Digital Signal Processing and its Applications
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Lecture No. 10 A
Introduction to DTFT, Inverse DTFT and Convergence of DTFT

A warm welcome to the 10th lecture on the subject of Digital Signal Processing and its Applications. We continue today with our discussion of the discrete time Fourier transform. We have just introduced the idea and introduced the term in the previous lecture, but we have promised that we would look at it in more detail in the lecture today and we do so. Let us recapitulate a few ideas that we have begun with in the previous lecture.

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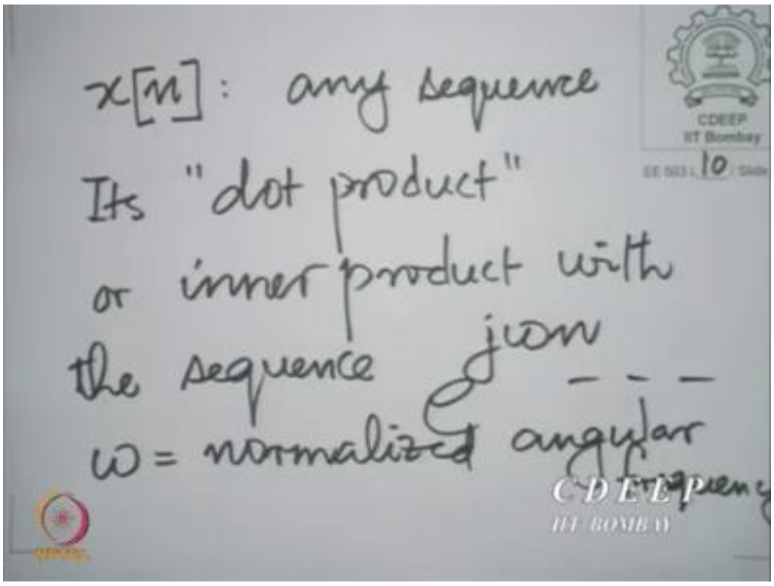

We have said that if we take an arbitrary sequence x[n] not necessarily the impulse response of a Linear Shift-Invariant system any sequence then its dot product or inner product with the sequence $e^{j\omega n}$where ωas you know is the normalized angular frequency. Its dot product or inner product with the sequence $e^{j\omega n}$was given a name of course if this converged.

So, when you take a dot product here unlike in the case of the finite dimensional space where a dot product of two vectors is bound to have a convergent value or finite value. Unlike the case of finite dimensional spaces in infinite dimensional spaces we do not have this guarantee. So we cannot rest assured that this dot product will converge, because there is an infinite summation involved, but whenever it converges we call it the Discrete Time Fourier Transform of the sequence.

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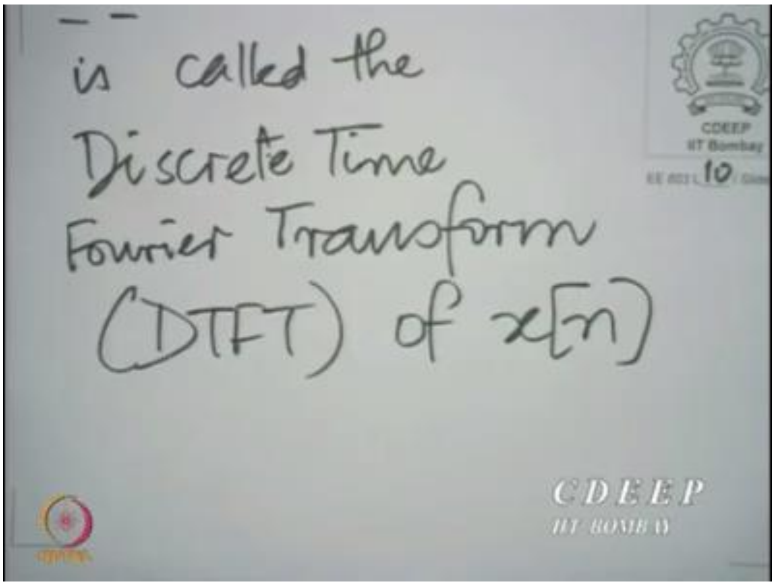

So, it’s dot product is called the Discrete Time Fourier Transform abbreviated by DTFT of x[n] and in fact we have given a name to the Discrete Time Fourier Transform of the impulse response. We have called it the frequency response of the system.

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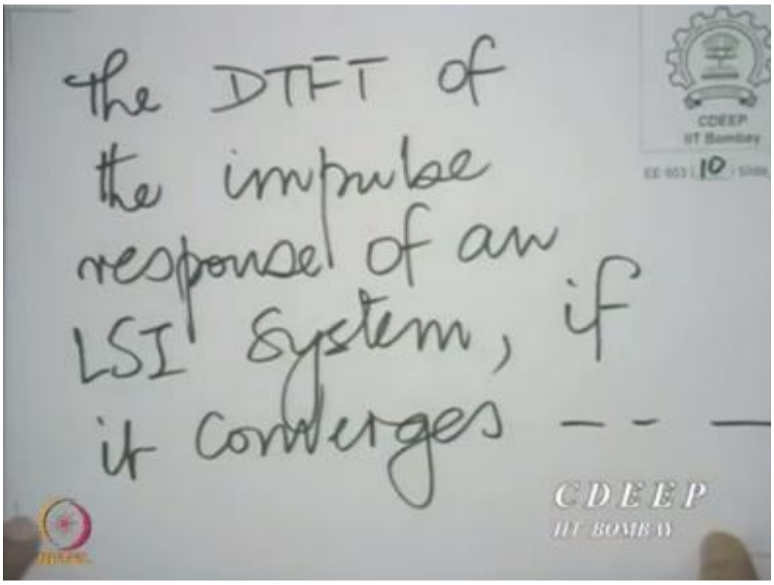

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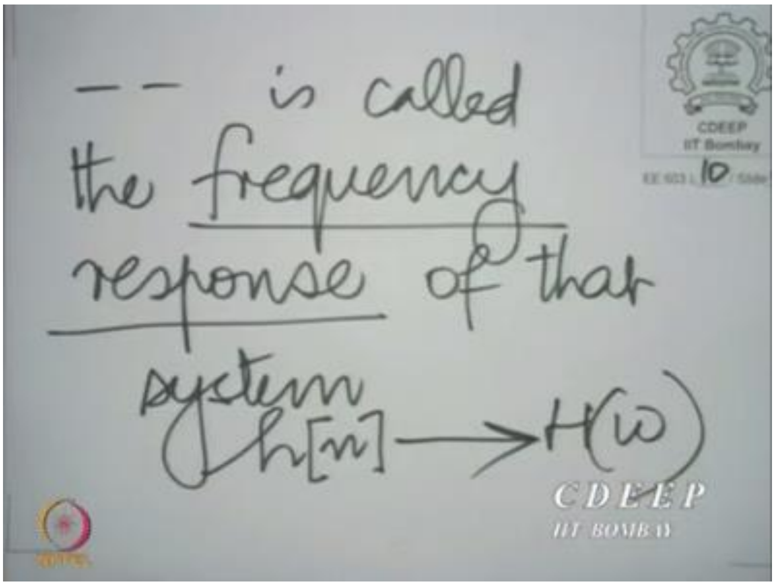

The DTFT of the impulse response of an LSI system if it converges,the DTFT of the impulse response of an LSI system, if it converges is called the frequency response of that system and in fact we have used h[n] normally to denote the impulse response and H(ω) to denote the frequency response. Now, we employ a combination of these ideas. So, we understand that X(ω) is like the projection of the sequence x[n] on the sequence $e^{j\omega n}$and therefore we would agree that you could possibly represent or reconstruct x[n] from its projections as you can do in the case of vectors.

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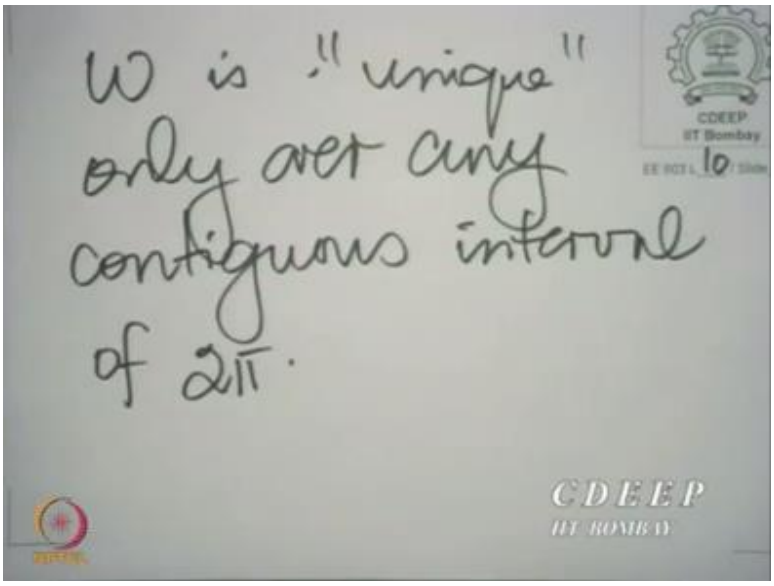

ω is "unique" so to speak only over any interval of 2π, you know why this is the case you see 2π denotes the sampling frequency on the normalized scale. So the maximum frequency component that could have been present in the original signal is not more than half the sampling frequency if you have taken care to avoid aliasing and if you have not taken care anyway it is indistinguishable now.

So, you need to do orderly with the frequencies from $-\pi$ to π if you talk about the original phasors or in fact if you look at the Discrete Time Fourier Transform it is very easy to see or it is going to be periodic with the period 2π. Let us take a minute to prove that, that is another way of saying this.

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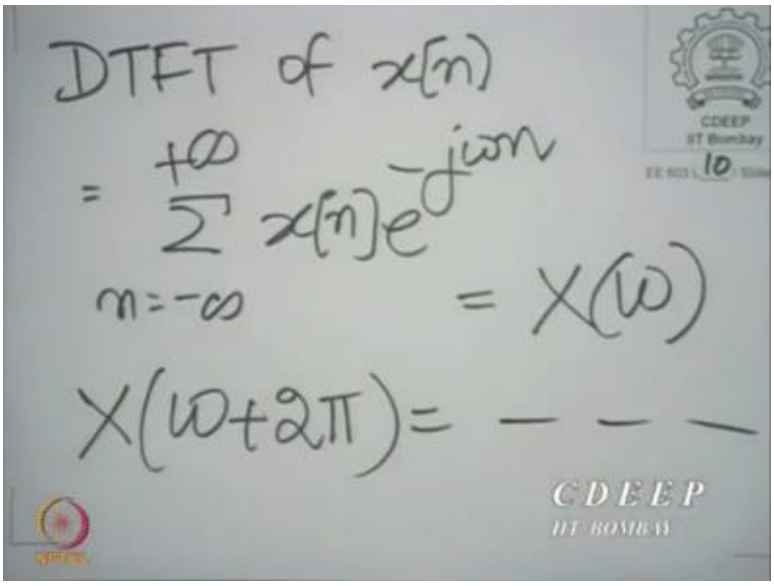

You see the DTFT of x[n] is usually defined as

$$\sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} = X(\omega)$$

and we use capital X (ω) to denote this. Let us consider X($\omega + 2\pi$).

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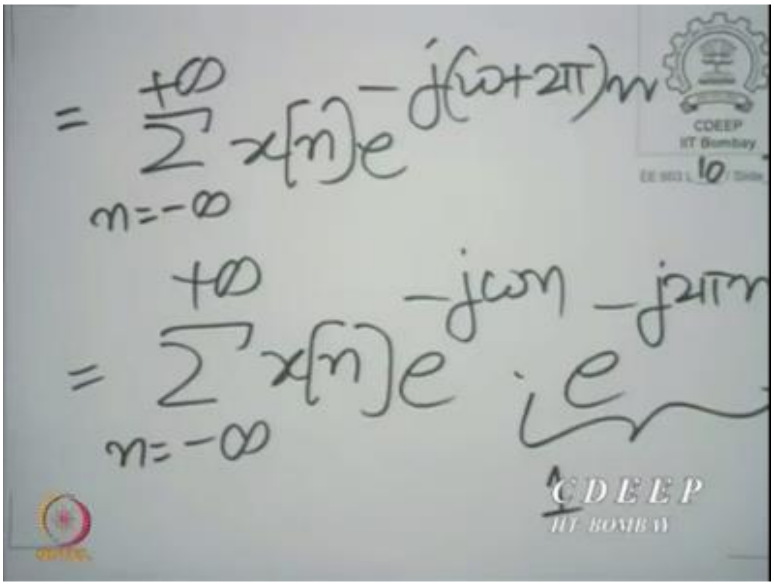

Clearly X($\omega + 2\pi$)is obviously going to be

$$\sum_{n=-\infty}^{+\infty} x[n]e^{-j(\omega+2\pi)n} = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}\, e^{-j2\pi n}$$

$e^{-j2\pi n}$ is identically 1 and therefore this is the same as X(ω).

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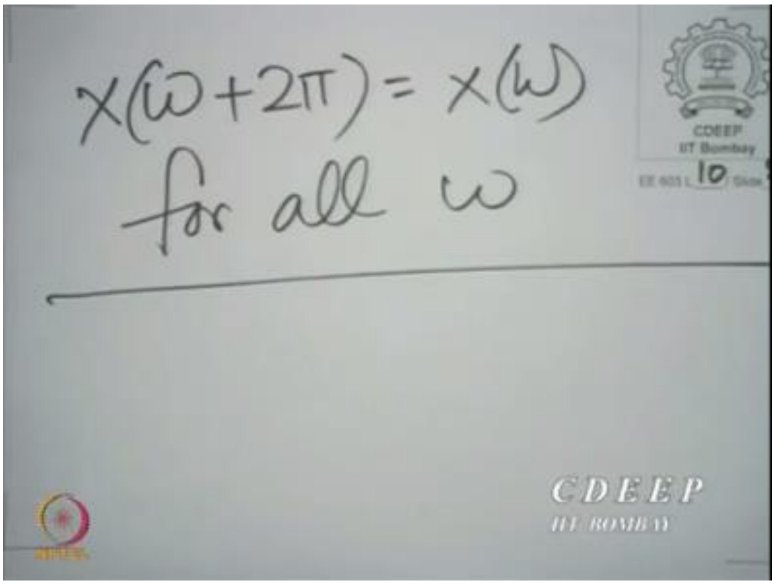

So, X($\omega + 2\pi$) is identically equal to X(ω) for all $\omega$and of course in particular for ωbetween minus $-\pi$ and $+\pi$. Now this is of course the mathematical way of demonstrating this, but the physical interpretation is that uniqueness is only over the region from 0 frequency to half the sampling frequency and for every rotating phasor with frequency ω you have a counter rotating phasor with frequency -ω they come together to form a sine wave.

So, you have uniqueness only over the region -π to πbeyond that there is periodicity. So, there is uniqueness only over any contiguous interval of 2π That is what we are saying. Contiguous means an unbroken interval of 2π. Now, of course this is true for any Discrete Time Fourier Transform and therefore we expect that we should be able to reconstruct x[n] from its components which are X(ω) over this unique interval. In particular you could take the unique interval from -π to π.

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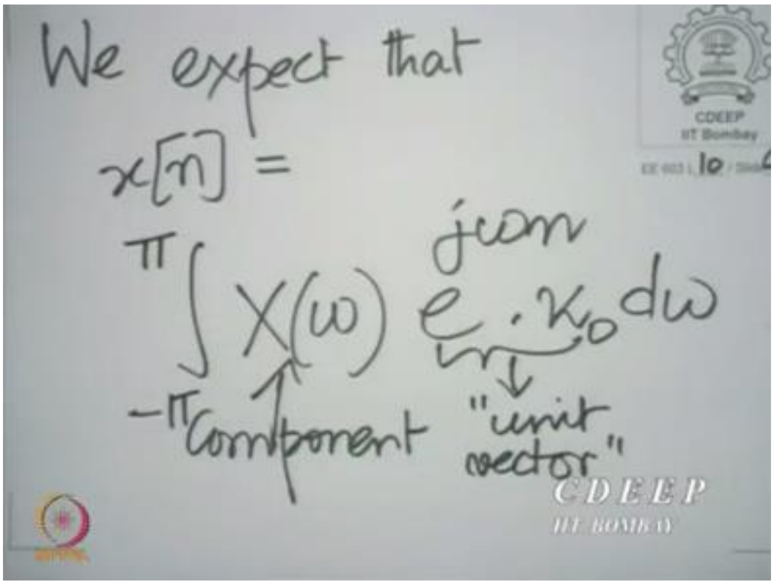

So, what are we saying in mathematical language? We are saying that we expect that x[n] should get reconstructed by taking these components. How do you reconstruct the vector from its components? You multiply the components by unit vectors in the direction of each of those components and add up these. So, if you have a three dimensional vector and if its components in the x, y and z directions are 1, 2 and 4. Then how do we construct the three dimensional vector? 1 time the unit vector in the x direction plus 2 times a unit vector in the y direction plus 4 times the unit vector in the z direction. So you multiply each component by unit vector in the direction of that component and add overall such components. Now, here there is a slight difference here the componentsωrun from -π to π and these components are not discrete.

They are continuous; there is a continuum of components. Now, you have a continuum of components you cannot add what should you do? You should integrate and of course we ask for a unit vector so I do not know whether $e^{j\omega n}$ is a unit vector or not. So I have to make a provision that if it is not a unit vector I should allow for a constant to divide or multiply that vector essentially constant multiplying that vector to make it a unit vector.

Hopefully the constant can be independent of ω. So, what I am saying is I am multiplying the components by the so called unit vector and the unit vector is $e^{j\omega n}$ amd some constant κ_0 let us call it and integrate over $\omega$for ω going from -π to π. I expect that this should be true. So, I already have a physical interpretation now I need to prove this mathematically.

The physical interpretation is I expect that I should be able to reconstruct the sequence here infinite dimensional vector by taking the product of components multiplied by the so called unit vectors and integrate it over all such components over the region of uniqueness of ω, but now we need to prove this mathematically. So, let us look at this expression and let us in fact forget about the constant.

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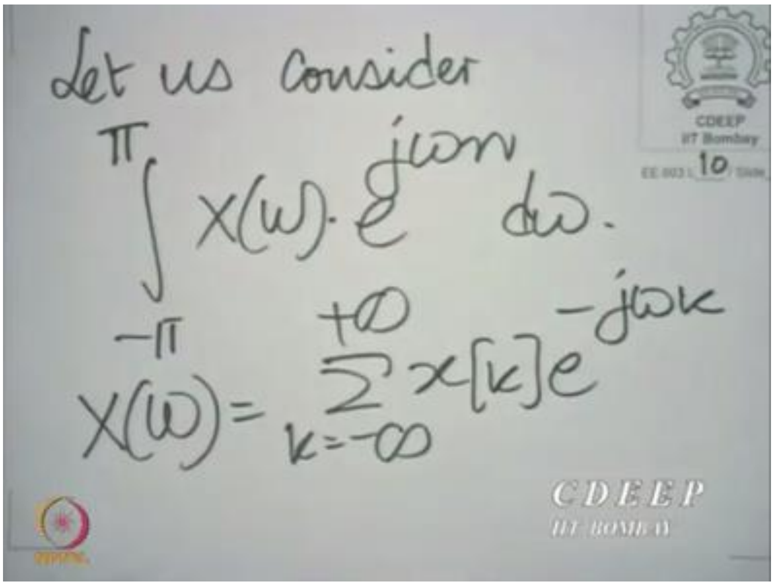

Let us consider summation or rather let us just consider $\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$. We will worry about κ_0 afterwards. Now $X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$. I am intentionally using a different variable of summation here to distinguish it from the index n and I substitute that.

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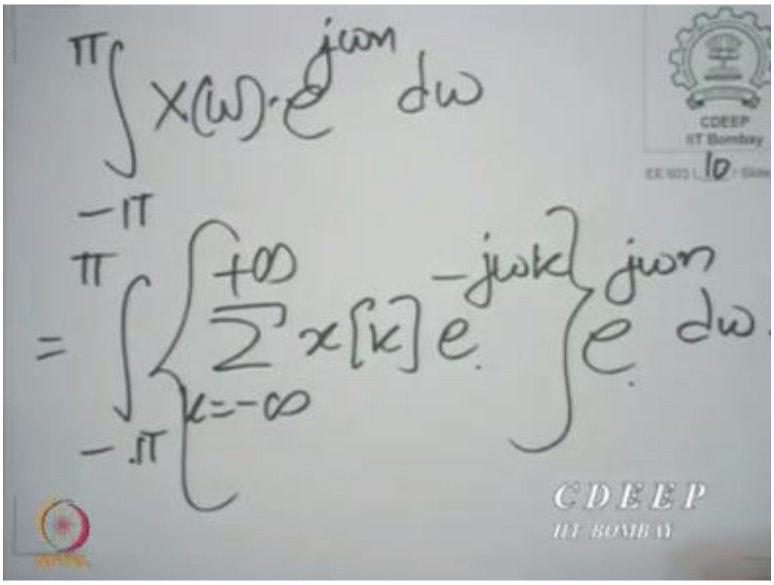

So, I have integral $\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{+\infty} x[k]e^{-j\omega k}\, e^{j\omega n}d\omega$. Now here I have a finite integral and of course I am assured that this infinite summation has converged.

So, I can bring the finite integral in and make it act only on ω. So, you notice it is only this and this that depend on ω and I can make the integral act on them and bring the summation outwards. I must remark here that it is of course an important technical point when you can make such interchanges of integrals particularly when the integrals or summation are infinite in length, but we shall not dwell on those technicalities here. Let us take it that in this context it is acceptable and one of the justifications is that we have assured ourselves of the convergence of X(ω) which is a good factor for it.

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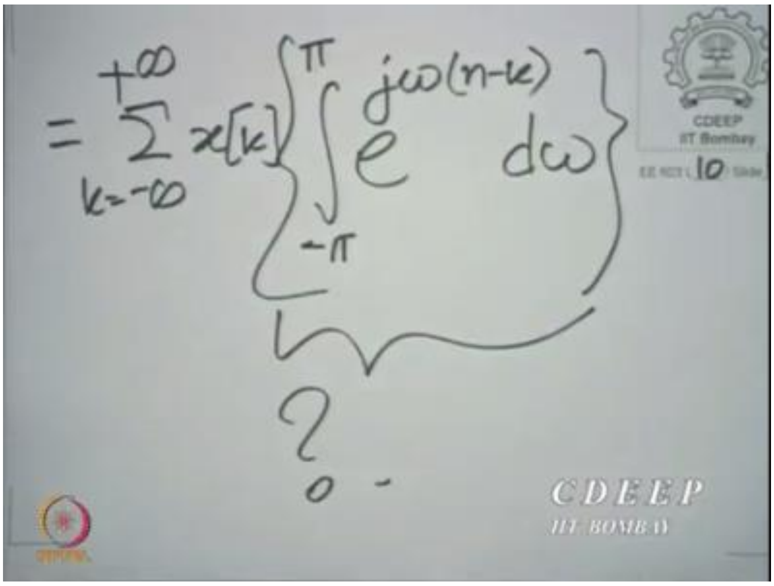

So, this is equal to $\sum_{n=-\infty}^{+\infty} x[k] \int_{-\pi}^{\pi} e^{j\omega(n-k)} d\omega$ and it is this that we need to study. What is this? That is very easy to evaluate. In fact we shall evaluate it mathematically, but let us evaluate it with some intuition here. What are we asking here? We are asking for the integral of a rotating number.

Note here see now for a moment you have to reverse the role of $\omega$and n. N - k now you see you are integrating with respect to ω. So, it is ωwhich is changing here not n and k. When ωgoes from -π to πsuppose n - k is equal to 1 now we are going through one complete cycle. If n - k is equal to 2 we are going through two complete cycles. Now, of course n-k can be 2 or - 2.

It depends on whether you are going clockwise or anticlockwise, but in any case we are always completing n - k cycles. When n - k is not equal to 0, but when n - k is equal to 0 what are you doing? You have $e^{j\omega 0}$ which is 1 so you are essentially integrating 1 from -π to πwhich simply becomes 2π.

So n - k is not 0 the integral must go down to 0 because you are starting from a point and coming back to the same point. In fact each time you are going through for each particular value in the cycle we are going for the negative value as well. So we expect that this integral is going to vanish when n - k is not equal to 0, but when n - k is equal to 0, but integral is going to be equal to 2π now it is very easy to show this mathematically.

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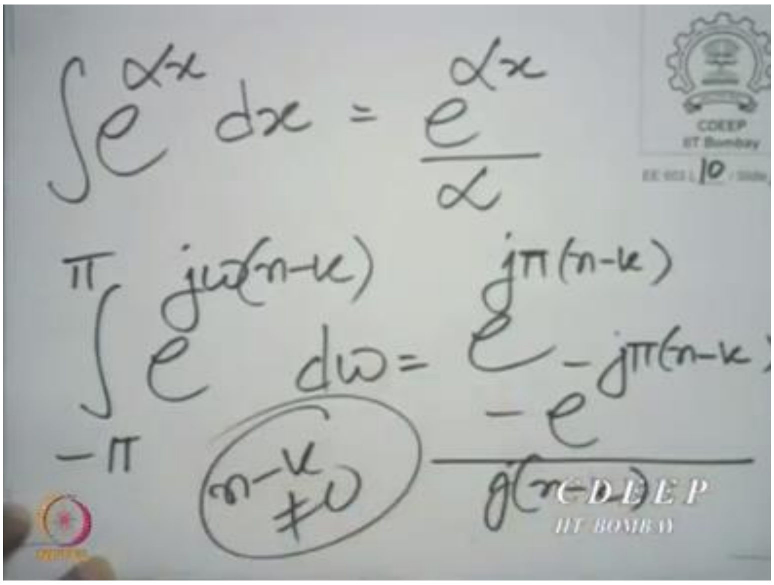

In fact it is very easy to see that $\int_{-\pi}^{\pi} e^{j\omega(n-k)}d\omega = \frac{e^{j\pi(n-k)}-e^{-j\pi(n-k)}}{j(n-k)}$ when n - k is not equal to 0.

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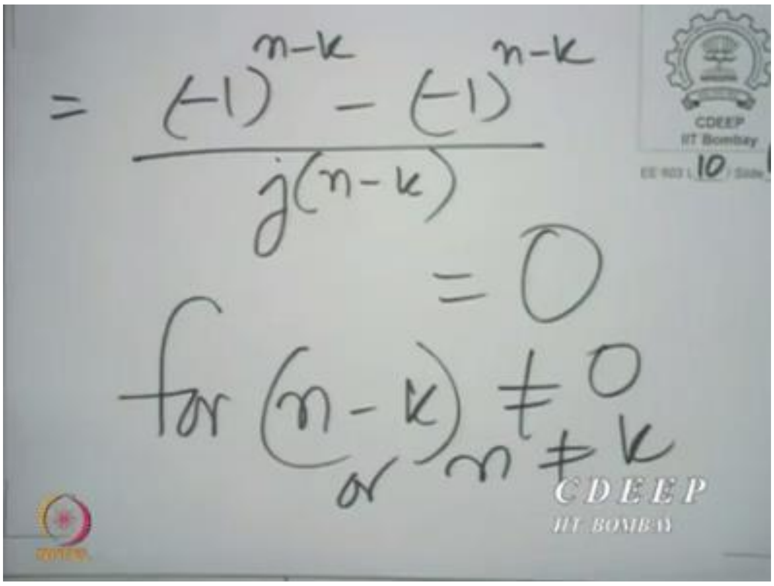

So for n - k ≠ 0 or n ≠ k that integral vanishes.

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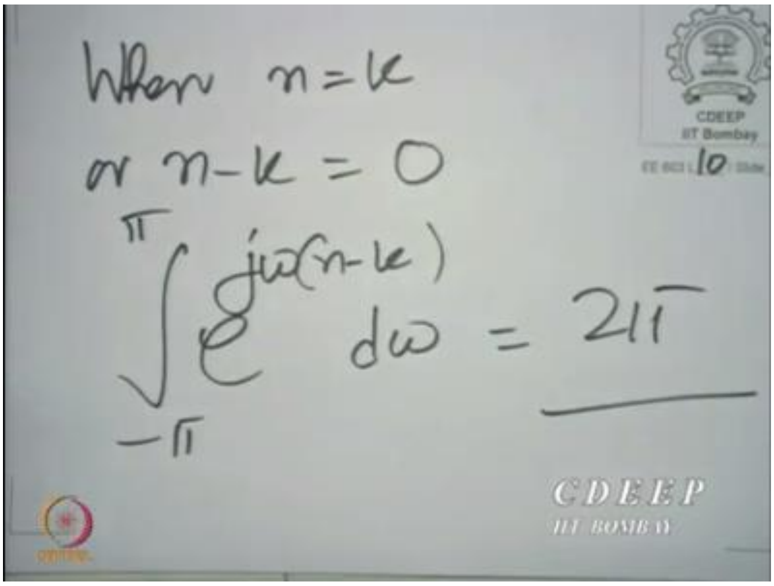

On the contrary when n = k or n - k = 0, $\int_{-\pi}^{\pi} e^{j\omega(n-k)} d\omega = 2\pi$ and therefore all that we need to do is to go back to that expression that we have that summation.

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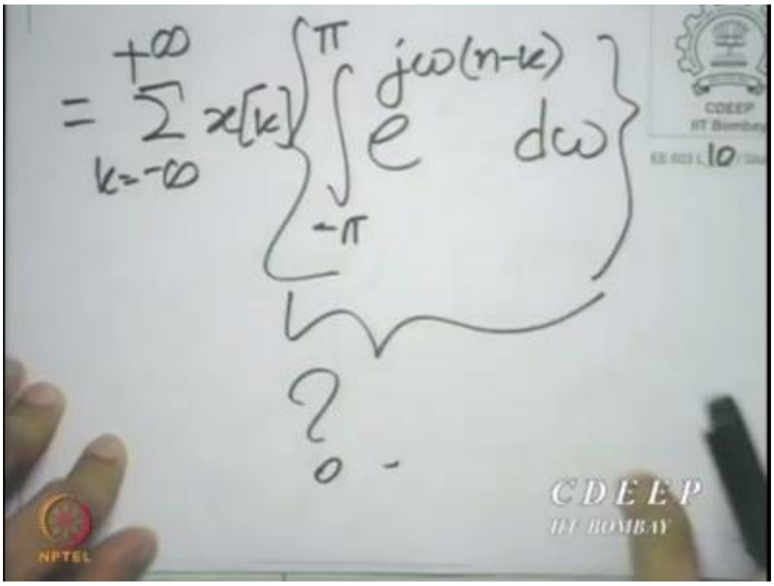

You see what we have said now is that in this summation for any given n all these integrals for k not equal to n vanish and they leave only the integral when k is equal to n and for k equal to n you get 2π here and therefore we are saying that in fact we have answered two questions at once. We have suddenly answered what is κ_0 as well.

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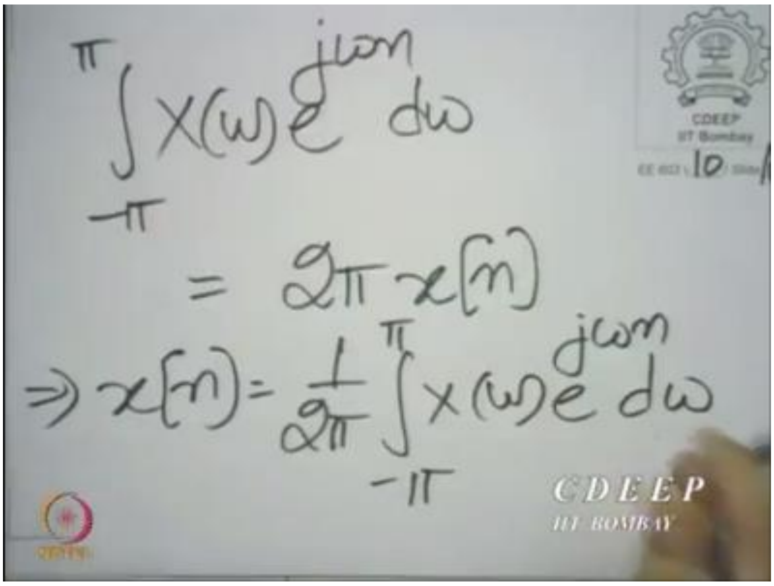

So, we have said $\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega = 2\pi x[n]$ and now it is very obvious that $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$. Now obviously we have this mathematically we already made an interpretation in the beginning of the class in terms of vectors it is a very interesting correlation that we see.

And in fact as I said we have already answered the question what is κ_0, must be $\frac{1}{2\pi}$ we did not need to work very hard to arrive at κ_0. Now this is called the Inverse Discrete Time Fourier Transform.

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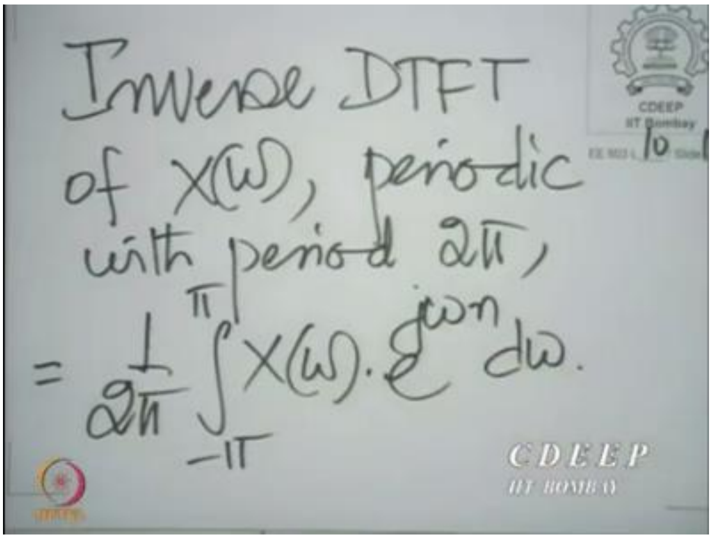

So, we say the Inverse Discrete Time Fourier Transform of $X(\omega)$ which is periodic with period $2\pi$is $\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$ of course we assume again that this should converge, but the good thing is that if $|X(\omega)|$ finite and if $X(\omega)$ is continuous then we expect that convergence will happen there should not be too much of trouble.

It is only in pathological cases that we would have trouble. So if you have given a Discrete Time Fourier Transform then going back to the sequence is possibly not too difficult I mean the ability to go back does not seem to be too much of a problem, but the other way is a problem. A sequence may not have a Discrete Time Fourier Transform. In fact let me immediately give you an example of a sequence which does not have a Discrete Time Fourier Transform.

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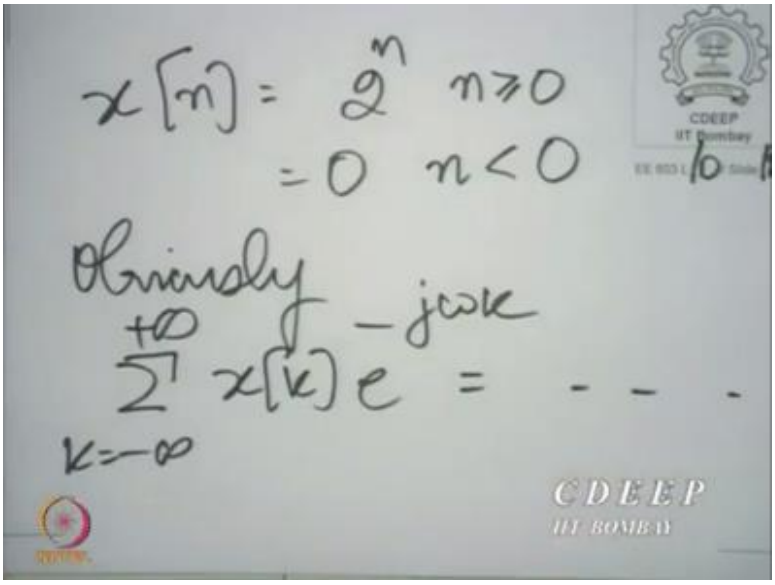

Let us take the sequence x[n] = 2^n for n ≥ 0 and 0 for n < 0. Obviously this summation, $\sum_{n=-\infty}^{+\infty} x[k]e^{-j\omega k} = \sum_{k=0}^{\infty} (2e^{-j\omega k})$. Now this is a geometric progression with common ratio $2e^{-j\omega}$ and obviously this common ratio has a magnitude > 1.

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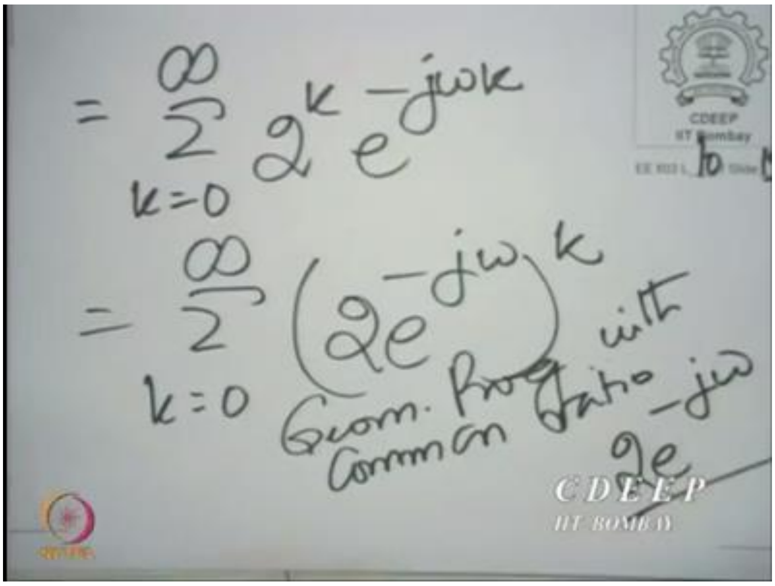

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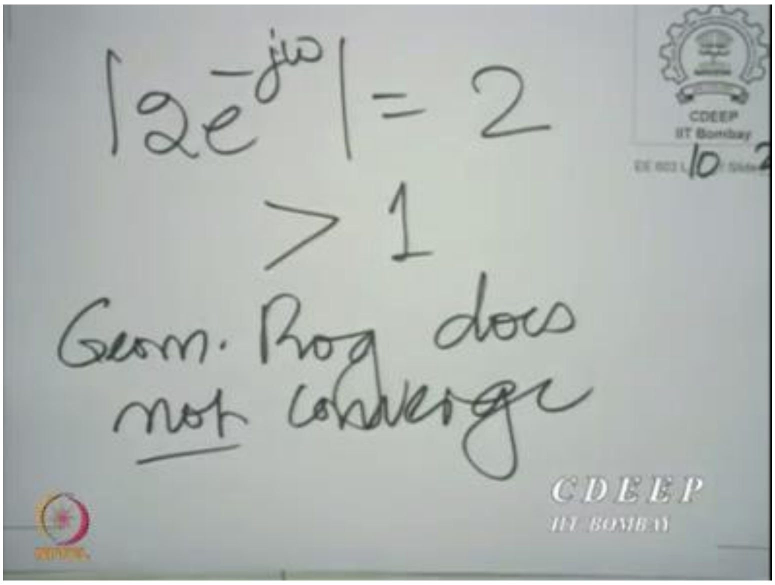

So the geometric progression does not converge and therefore this x[n] does not have a Discrete Time Fourier Transform. So you do not have to go very far to see examples of frequency that do not have a Discrete Time Fourier Transform. So of course we need to confine ourselves if we want to deal with the Discrete Time Fourier Transform we need to confine ourselves to sequences that do for the moment at least.

Now, this also illustrates an example of an impulse response which would not correspond to a frequency response. Of course, it is obvious because this particular impulse response also would correspond to an unstable system. Impulse response is not absolutely summable and this LSI system if it had this impulse response would be unstable.