Digital Signal Processing & Its Applications
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Lecture 07 A
Characterization of LSI systems,
Convolution Properties

Very good morning, and a warm welcome to the 7th lecture on the subject of Digital Signal Processing and its Applications. We will spend a few minutes in recalling what we did in the previous lectures. In the previous lectures, we were looking at what an LSI system is characterized by. So, we noted that for a linear shift-invariant system, it is adequate for us to know what is called the impulse response or the response of the LSI system to a special chosen input namely the unit impulse.

Once we know the output of an LSI system, given a unit impulse input, we know everything about the system. In other words, we know what the output could be for any input. And in fact, we proved last time that the output, the input, and the impulse response are related by an operation called convolution. The convolution of the input sequence with the impulse response sequence results in the output sequence, convolution is an operation in its own right.

What I mean by that is, independent of the fact that it is an important operation in the context of linear shift-invariant systems. convolution can be thought of as an operation between two sequences, irrespective of where the two sequences come from. In fact, there are reasons why we may want to think of convolution in that manner. And let us take a couple of situations where we need to deal with convolution as an operation independent of the context of LSI systems. In fact, let's take a situation where we have cascaded two LSI systems.

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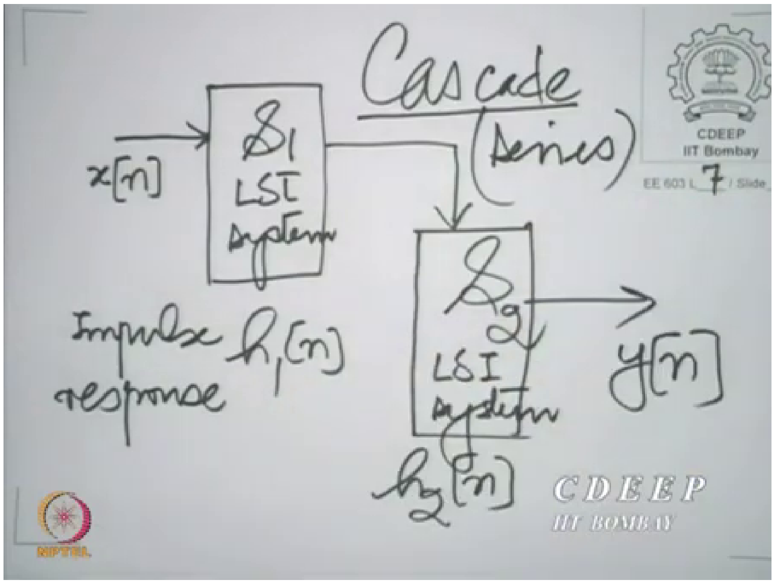

So, let us consider a situation where we have an input x[n] being applied to one LSI system, lets call it S_1 with impulse response $h_1[n]$. The output of which goes to a second LSI system. Let us give it to the name, give to it the name S_2 and here the impulse response is $h_2[n]$ and this results in the output y[n]. This is called a cascade interconnection, a cascade or a series connection of two LSI systems.

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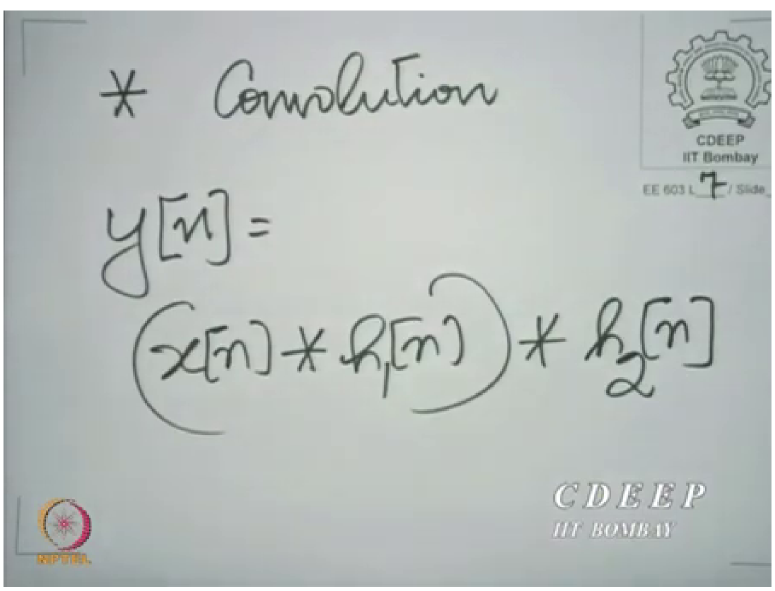

Now, it appears that if we use the symbol * to denote convolution, then

$$y[n] = (x[n] * h_1[n]) * h_2[n]$$

So, here we have two convolution operations, following one another to relate the input to the output. And there are several questions that we can ask. One question is what would happen if I interchange the order of these systems? So, if I were to interchange S_2 with S_1, would it yield something different. The other question that we need to ask is, can I replace this combination, this cascade combination by one single LSI system. So, let us put these two questions down.

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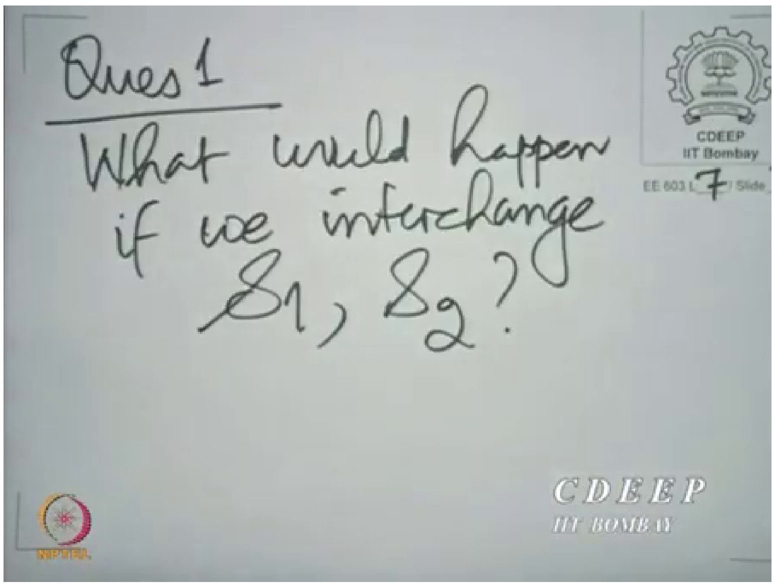

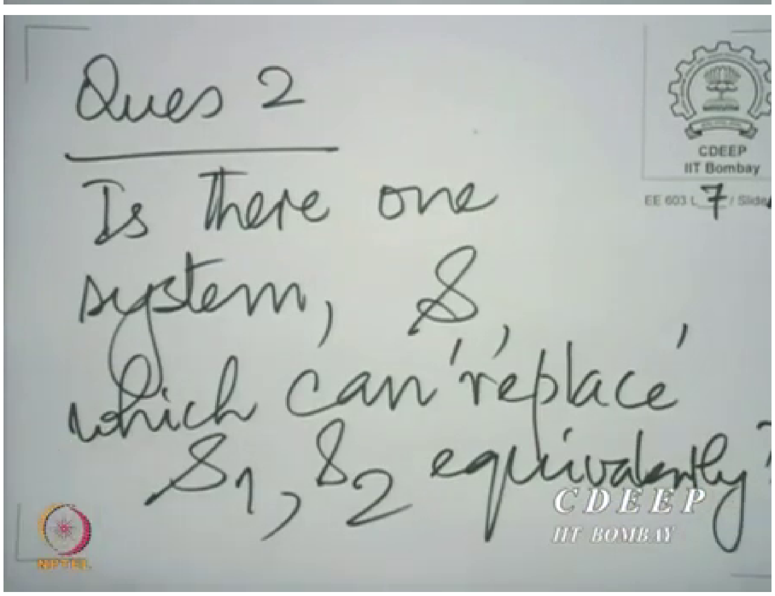

Question 1, what would happen if I interchange S_1 and S_2? And question 2, is there one system, let us call it S, which can replace S_1 and S_2 equivalently. Now, we will answer the second question first.

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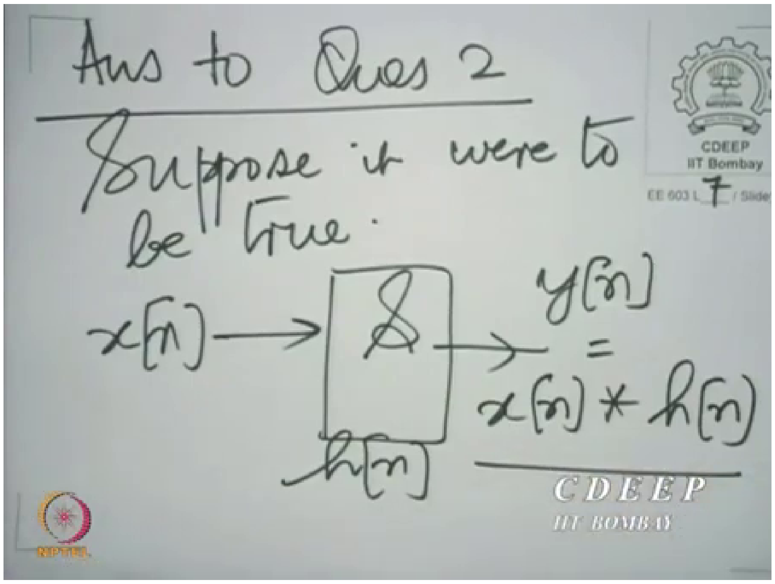

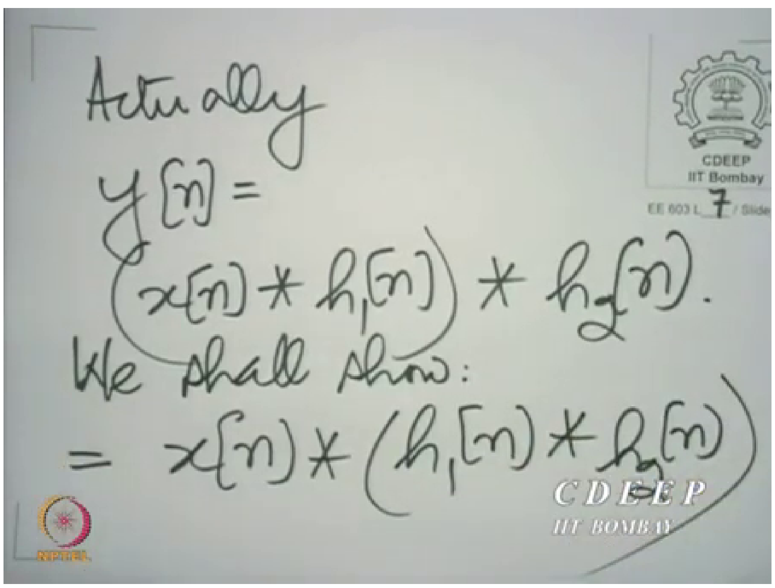

In fact, the answer to the second question would emerge from trying to find out whether I can change the location of the bracket. What I mean by that is, suppose it were to be true that means, I could replace S_1 and S_1 in cascade by a single system S perhaps with impulse response h[n]. Obviously, the output would be $x[n] * h[n]$ and we need to find out this h[n]. and what do we have, y[n] to be actually:

$$y[n] = (x[n] * h_1[n]) * h_2[n]$$

So, my difficulty is the place where the bracket lies. We shall show now that this is equal to $x[n] * (h_1[n] * h_2[n])$. In other words, we can change the position of the bracket for convolution. We shall prove this. This is what we shall now prove and this property is called the property of associativity.

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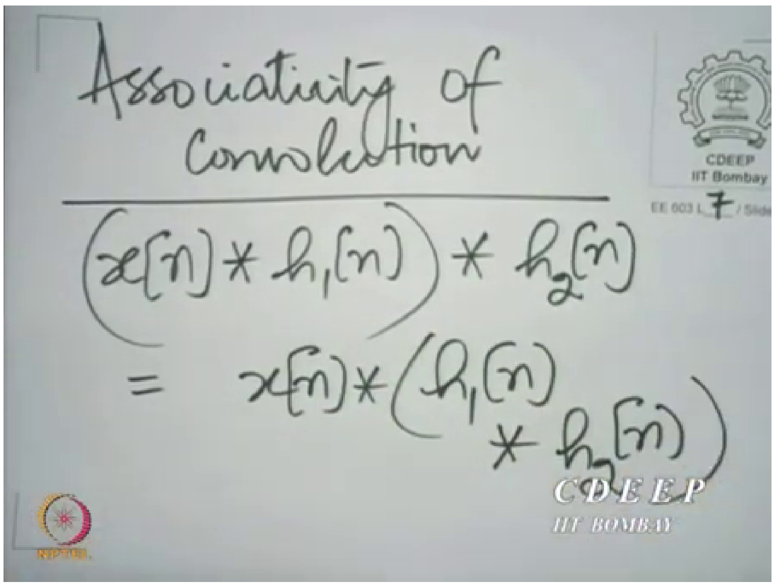

Now, we are looking at convolution in its own right, Associativity of convolution. So, in general, what we need to show is that $x[n] * h_1[n]$and here we do not necessarily imply x[n], $h_1[n]$ or $h_2[n]$ to be inputs or impulse responses, they are any sequences really. Now, lets take the left-hand side. I should indicate to you the modus operandi of the proof and I shall leave some steps of the proof to you to complete, is that right?

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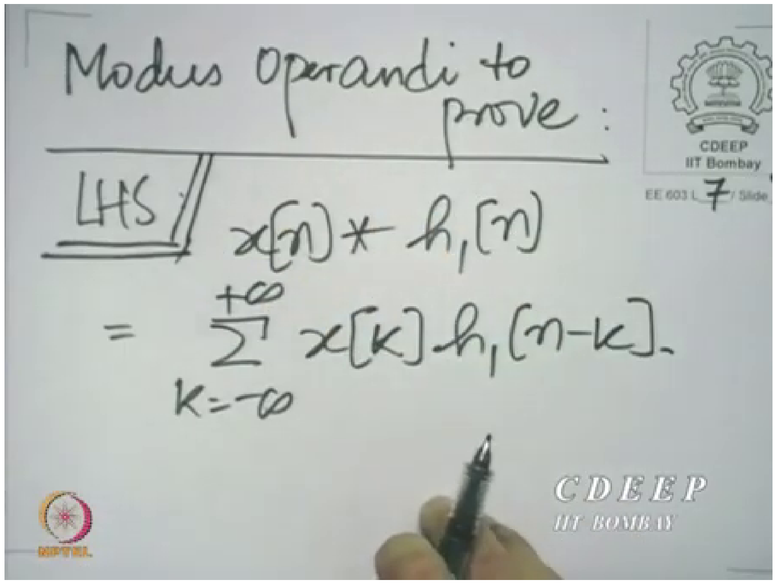

The modus operandi of the proof or the method the process to be followed to prove this is as follows, consider the left-hand side. Now, $x[n] * h_1[n]$, can of course be written as

$$\sum_{k=-\infty}^{\infty} x[k]h_1[n-k]$$

Essentially this is a function of n. This is not the whole left-hand side, I mean, this is just a part of it, so I am just, let me, so we are processing the left-hand side. That is what I mean by putting this double line here, this is not done, we have just written one part of the left-hand side. Now, we will complete this.

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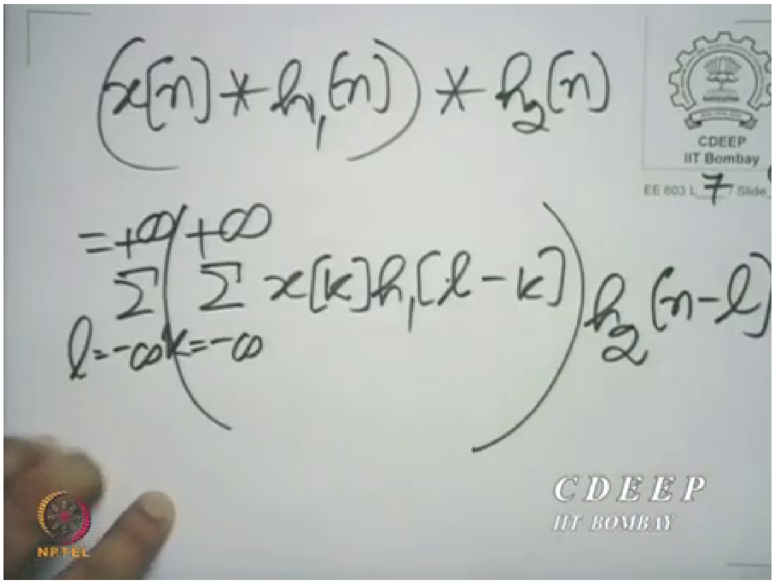

So, we have $(x[n] \ * h_1[n]) \ * h_2[n]$ to be

$$\sum_{l=-\infty}^{\infty} (\sum_{k=-\infty}^{\infty} x[k] h_1[l - k]) h_2[n - l]$$

now you see, here, I need to bring in one more variable. So, I won't write n now, this is k, but here, I need to introduce one more variable, that is called l, is that correct? This is the expression this is the complete left-hand side. Any doubts so far? Yes.

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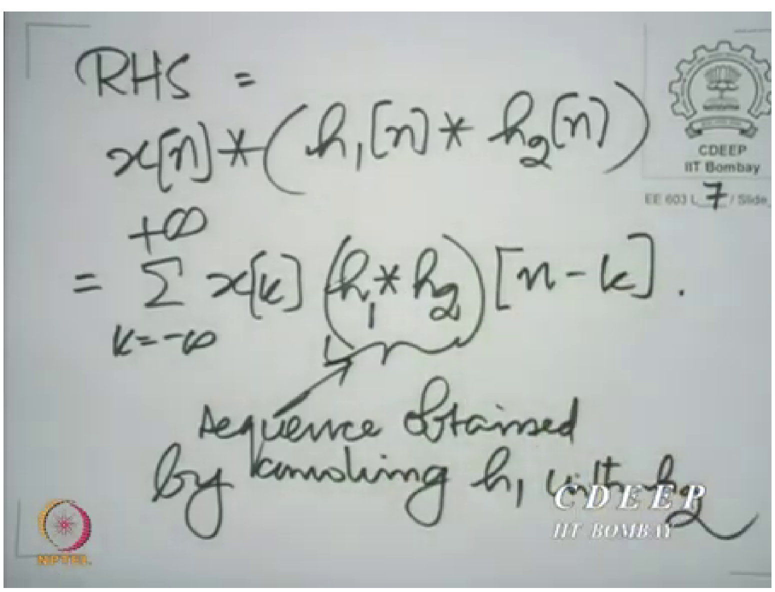

Now, we write the right-hand side. And the right-hand side is, of course, $x[n] * (h_1[n] * h_2[n])$. And this is very clearly

$$\sum_{k=-\infty}^{\infty} x[k](h_1 * h_2)[n - k]$$

I use this to denote the sequence obtained by convolving $h_1[n]$ and $h_2[n]$. This sequence evaluated at n-k, whatever that sequence be. Now, when you evaluate that sequence at n-k, we know how to write an expression for the (n-k)th sample of that sequence, we need to use a summation to write that sample as well, the (n-k)th sample of the convolution can be obtained as follows.

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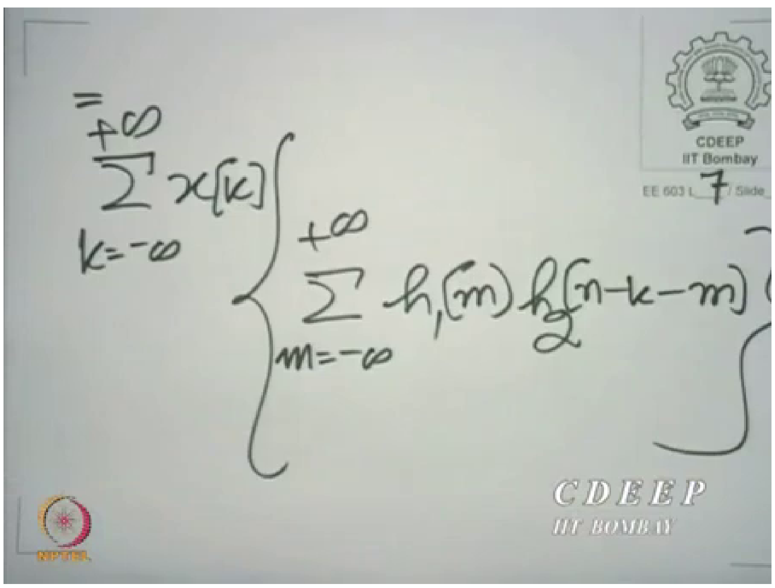

$$\sum_{k=-\infty}^{\infty} x[k]\left(\sum_{m=-\infty}^{\infty} h_1[m]h_2[n - k - m]\right)$$

So, what I need to do is to now bring these two expressions together here.

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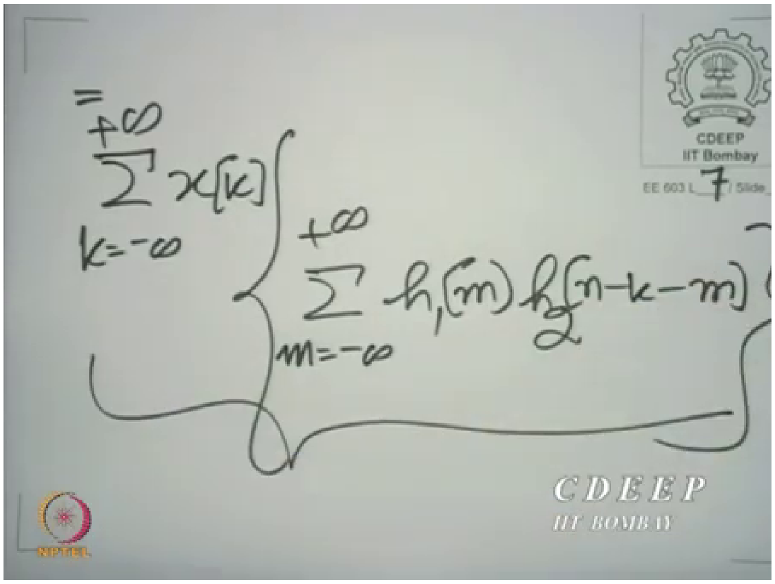

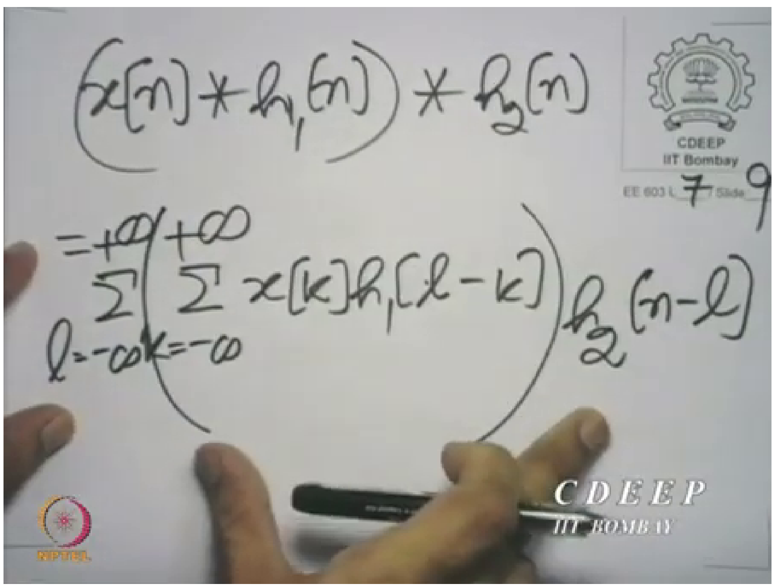

I have this expression for the right-hand side, and I have this expression for the left-hand side here, and I need to compare them. Lets write them on the same page.

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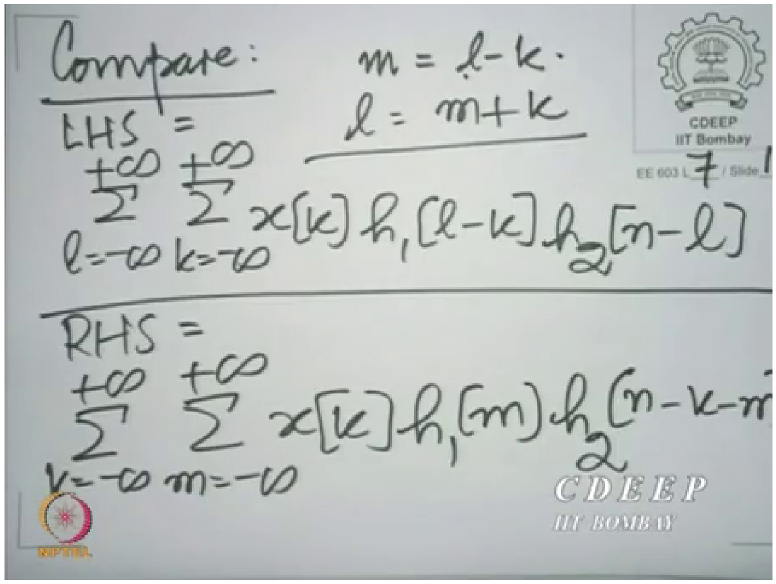

With RHS, which is $\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x[k]h_1[m]h_2[n-k-m]$

And now it is very clear to us how to prove the equivalence of these two expressions. You see, what we really need to do is to make a substitution of variable here, if we replace the variable l-k with the variable m, the summation of k is common to both.

As far as the summation of l here is concerned for any fixed k, you remember, the summation is $\forall\, k \in \mathbb{Z}$ that means the summation is on all integers k and all integers l in the left-hand side. And all integers k and all integers m, on the right-hand side is that right? And x[k] is the term common. Now, if we replace m for l-k, then it is very clear that n-l, what we need to do is to replace l.

So, l is then m+k. And, of course, now it is very easy to see that n-l would then become n-m m-k. But the only thing we need to justify is that the summations are also correct. And that it is very easy to see. You see, k is as it is k runs independently from $-\infty$ to ∞. And the question is, what happens to m? For any fixed k, when l runs over all the integers, m also runs over all the integers.

So, m, which is l-k here, runs over all the integers, when l runs over all the integers, or vice versa. You see, for a fixed l and m would concurrently run over all the integers, of course, not the same integer. But they will run over all the integers independently. And therefore, one can replace this summation,

$$\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} x[k]h_1[l-k]h_2[n-l]$$

by the double summation on k and m also, overall, the integer.

$$\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x[k]h_1[m]h_2[n-m-k]$$

And therefore, the left hand and the right-hand sides are equal and that proves the property of Associativity of convolution.

Now, we need to spend a minute in ensuring that we have no doubt about this proof. Do we have any doubt about the proof? Because it is a very important conclusion that we have drawn. So, we have concluded that convolution is associative and that answers our first question.

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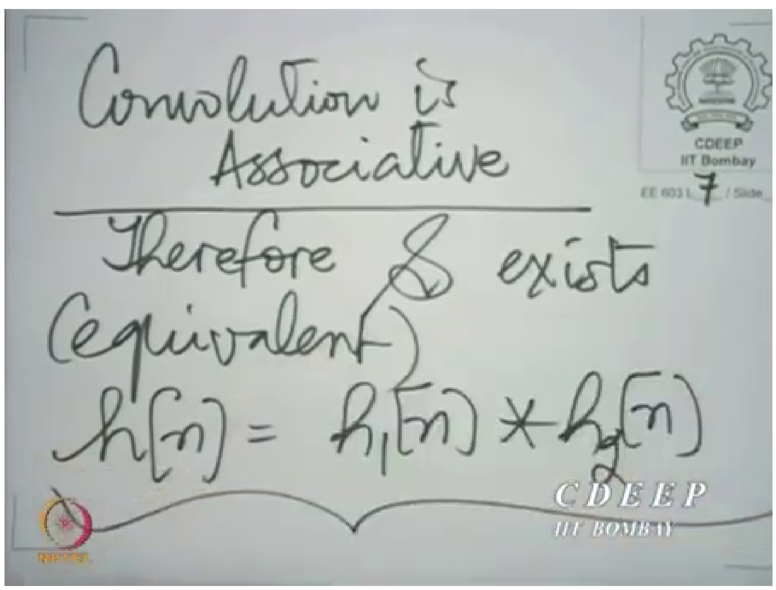

Convolution is Associative. This is a more general conclusion we have drawn. And of course, its implication for the context of LSI systems is that S indeed exists, the equivalent system exists and that equivalent system has the impulse response, $h_1 * h_2$. Very interesting. Now, we need to answer the second question. The second question is, can I interchange the order? And that amounts to asking, see now, in a way, we have also got an interpretation.

We know that together, the impulse response of equivalent system is $h_1 * h_2$. And if you could interchange the order, then the impulse response to the $h_2 * h_1$. So, in other words, we are asking whether convolution is commutative.

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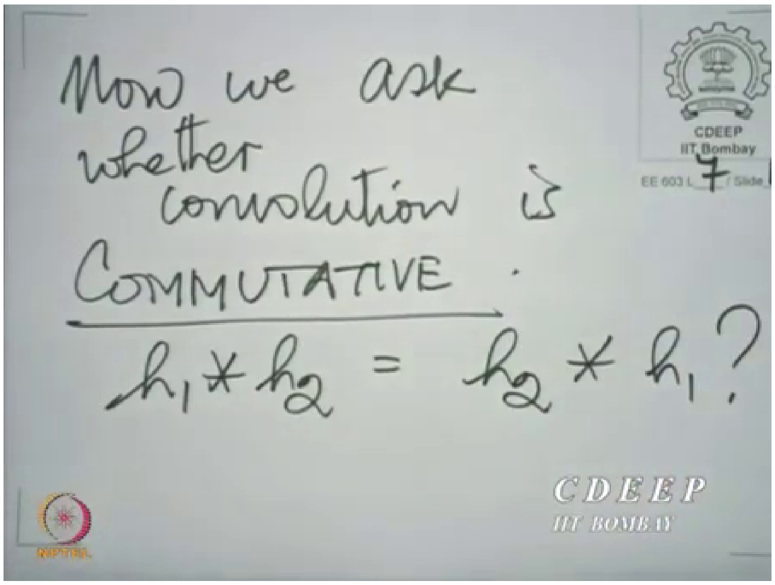

In other words, would $h_1 * h_2$, be equal to $h_2 * h_1$. Now, please note that here, I am abusing notation a little bit, by suppressing the dependence on the integer index, I am just writing. And anyway, in a way, it is not too much of an abuse because ultimately, convolution is an operation between sequences not between numbers. It is not too much of an abuse of notation. So, let us answer this question. Once again, let us look at the left-hand side first.

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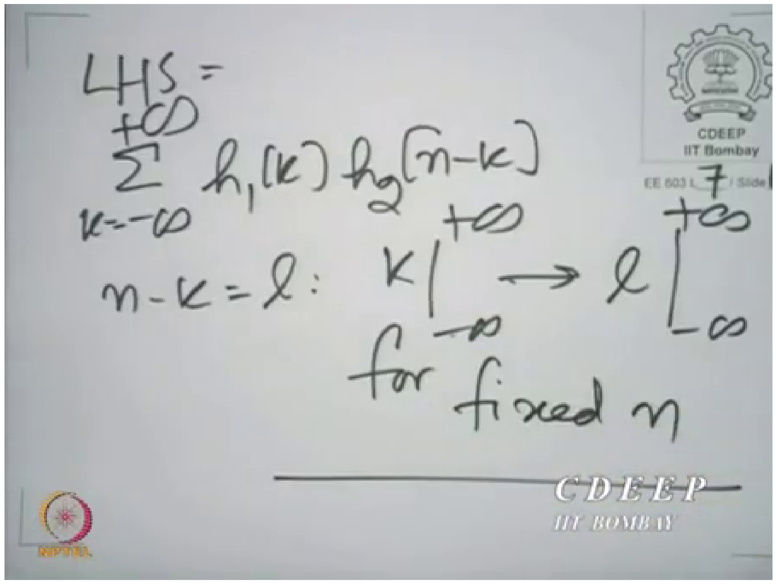

So, left-hand side is essentially

$$\sum_{k=-\infty}^{\infty} h_1[k]h_2[n-k]$$

.And here again, we make a very simple substitution, we substitute n-k by l and we note, that k running from $-\infty$ to ∞ leads to l running from $-\infty$ to ∞ for fixed n. This is easy to see, for a fixed n, if k runs over all the integers and n-k, which is l also runs over all the integers. And therefore, we can make a replacement here we can replace m-k by l. And that is very easy.

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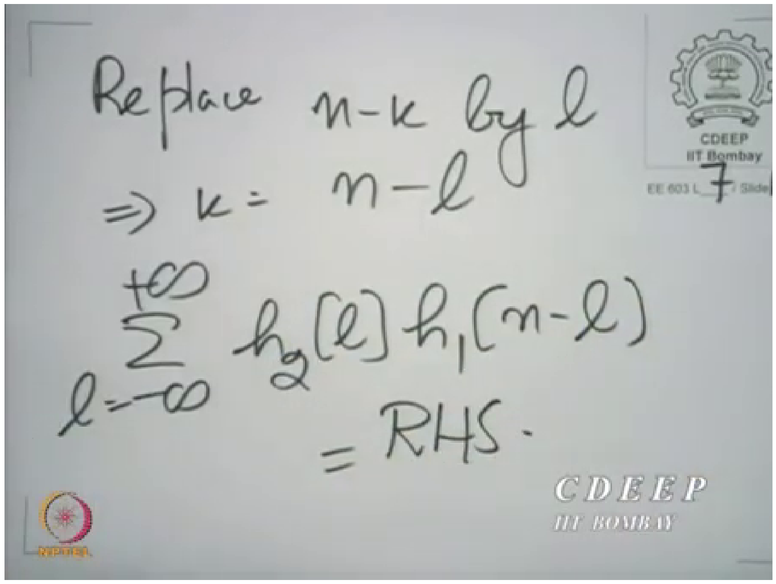

What we want on the right-hand side, so replace n-k by l, which means k is n-l. And of course, then we have

$$\sum_{k=-\infty}^{\infty} h_1[n - l]h_2[l]$$

which is essentially the right-hand side. And therefore, we have concluded that the left hand and the right-hand sides are equal and convolution is commutative. Yes, there is a question.

Student: Sir, when k runs from $-\infty$ to ∞ (())(20:43).

Professor: Yes. So, I will repeat the question. The question is, we said that, in this case, for example, when k runs from $-\infty$ to ∞, n minus k would invert, so to speak, or in other words, it would seem to run from ∞ to $-\infty$. Well, that is correct. I mean, what I am trying to say is, the set of integers, the entire set of integers is covered by k and it is also covered by l.

Though not necessarily in correspondence. That means when k is at a particular integer, l is not the same integer, that is true. But there is a one-to-one correspondence. So, for every k there is a unique l, and for every l there is a unique k. And not only that, the entire set of spans, when you are dealing with infinite sets, you have to be careful.

So, in fact, the question has raised an important point, when you are dealing with infinite sets, one must be sure that the infinite sets are equivalent, I mean, infinite sets, you cannot simply conclude they are the same unless you can draw correspondence. That means, you must be able if you see, if you claim that two infinite sets are the same or that are equal, then you must be able to draw a one-to-one correspondence between those two sets, I must be able to make handshakes between the elements of the two sets and here you can do that, you can make a handshake between elements of the set k and elements of the set l, a one-to-one handshake.

So of course, the handshake is not between the same elements, but between different elements. Is that clear? Are there any other questions about this proof? We must, it is very important that we clarify questions as we go along. Otherwise, it would lead to gaps in understanding and weaknesses on foundation as we proceed and a weak foundation leads to cracks on the upper storey, upper storeys, upper floors, is that right? So, foundations must be strong and then that does not allow cracks to seep in on the upper floors of a building. So, are there any other questions on the proof so far? Yes, is the proof very clear to everybody?

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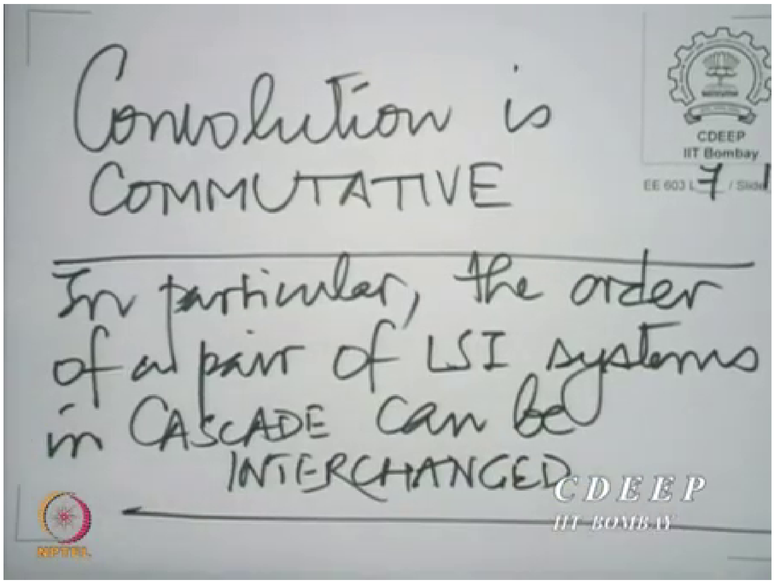

So, what is our conclusion? The conclusion is that convolution is commutative. And in particular, what does it mean for LSI systems? The order of a pair of LSI systems in cascade can be interchanged, that answers question one. Now, I leave it to you as an exercise to prove the following.

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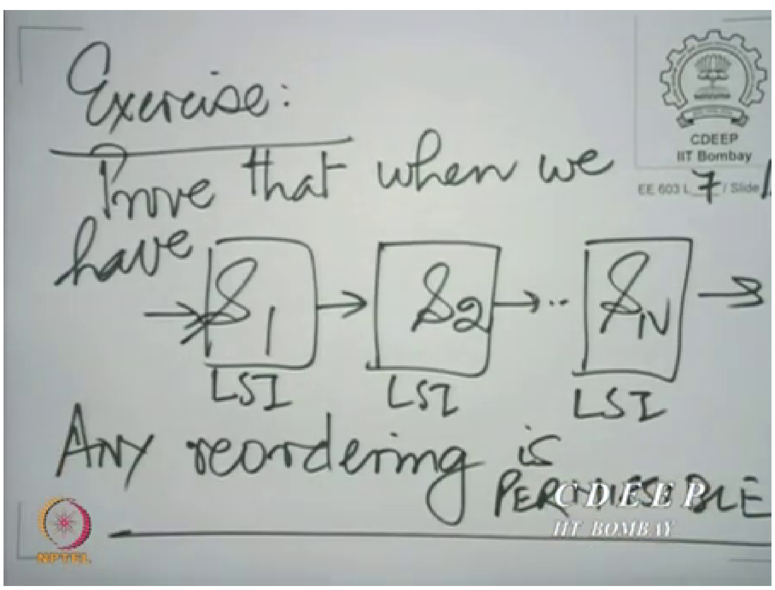

Prove that we have, whenever we have multiple LSI systems in cascade. Let us say N, capital N subsystems in cascade. Any interchange, any reordering is permissible. In other words, the order does not matter. If you have a set of LSI systems in cascade, not just two of them, any number of them, then any reordering of those LSI systems in cascade does not influence the overall input-output relationship and to prove this you shall require both commutativity and associativity of convolution. So, please use commutativity and associativity to prove this by mathematical induction or any other method that you choose, is that right? So, I leave, give this to you as an exercise.