Digital Signal Processing and its Applications
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Lecture 08 C
Causality and Memory of an LSI System

We now come to one more such independent property that we need to discuss in the context of systems, and that is relevant when we are talking about the independent variable of time, but not so much when we are talking about the independent variable of space or any other kind of independent variable. When the independent variable is time, we have a notion of fixed directionality; you cannot move backward in time, you can only move forward. If you are dealing real time, then you need to move only forward. If you are dealing with offline systems (where you work with stored signal data), then you can use samples from the future too.

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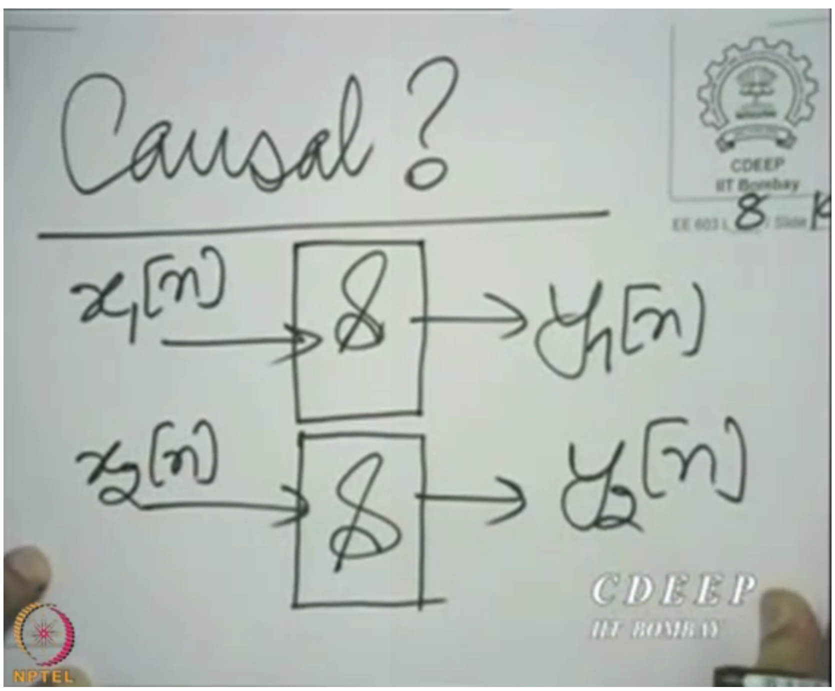

So, we ask whether a system is causal or not causal, which is another independent property. We say a system is causal if the following happens. Suppose we have a system S, which is not necessarily LSI. You perform two experiments on the system, with two different inputs. You give it an input $x_1[n]$ and you give an input $x_2[n]$ and record the two corresponding outputs. The only catch is that x_1 and x_2 are identical up to some $n = n_0$.

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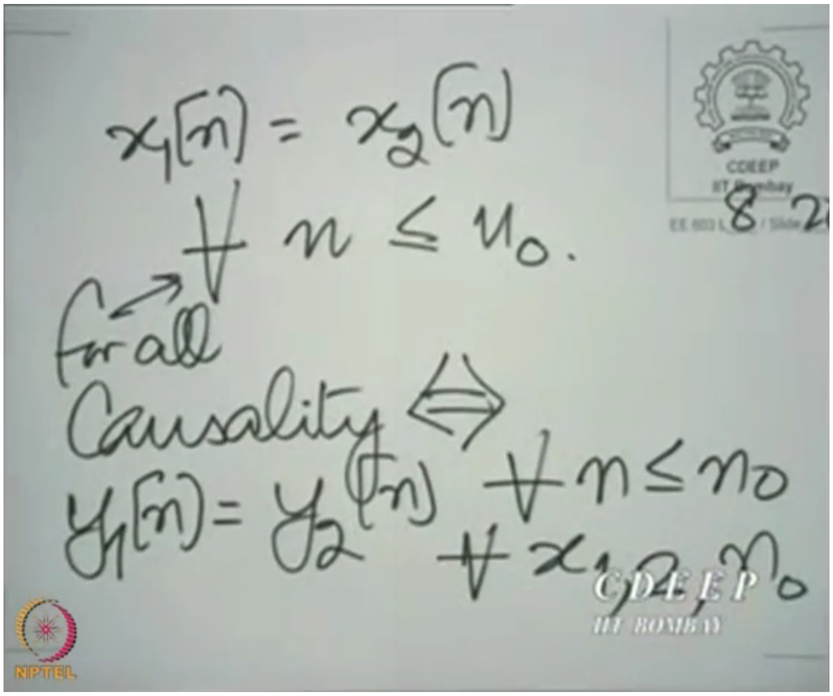

We denote "for all" with the symbol ∀. We have $x_1[n] = x_2[n]\ \forall\, n \le n_0$, for some integer n_0. Causality implies and is implied by:

$$y_1[n] = y_2[n]\ \forall\, n \le n_0 \text{ and for all such } x_1, x_2 \text{ and } n_0$$

That means, take any such pair of inputs which are identical upto some $n = n_0$, apply them to the system in two different experiments, and study the output. The output is identical up to that point n_0 if the system is causal and vice-versa. The system is causal only if these outputs are identical for all $n \le n_0$ and this happens for any such choice of inputs x_1, x_2 and any point n_0. What this means is, if I have two inputs which are identical in all respects up to a point in time, a causal system does not show any difference in its output up to that point in time.
Needless to say, there could be differences afterwards if there are differences in the input.

Another way of understanding this is the system never looks into the future. The system has no idea whether the inputs could be different in the future. And therefore, in the benefit of doubt, it remains identical in its output up to the point where the inputs are identical. Now of course, the word causal suggests that; 'causal' refers to cause and effect.

So an identical cause produces an identical effect. You can talk about cause and effect only if there is an ordering, a one-after-the-other relationship in time. Then you can talk about cause and effect, otherwise cause and effect is not very well understood.

If an effect comes before the cause, then it is not an effect at all. That is why we say a system is causal if it follows the principle of cause and effect. Again, causality is independent of linearity, shift invariance or stability. And that is therefore a fifth possible property that a system could have or not have. We can see that in a causal system we want dependence only on the past. That is what we are trying to say in effect.

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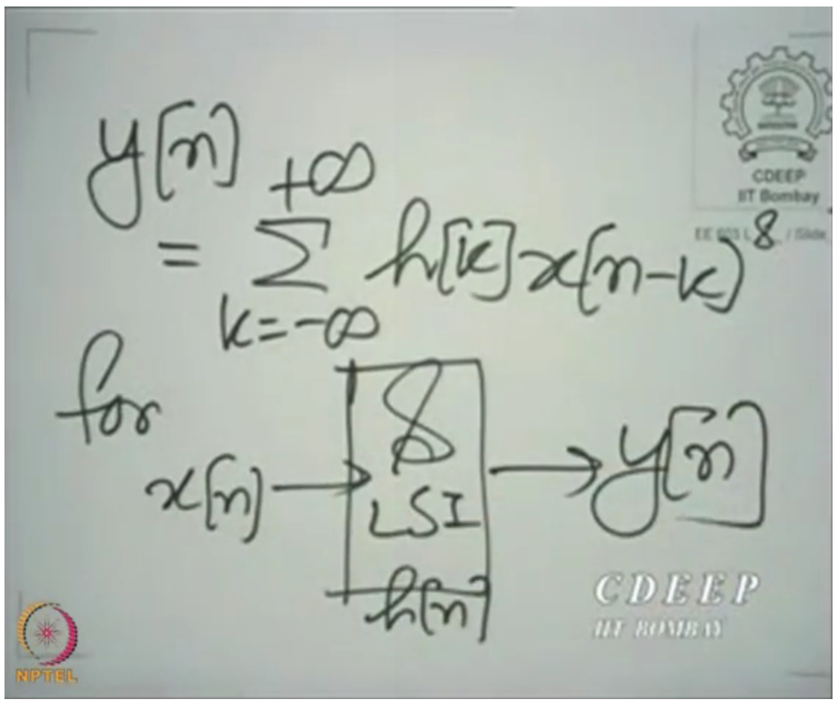

Let us look at the convolution expression for an LSI system S with impulse response $h[n]$:

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k]$$

If we wish that $y[n]$, at any point n, have nothing to do with future samples, it means that the sum should have nothing to do with negative ks. You see, when would $y[n]$ involve future samples? When $-k$ is positive, or equivalently k is negative. And that means for all negative k, $h[k]$ needs to be 0.

It is very easy to see that if $h[k]$ is equal to 0 for all negative k, then $y[n]$ depends only on $x[n]$ and $x[n-k]$ for positive k, which means all past samples, so to speak. I have given you an informal argument, but I leave it to you as an exercise to prove this more formally.

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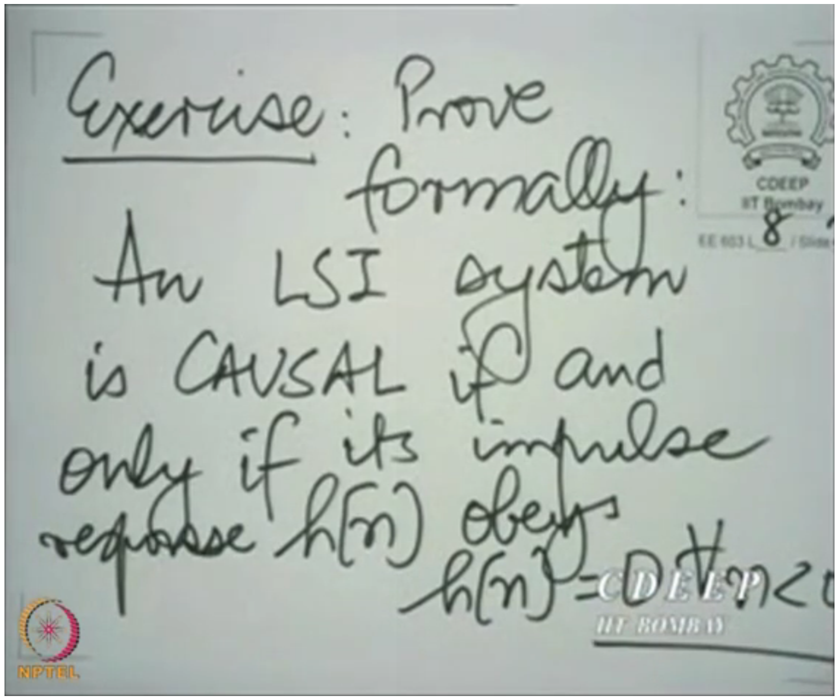

That means you should formally prove that an LSI system is causal if and only if its impulse response obeys $h[n] = 0 \; \forall \; n < 0$. By “proving it formally”, I mean that you must show it is necessary and sufficient. That means, you must first assume this condition holds and show that it's sufficient for two identical inputs to produce an identical output up to that point in time and then, also prove its converse, namely if I have any set of inputs that are identical up to a point in time and if I want the outputs to be identical up to that point, that cannot happen unless all the impulse response samples at negative indices are 0. I leave it to you as an exercise to prove this formally.

Anyway, we have now identified 5 independent properties of systems: additivity, homogeneity or scaling, shift invariance, stability and causality. Needless to say, for a system that is linear and shift invariant, looking at the impulse response should tell us everything and therefore it tells us whether the system is stable and whether it is causal. And we have also identified how we can do so.

Look at the impulse response, look at all the negative located samples, if they are all 0, the system is causal. Look at the impulse response, look at its absolute sum, if the absolute sum converges, then the system is stable. And finally, we take one more property of systems, namely, the property of memory, which is not quite independent, that is why I did not say sixth property.

Now we say a system has no memory, if and only if, $y[n]$ has to do only with $x[n]$ and no other $x[n - k]$ for $k \neq 0$. So, it is a point-by-point relationship. That is called a system without memory.
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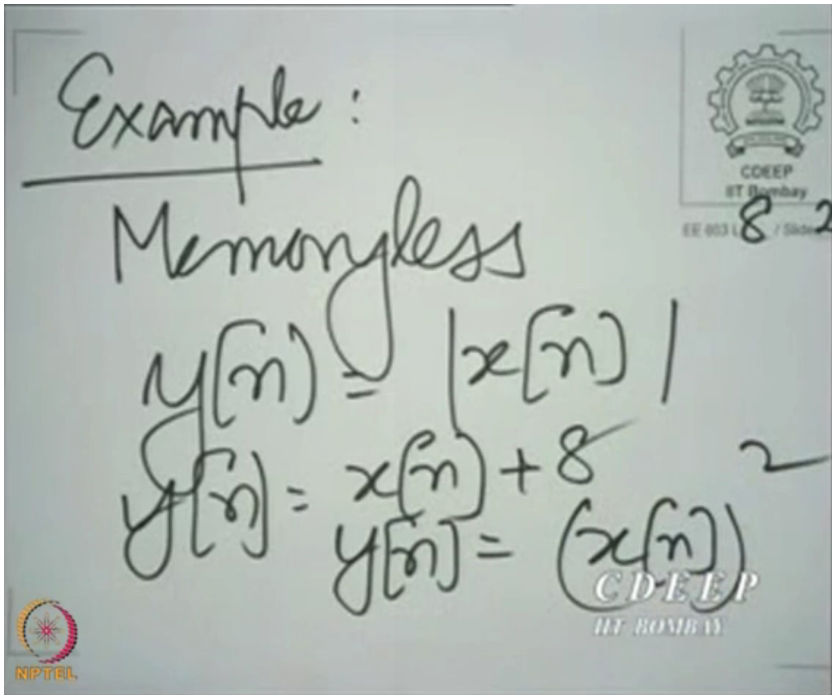

An example is, $y[n] = |x[n]|$ or $y[n] = x[n]$, of that matter even $y[n] = x[n] + 8$, if you like. Another example is $y[n] = (x[n])^2$. These are all systems without a memory. And of course, it is very easy to give examples of systems with memory, you just have to involve some other term like $x[n - 1]$ or $x[n + 1]$ and you get a system with memory.

As I said, memory is not entirely independent of the other properties. In fact, if a system is memoryless, it is automatically causal. So they are not entirely independent. Systems without memory, or memoryless systems, are one class of causal systems.