Digital Signal Processing and its Applications

Professor Vikram M. Gadre
Department of Electrical Engineering
Indian Institute of Technology, Bombay

Lecture 08 B

BIBO Stability of an LSI System

We will use the notation ∃ to write 'there exists'. We said that an input $x[n]$ is bounded if $\exists\ M_x \geq 0$ such that $|x[n]| \leq M_x$ for all integer n of course. And of course, M_x is independent of n, that is obvious.

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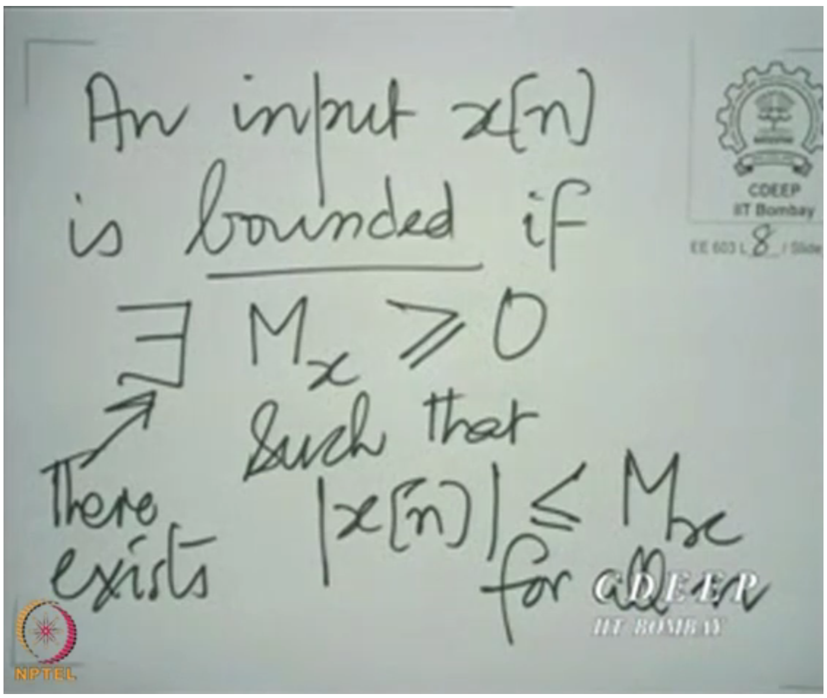

So, it should be noted that we are talking about the boundedness of the samples here. We are not asking that the input be absolutely summable. We are just saying that the input needs to be bounded. Now we are saying that, given that the input is bounded, the output is also bounded.

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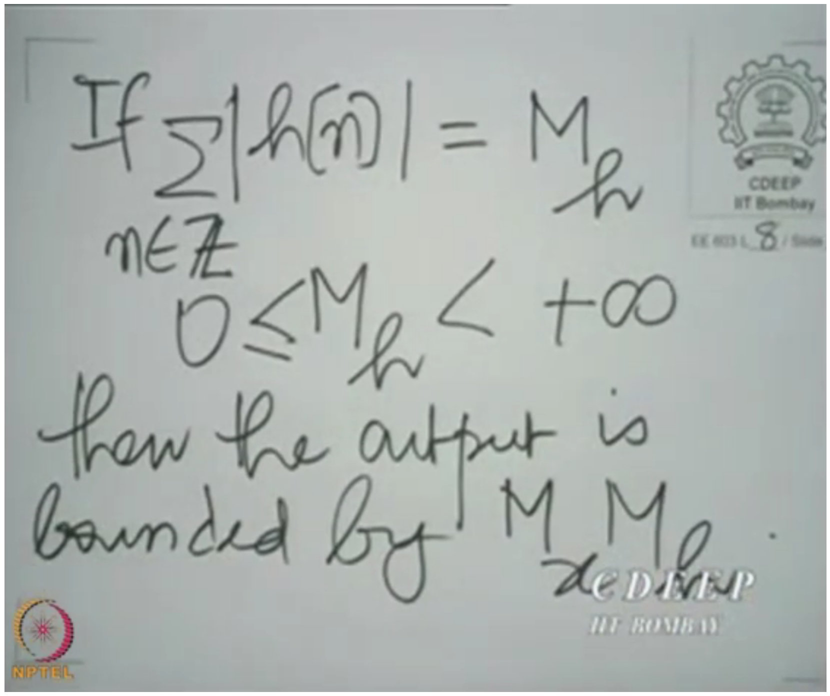

Let $\sum_{n\in Z} |h[n]| = M_h$. Of course, $0 \le M_h <+ \infty$. If the summation is M_h, then the output is bounded. Not only is the output bounded, we know a bound on the output, the output is bounded by $M_x M_h$, that is very interesting.

So, we said that we have a constructive conclusion here, not just an existential conclusion. A constructive conclusion is that, if I know the bound on the input and I know the absolute sum of the impulse response, the bound on the output is equal to the bound on the input multiplied by the absolute sum of the impulse response.

What we were asking yesterday is whether the condition on M_h is necessary for this to happen and that is what we will now need to answer but with a little bit of hard work. Before we embark upon that question, I would like to allow a couple of questions. Yes, there is a question from here.

[Student asks a question]

Professor: So, the question is, why should M_x be non-negative? You see, M_x is a bound on $|x[n]|$, which can never be negative. So, the least value of $|x[n]|$ is 0. Of course, $|x[n]|$ must be non-negative. The non-negativity of $|x[n]|$ is trivial, the catch is that it is finite. Is that right?

Any other question before we work towards the necessary part? Any other questions so far? Because so far, we have drawn some important conclusions.

We now embark on the very important question of whether this condition is necessary. So, let us pose the question first. The question is, is it necessary for the impulse response to be absolutely summable, if every bounded input produces a bounded output? So, you see now we are asking the converse. We are saying, let the system produce a bounded output for every bounded input.

If every bounded input results in a bounded output, the system is what is called bounded input bounded output stable. Before we start answering this question, we would like to understand some of the terms that we are using. We call a system BIBO stable if every bounded input results in a bounded output.

Now, remember, we are saying every bounded input results in a bounded output. In a stable system, every bounded input results in a bounded output. So, it is not adequate to consider a specific example of a bounded input and look at the output and see that it is bounded and conclude the system is stable.

Now, I always like to explain the idea of stability of systems by drawing a parallel to the sanity of human behavior. So, we take a parallel between stable systems and sane humans. When would you say a human behaves sanely? When, in every instance he has a sane behavior from another human being, he responds sanely.

So, if you behave sanely, if you behave properly with a sane human being, he or she responds with sane behavior. That is what a stable system does; give it a bounded input, it responds with a bounded output. But you see, now take the case of a sane human being, suppose an insane person comes in front of a sane human being, it is not necessary the sane human being would behave sanely. You see, it all depends on how insane that person is.

If the person is all out to attack him, that person might also behave equally insanely. So, even if a system is stable, and if you give it an unbounded input, it is not necessary the output needs to be bounded. A BIBO stable system, when given an unbounded input, could produce either a bounded output or an unbounded output.

So, what are sane, what are stable systems? Those that produce bounded outputs when they are given bounded inputs, like sane human beings. Now, what about unstable systems? They are also like insane human beings. Insane human beings at times can behave very sanely even if you behave insanely with them. Insane human beings are unpredictable. Similarly, unstable systems are unpredictable. If you have an unstable system, give it a bounded input, you know nothing, it may produce a bounded output, it may not produce a bounded output. Give it an unbounded input, it may or may not produce a bounded output.

It should be clear that one should not conclude that if a system is stable, then giving it an unbounded input will necessarily produce an unbounded output, or giving it an unbounded input will necessarily produce a bounded output. All we can say is that the output is unpredictable. That is for stable systems.

For unstable systems, nothing much can be said at all. All that can be said for insane people is that there is at least one instance when somebody behaves sanely with that person and he responded insanely. That is why you call him insane. If every instance of sane behavior elicited a sane response, you wouldn't call the person insane. You call a person insane because some sane behavior elicited an insane response. And the same is true of unstable systems. I am specifying this because sometimes, these finer if-but, if-then are not properly obeyed.

Now, coming back to the question of stability and instability, we have shown that it is sufficient for the impulse response to be absolutely summable for an LSI system to be BIBO stable. But we have to now ask, if a system is BIBO stable, can we immediately conclude the impulse response is absolutely summable?

And again, to answer this question, we are again going to take the help of a famous story, which occurs in many legends, in many countries, in various forms. And the story goes, that there was this gentleman who knew many languages, and only one of them was his mother tongue. That man posed the challenge that anybody who could find out his mother tongue, would have him as his servant for a certain period of time.

This person was, other than being a polyglot, very knowledgeable in many ways. So the king wanted to have him in his council of ministers, if possible at least for a while. But the condition for that to happen was the deciphering of his mother tongue. So the king posed the challenge before a very witty and intelligent minister.

So, what this wily minister did was to let that person rest very peacefully on a particular pleasant night. And then in the middle of the night, this wily minister sent a couple of soldiers and threw absolutely cold and unpleasant water on this gentleman, rudely awakening him. You can quite guess the language in which he uttered the vindictive statements and the unpleasant things that he wanted to. His mother tongue was very quickly known.

This teaches a very interesting lesson: if one really wants to drive a system to the brink—and a human system is no exception—use a situation which is extremely troublesome. And we'll do the same for a stable system. If we wish to prove that the impulse response must be absolutely summable, we try and drive that stable system almost to the brink of instability, like you drive a sane human being almost to the brink of insanity to reveal or to test the person's sanity.

Now here, you drive the stable system almost to the brink of instability by forcing upon it an input which is bounded in principle, but which forces the absolute sum of the impulse response to emerge. That's about the worst that can come out. So let us then see which input to a stable system would force the impulse response to emerge.

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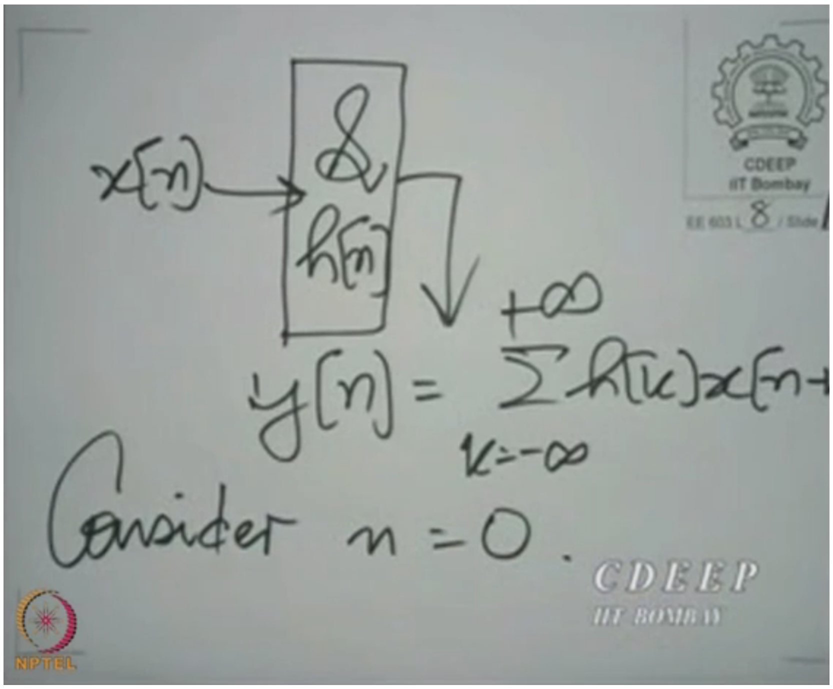

The proof of necessity hinges on the principle of a "troublesome input". And that troublesome input is the following. We know that for an LSI system with impulse response $h[n]$ and input $x[n]$, the output $y[n]$ is given by:

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n - k] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n - k]$$

In particular, we are going to consider $n = 0$.

Now we are going to give a particular troublesome but bounded input $x[n]$. Even in the story we discussed, after the gentleman was rudely awakened from his sleep, he asked who had thrown the water, and the minister very politely told him that it was raining and there was a little gap in the roof.

Now, what I am trying to say is that here we are going to drive the system to the brink of instability, but we are going to do it with a sane input, a stable, a bounded input. So, we'll keep to the principles of boundedness, but we'll still twist and turn our input to bring out the worst. Assume that you are able to such a troublesome bounded input to the system. If the system is stable, the output needs to be bounded. And if the output is bounded, every sample is finite in magnitude. And therefore, in particular, $y[0]$ must also be finite.

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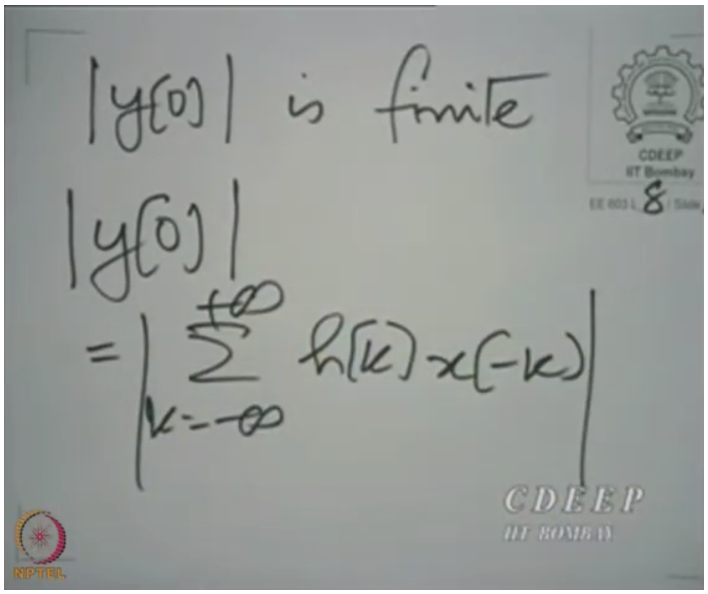

So, we shall strategically choose a bounded input $x[n]$. If the system is stable, the output must be bounded. As the output is bounded, $|y[0]|$must be finite in particular. We have:

$$|y[0]| = \left| \sum_{k=-\infty}^{+\infty} h[k]\, x[-\, k] \right|$$

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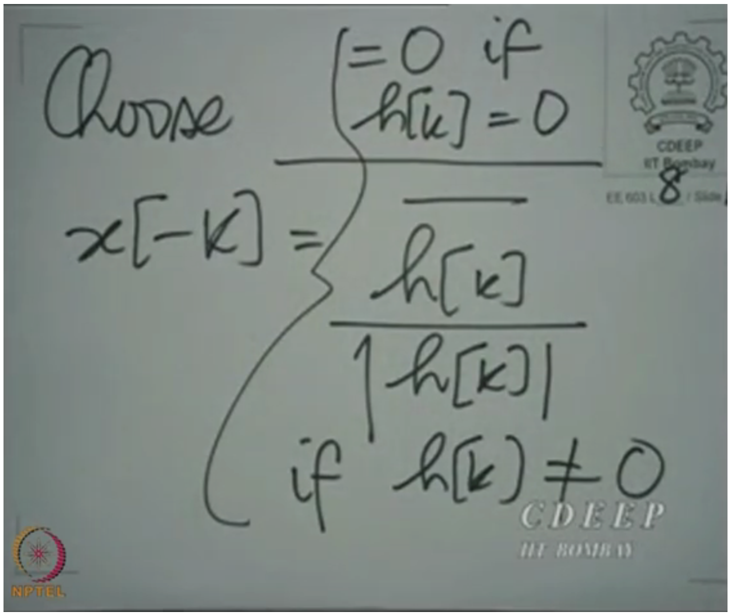

And now, let us very wickedly choose $x[-\ k]$ in the following way:

$$x[-\ k]\ =\frac{\overline{h[k]}}{|h[k]|},\ \text{if}\ h[k]\ \neq\ 0$$
$$=\ 0, \qquad \text{if}\ h[k]\ =\ 0$$

So the input is constructed as follows. At every point of the impulse response, if the impulse response sample there is non-zero, take its complex conjugate. Here we are allowing for a complex impulse response in general. So, take its complex conjugate and divide by its magnitude, and put that in the location $-\ k$ of the input. If that impulse response sample happens to be 0, then simply put a 0 in the same negative location on the input.

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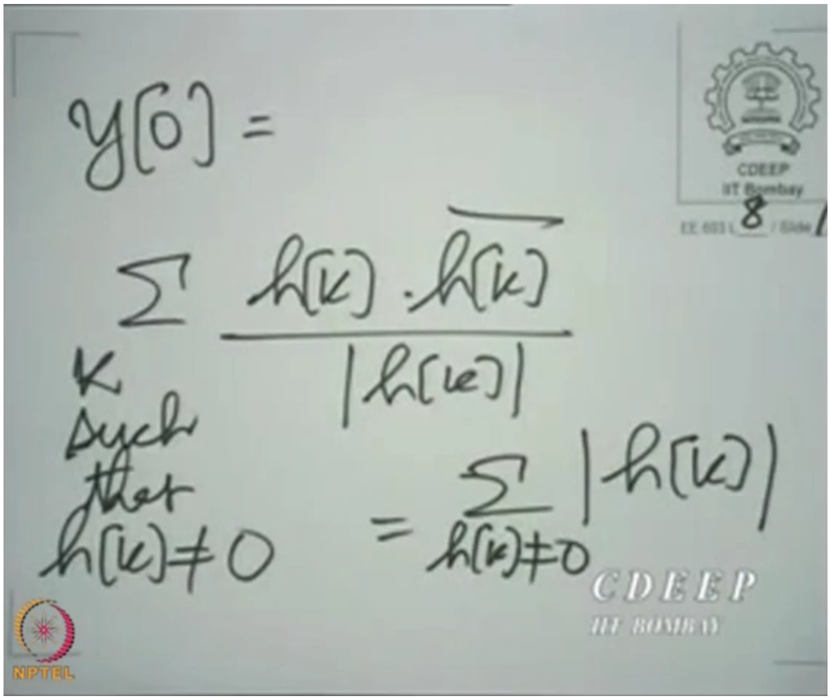

If you do this, the output $y[0]$ can be written as:

$$y[0] = \sum_{\substack{k \text{ such that} \\ h[k] \neq 0}} \frac{h[k] \cdot \overline{h[k]}}{|h[k]|} = \sum_{h[k] \neq 0} |h[k]|$$

You don't even have to worry about the non-negativity of $|h[k]|$. You know that the condition that $h[k] \neq 0$ for each term in the sum is not serious at all, because wherever $h[k] = 0$, it does not contribute to the summation.

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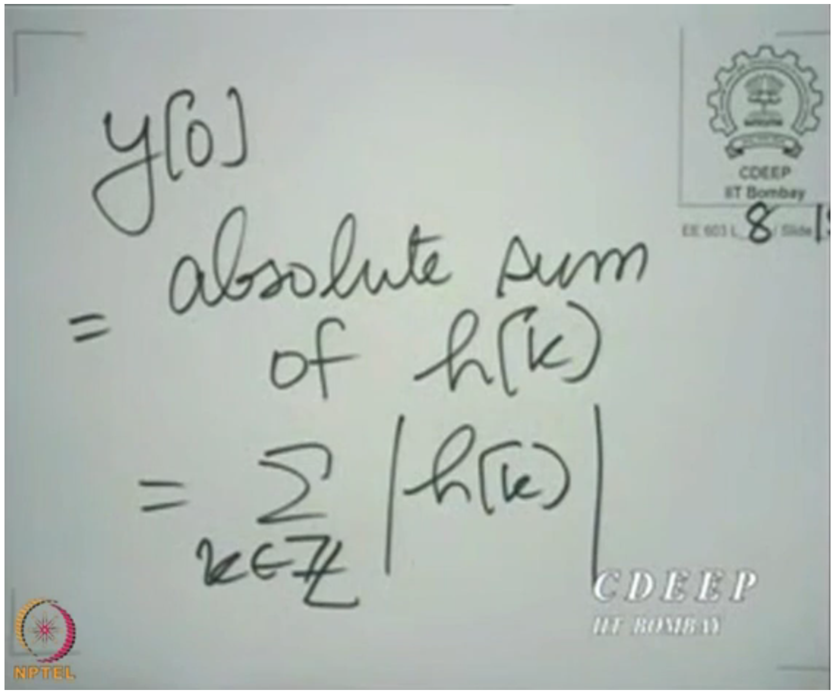

So, it's quite correct to write the following:

$$y[0] = \text{absolute sum of } h[k] = \sum_{k \in Z} |h[k]|$$

We want $y[0]$ to be finite. And therefore, its absolute sum must be finite. There we have the cold water thrown on the system. Although I certainly recommend such approaches to identifying properties of systems, I do not recommend that you take this as a lesson in dealing with people. People need to be dealt with differently from systems. One doesn't always have to drive them to the brink of insanity.

We note that in the future, unless otherwise stated, we shall use "stable" to denote BIBO stable. Let us now state the theorem. The necessary and sufficient condition for BIBO stability of an LSI system is that its impulse response must be absolutely summable. Here again, I must bring a word of caution. We have drawn this conclusion for an LSI system. For a system which is not LSI, this conclusion is incorrect. In that case, anything can happen. It is quite possible that the impulse response be absolutely summable but the system be unstable. And we do not have to go very far to draw a conclusion that the condition on the absolute sum of $h[n]$ is not adequate.

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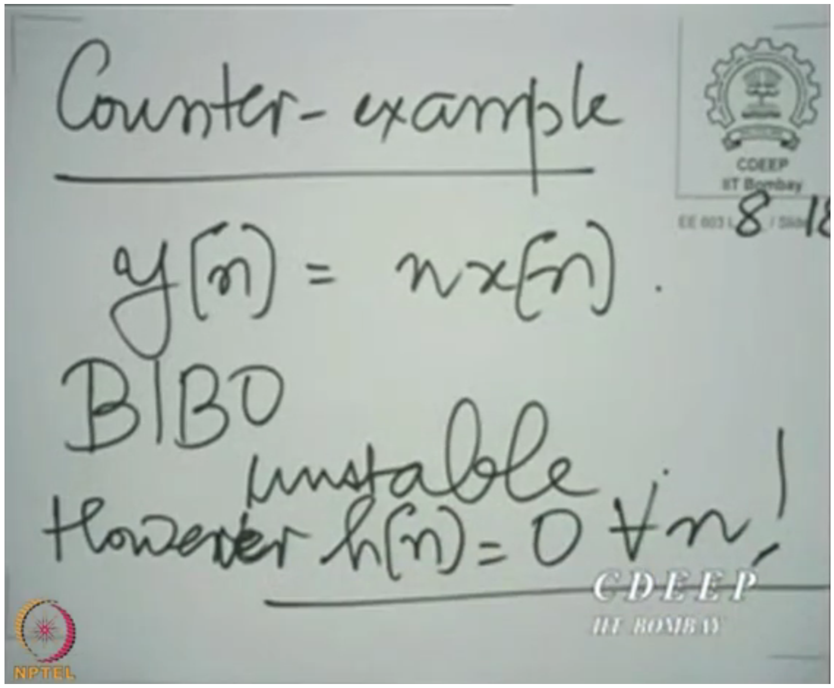

Take for example the system for which the output $y[n] = n\,x[n]$ for an input $x[n]$. Is this system BIBO stable? Please remember BIBO stability is independent of the other three properties. Now we have brought in a fourth property of systems. We had additivity, we had homogeneity, we had shift invariance and now we have BIBO stability. BIBO stability is not dependent on the other 3 properties, it is independent, it is a fourth independent property in its own right.

So we can ask for any system whether it is or is not BIBO stable. And we can ask for this system as well. Is this system BIBO stable? It is very easy to see it is not. In fact, all that you need to do is to give it an impulse at different places. If you give it a unit impulse sequence located at 0, the output is identically 0.

Give it a unit impulse located at location $n = 10^6$ and you get 10^6 at the point $n = 10^6$ and 0 everywhere else. And since a simple bounded input like a shifted unit impulse does not result in a bounded output, therefore the system is BIBO unstable.

However, its impulse response is identically zero, that is $h[n] = 0$ for all n. And of course, it is absolutely summable. If I give it a unit impulse sequence at the location $n = 0$ as the input, the output is 0 for all n. And it is obviously absolutely summable. So, absolute summability is not enough when the system is not LSI.