Digital Signal Processing and its Applications
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Lecture 07 C
Convergence of convolution and BIBO stability

And we need to answer whether there is something that we should impose upon the system to make the summation convergent.

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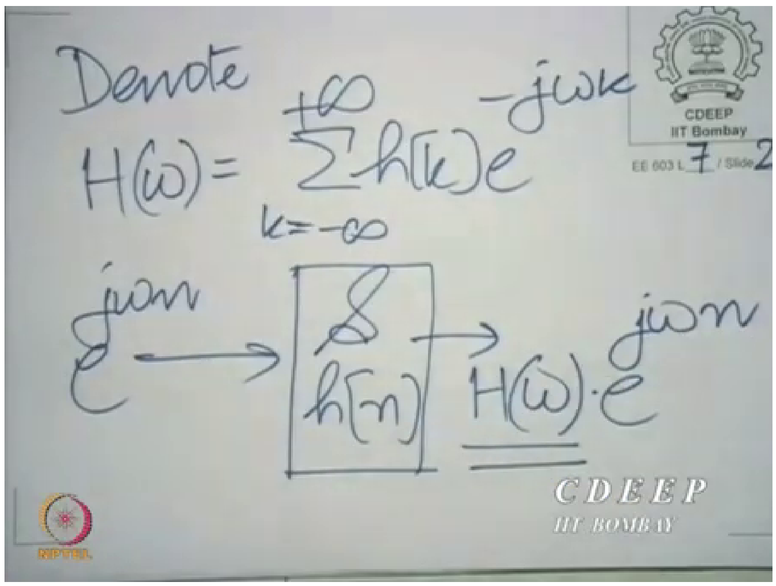

Now, one thing is clear if this summation converges, so let us see, if this summation converges, then we have a very interesting, in fact we have done what we wanted to, when we give in a complex exponential of frequency ω angular frequency of normalized angular frequency ω to a system, what we get out is the same complex exponential but multiplied by a complex constant.

There is no other change. That means, each complex exponential is dealt with in a decoupled way. So, now, if I have a sum of complex exponentials, of course, it is a minor, it is a very simple thing to see that if I multiply, I can multiply both sides by any complex number here. So, I can multiply, let me in fact, rewrite this.

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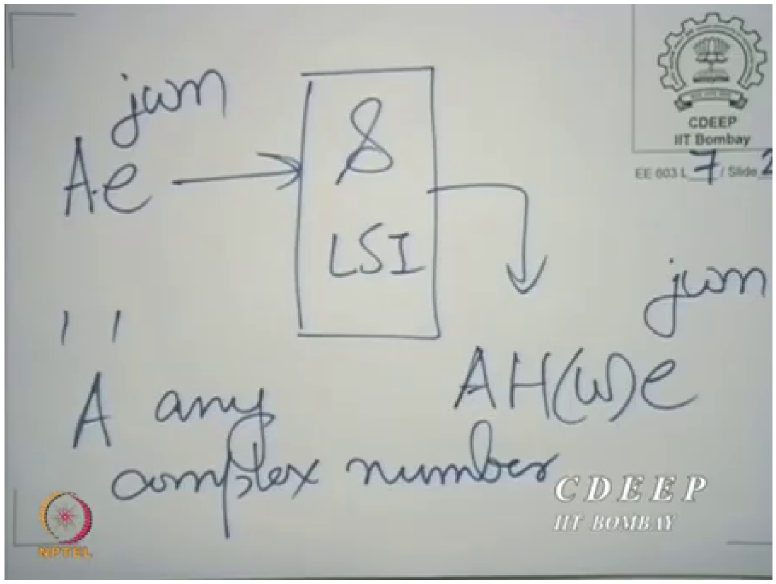

So, if I take $Ae^{j\omega n}$ and give it to the same LSI system, I would of course get $AH(\omega)e^{j\omega n}$, A is any complex number provided $H(\omega)$ convergence. That is the million-dollar question, when would it converge.

But at least, it is soothing to see that, provided we have convergence, we have got what we wanted. If I have a linear combination of complex exponentials with different angular frequencies, what is going to emerge is the output to each of these complex exponentials treated independently. And the output to each of these complex exponentials is going to be the same complex exponential multiplied by an appropriate constant which depends only on the angular frequency.

Of course, it depends on the impulse response. But since the system is the same, the response is the same for that angular frequency. So, we have got what we wanted. With the only catch that we do not know when this would converge, can we guarantee that the summation converges all the time. Now, to check convergence, let us explore the expression itself.

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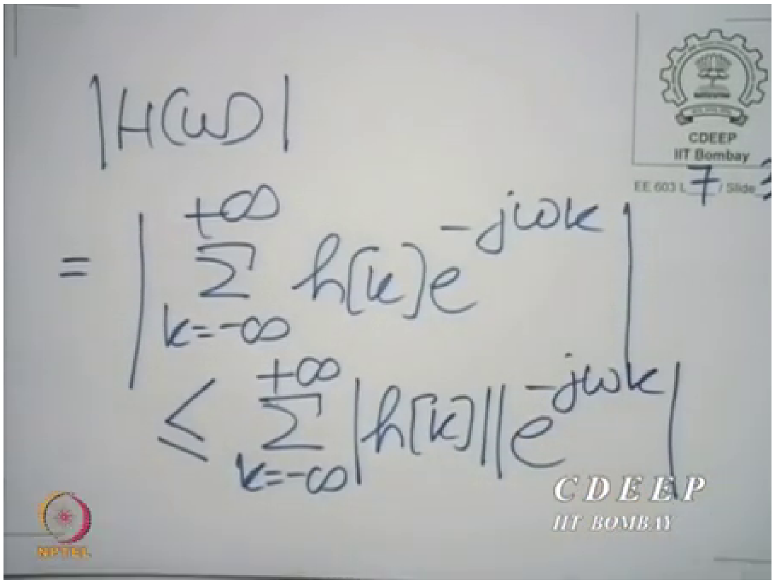

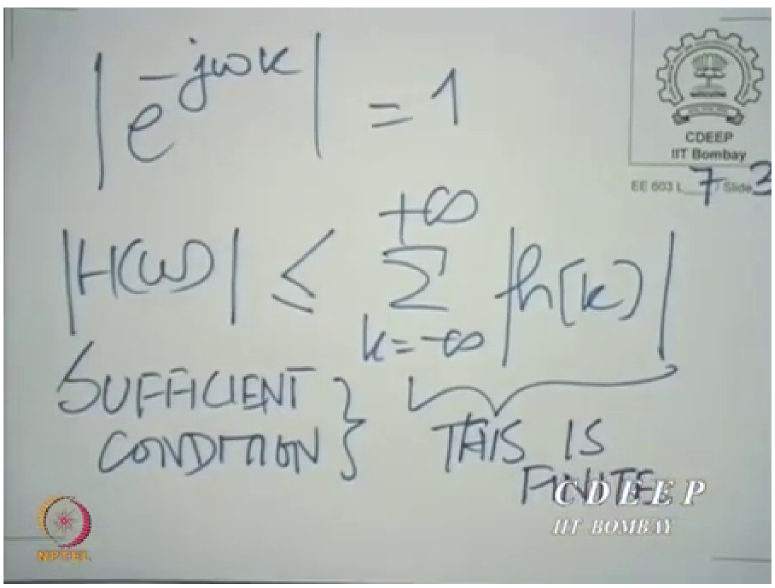

If a convergence ultimately has to do with the magnitude, the phase is irrelevant. So, let us look at the magnitude of $H(\omega)$, the magnitude of $H(\omega)$ is obviously the magnitude of $\sum_{k=-\infty}^{\infty} h[k]e^{j\omega n}$.

And this is obviously less than or equal to the sum of the magnitudes, that is, you see, $|a + b| \leq |a| + |b|$

And you can keep extending this to an infinite summation. This is less than or equal to the $\sum_{k=-\infty}^{\infty} |h[k]||e^{j\omega n}|$. But of course, $|e^{j\omega n}| = 1$ and therefore,

$$|H(\omega)| \leq \sum_{k=-\infty}^{\infty} |h[k]|$$

So, we have what we clearly see the sufficient condition.

The sufficient condition is that this is finite. So, it means if the impulse response sequence is absolutely summable, absolutely summable means the sum of the absolute value of the samples is finite. If the impulse response is absolutely summable then we shall make a remark.

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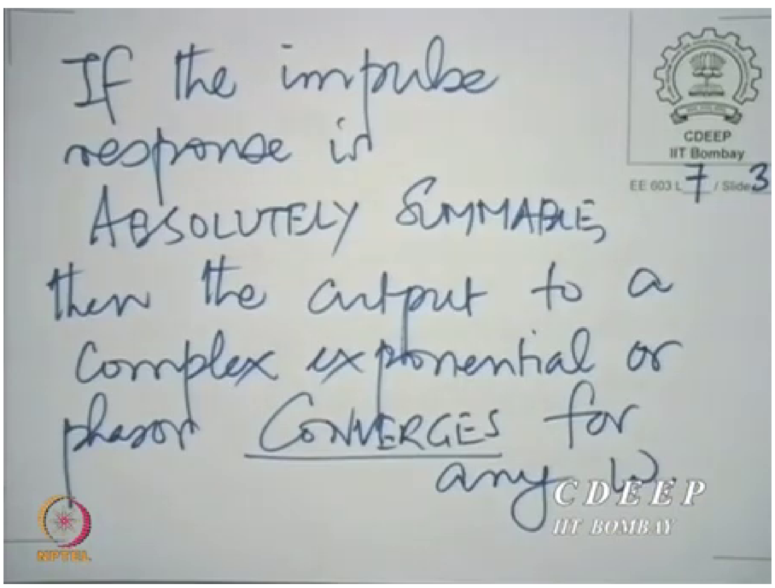

Then the output to a complex exponential or phasor converges for any angular frequency. You see there are, please note that this condition is sufficient the way we proved it, we have only proved sufficiency. In fact, it is indeed just sufficient. The fact that it is only sufficient is a very subtle point. We will understand that better later. But we need to appreciate the meaning of this condition a little better.

Mathematically, of course, we see it is an absolute sum. It is a sum of absolute values. Yes, there is a question. Yes. So, the question is in this page here, how would I conclude? No, no, what I am saying is, if I want, I would have shown is that $|H(\omega)|$ is less than or equal to this absolute sum here, so I am saying a sufficient condition for this to converge, is that this is finite, this is a sufficient condition. I am not saying that this is necessarily finite, but if this is finite, then this converge.

So now, we are saying we need to now answer what is the physical interpretation, what do we mean by this being absolutely summable. And in fact, we will get a hint, if we only look at the general convolution expression once again, we shall do that and we shall carry on the discussion in this and the next lecture.

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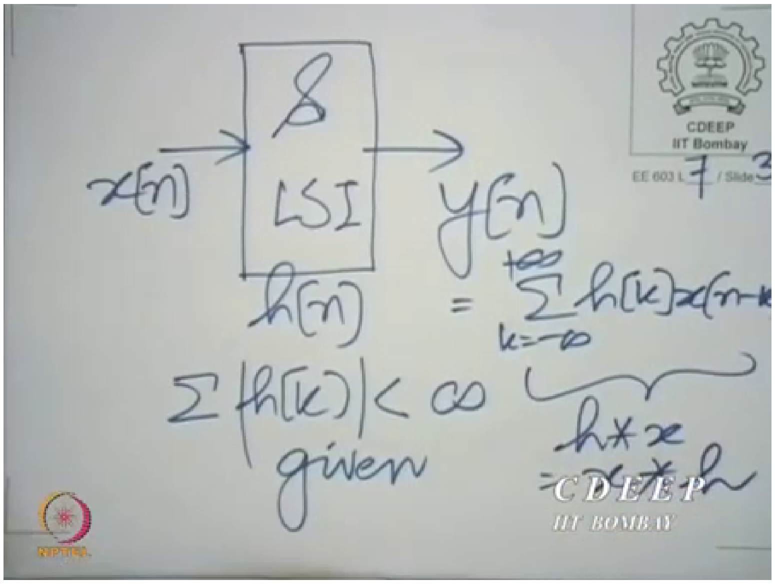

So, let us take that LSI system with a general input, we have x[n] we have the same impulse response h[n] we are asking what is the fact that and we of course, we have been told that this is true, given, so of course, we know what y[n] is, of course $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$. But now, you will agree with me that I can also write this down as $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$ because convolution is commutative.

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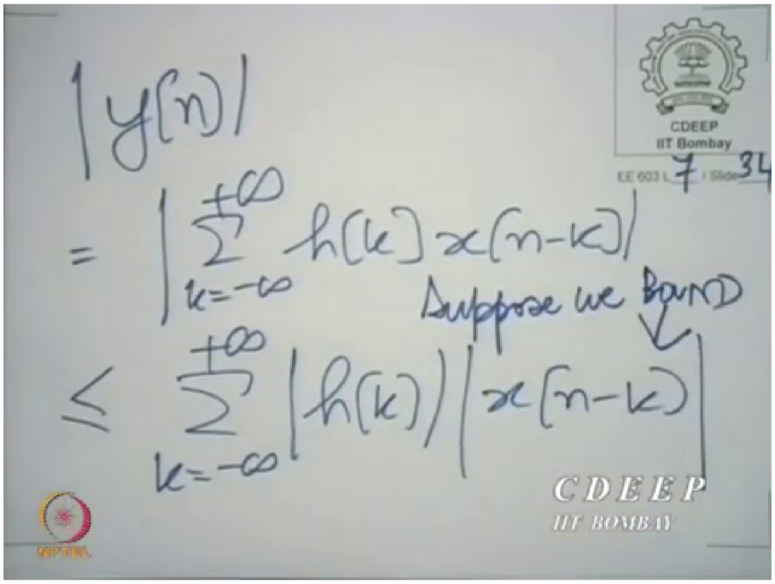

So, let us take the modulus of y[n], $|y[n]| = |\sum_{k=-\infty}^{\infty} h[k]x[n-k]|$and of course, this is less than or equal to $\sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$ as usual, because$|a + b| \leq |a| + |b|$.

Now, let us look back at this, you see here. Suppose we managed to put a bound on this, bound means we put a supremum on it. Supremum means, we identify a finite, non-negative number, such that none of these magnitudes can be more than that number.

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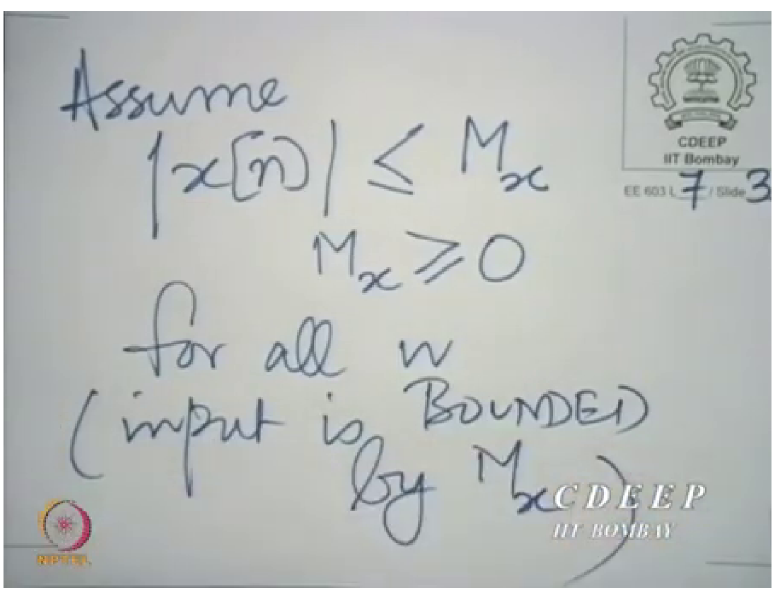

In other words, let us assume that $|x[n]| \leq M_x$, where M_x is a quantity greater than or equal to 0 for all n. In other words, the input is bounded by M_x, we say this is the terminology that we use, we say the input is bounded by M_x.

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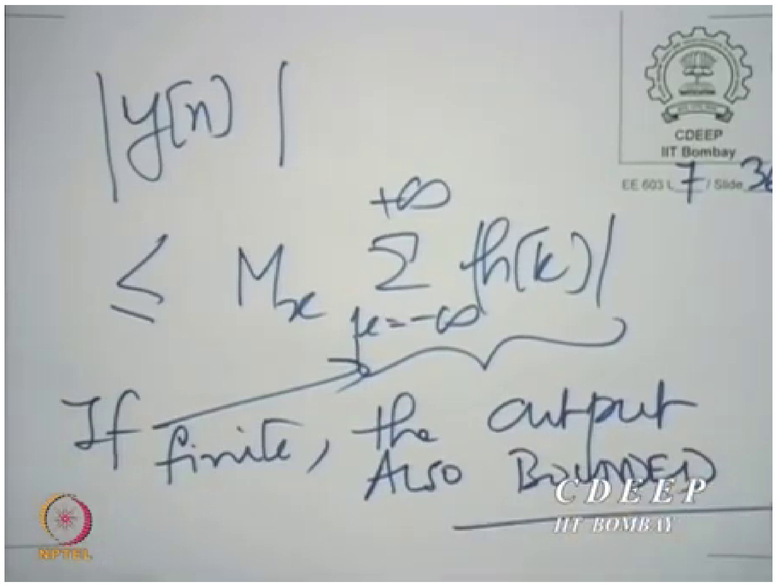

Obviously, the output is also bounded now, $|y[n]| \leq M_x \sum_{k=-\infty}^{\infty} |h[k]|$. And if this is finite, the output is also bounded, and in fact here too we have what is called a constructive proof, not just an existential proof.

So, what we have shown is that, if that condition absolute summability of impulse response is satisfied, then a bounded input results in a bounded output, that is the serious conclusion. And not only have we concluded that we have also shown what the bound, what the output bound is. That is what I mean by constructive you can calculate the output bound at or at least you can calculate one output bound from the input bound could be better.

Now, this leads us to one more property that may or may not be possible in systems, in LSI systems and that is called the property of stability. In fact, we define in the next lecture, the idea of stability in terms of inputs and outputs being bounded, there are different notions of stability. And we shall talk more about these notions and the connection to the impulse response in the next lecture.