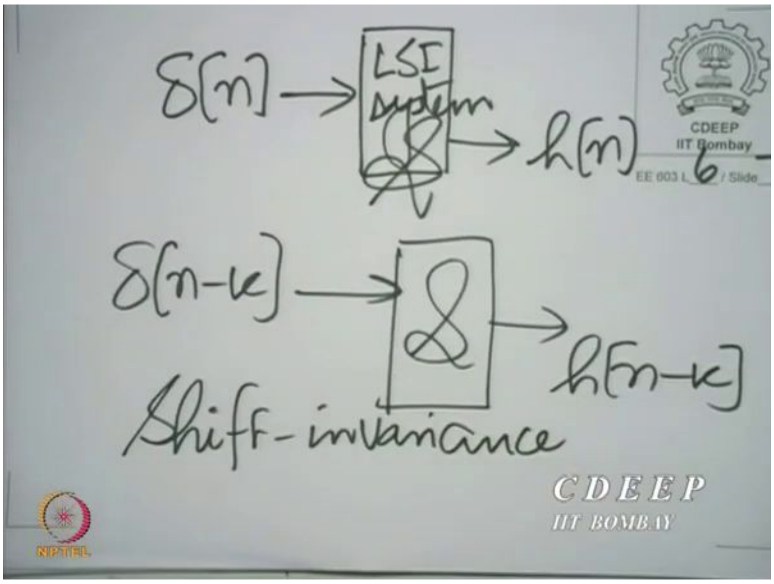

Now, I first invoke the property of shift-invariance. So, shift-invariance tells us that if I were to give $\delta[n-k]$, for a fixed *k* in place of $\delta[n]$, the only result on the output would be a shift of the same *k*. And therefore, if $\delta[n]$ produces, let us denote the system by *s* now, if delta n produces *h[n]*, then $\delta[n-k]$ is going to produce $h[n-k]$, that does half our work. So, here we have invoked shift-invariance.

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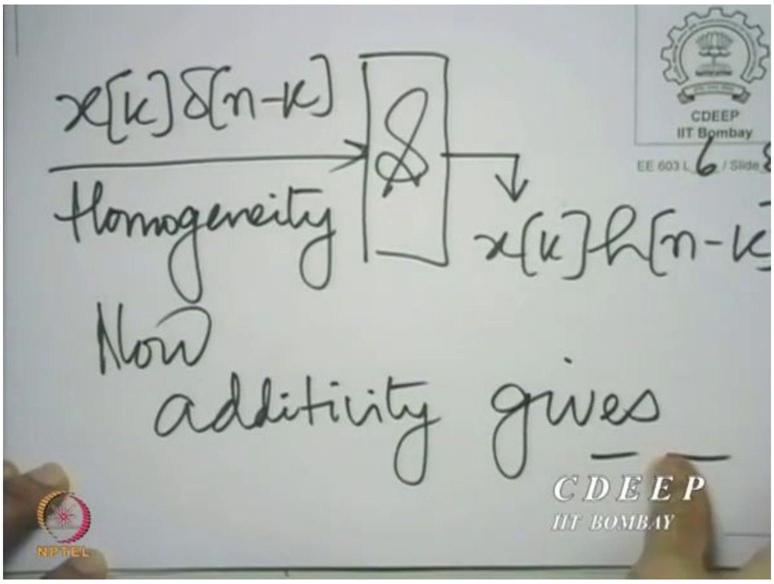

We now invoke homogeneity, $x[k]\delta[n-k]$, remember *x[k]* is a constant, $x[k]\delta[n-k]$, when applied to the same system is therefore, expected to produce $x[k]h[n-k]$, and note again here $h[n-k]$ is a sequence, *x[k]* is a constant. The interpretation must be very clear at every step. So, here we have invoked homogeneity, a scaling.

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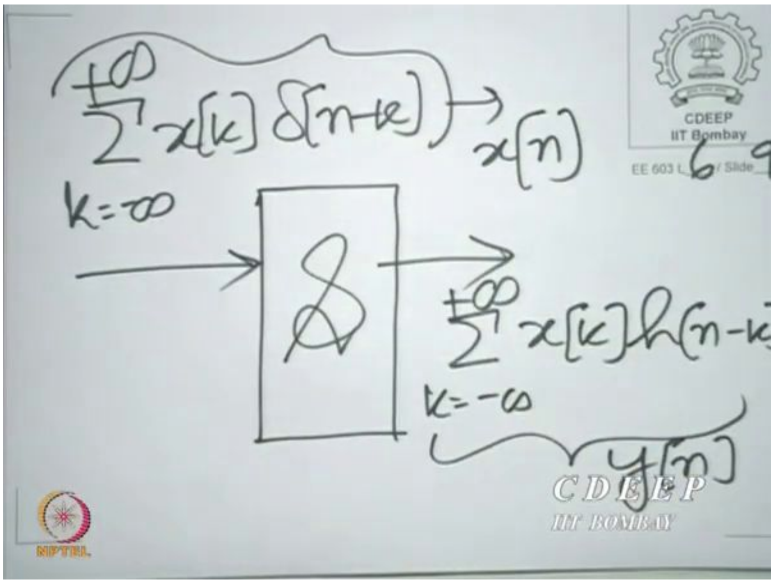

And finally, we invoke additivity. Now, additivity says summation over all *k*, what we are saying is invoke additivity pair by pair, motionally, keep taking pairs and go on constructing the whole sum here right? So, summation k over all integers $x[k]\delta[n-k]$ would then be expected to produce $\sum x[k]h[n-k]$.

In fact, this is really the output *y[n]* because this is the input *x[n]*. And what is significant is that here we have put no restriction on *x[n]* at all. For any arbitrary *x[n]* that we gave, we have an expression for the output in terms of the impulse response *h[n]* and the input *x[n]*. So, in fact, not only have we proved the theorem.

But we have also given what is called a constructive proof. A constructive as against an existential proof. I must distinguish, in some situations, one can only give an existential proof, that means one can prove that a certain solution exists. But it is not easy to construct the solution, or the proof itself provides not much of a cue on how one might construct the solution.

But here, we are fortunate to have a proof that is not just existential but constructive, in that it actually gives you a process for construction of the output given the input and the impulse response. In fact, what we have even now, is the construction in a rather raw form, in the sense that we have what seems like a fairly complicated expression. Let us spend a few minutes interpreting what we have got. So let us first draw a couple of conclusions.

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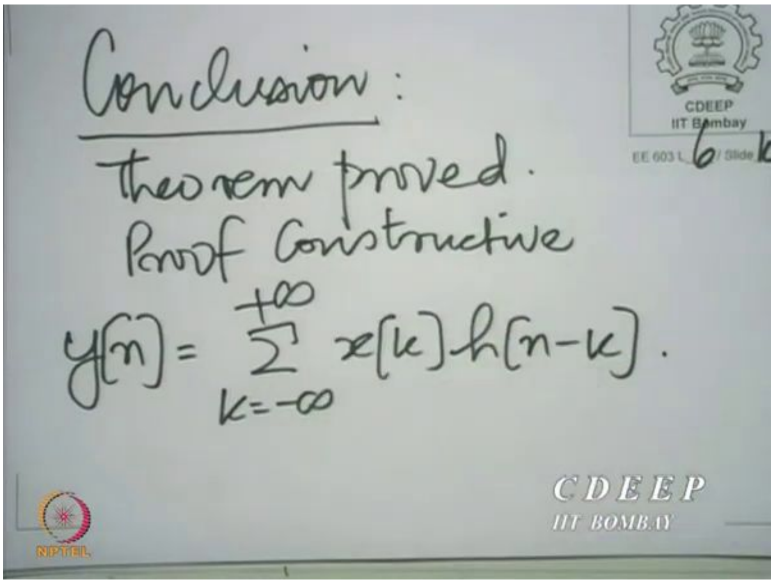

So the conclusion is that the theorem is proved. And the proof is constructive. In that you can actually explicitly write the output in terms of the input and the impulse response. And when we write it like this, the interpretation is very easy. What we are saying here is, essentially the output is a linear combination of several shifted versions of the impulse response.

The impulse response shifted by k samples, weighted by the input sample at the point k and then added over all such k. That is what the output is. So you can visualize this, you can visualize that if the system were to respond to unit impulse with a sequence h n, and what you have is the output is essentially this sequence $h[n]$ shifted by all possible shifts and the particular shift by k is multiplied by the input sample at the point k and all such shifts are added together. This is one interpretation.