Digital Signal Processing & Its Applications
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Lecture 10 b
DLFT in LSI system and convolution theorem

Now, let us assume that the impulse response and the input both do have a discrete time Fourier transform.

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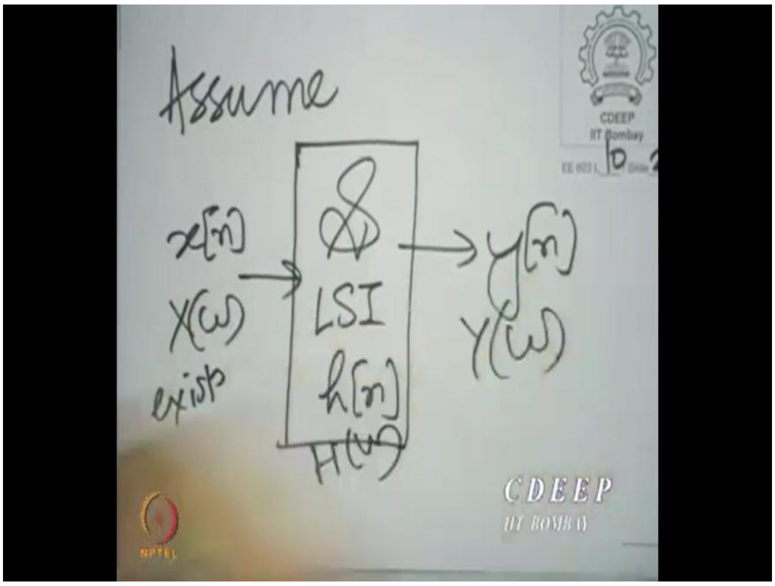

So, let us assume this situation you have an LSI system impulse response hn input x[n] output y[n], and we assume that $X(\omega)$ exists, $H(\omega)$ exists, and $Y(\omega)$ also exists, all of them exist. We ask what is the relation between $X(\omega)$,$Y(\omega)$and $H(\omega)$. Now, we would again first like to obtain this relation without the mathematics, and then we shall do the mathematics. Without mathematics, what is the interpretation of $X(\omega)$? $X(\omega)$ is nothing but the component of the sequence along $e^{j\omega n}$. So, focus your attention on that particular component of the input along the angular frequency ω. When you apply x[n] to this LSI system.

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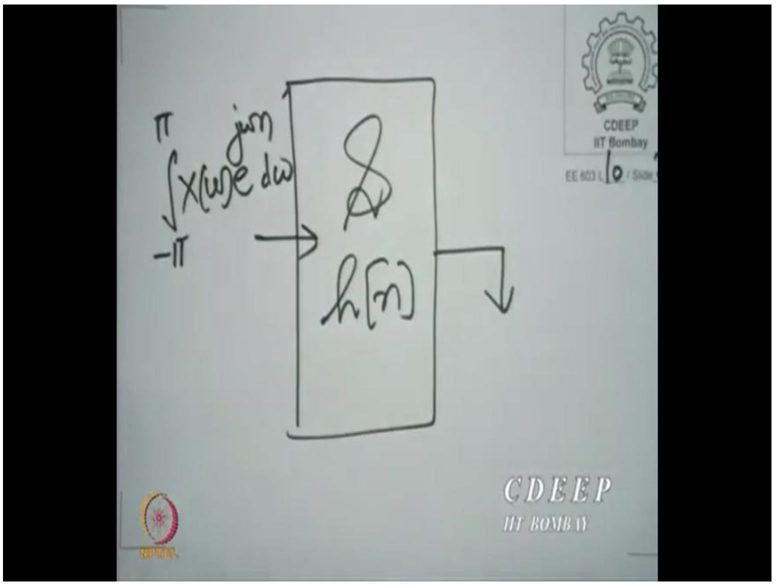

You agree that when we apply x[n], we are actually applying $\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$ to the system. Now, invoke the property of homogeneity and additivity or linearity if you like. You can think of $X(\omega)$ as a constant. And the integral is required or rather additivity is required to act on the integral.

So, you are saying take a combination of many different $e^{j\omega n}$ here for different values of ω, and that combination is taken by the integration. Now, X(ω) requires homogeneity. It is a constant, so multiply each of these vectors by a constant X(ω) and integrate or add over all such components, take the addition to its limit and make it as integration.

You know the property of LSI systems, what will be output of $e^{j\omega n}$? It is going to be $H(\omega)e^{j\omega n}$. Because of homogeneity when you multiply it by X(ω) the output is going to be $X(\omega)H(\omega)e^{j\omega n}$ because of additivity when you integrate this over all $\omega$the output will also get integrated over all ω.

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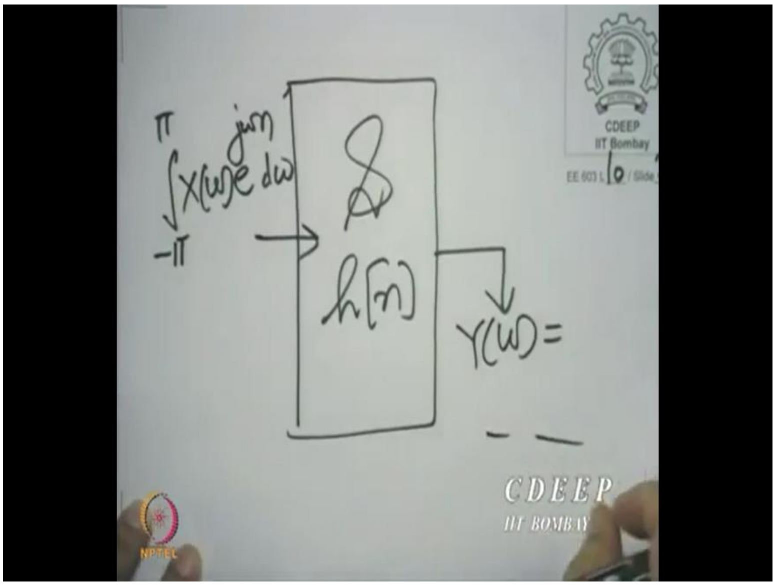

So, it is very clear that Y(ω) is going to be integral in fact let us write on the next page.

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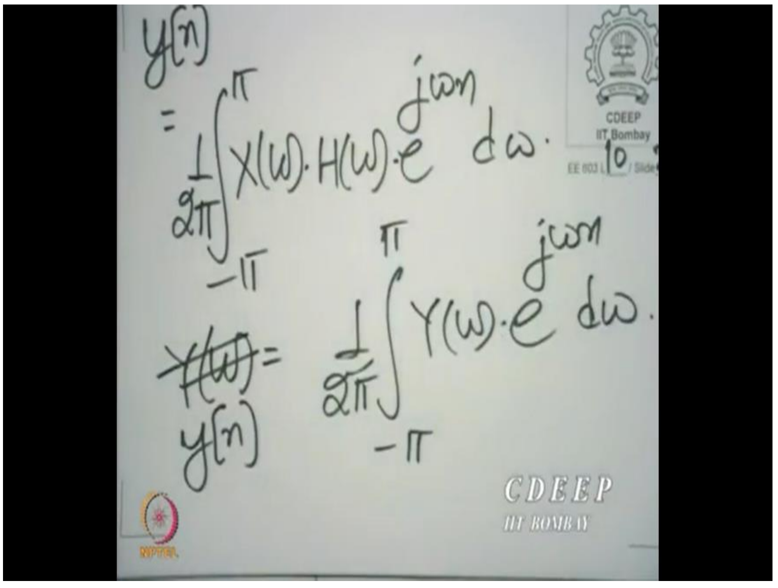

Of course, you know I have forgotten a factor of 2π. I did not pay too much attention to that. So, let me put that factor of 2π. Back again here $Y(\omega)$ $Y(\omega)$ must be equal to $\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)H(\omega)e^{j\omega n}d\omega$. y[n] is also equal to $\frac{1}{2\pi}\int_{-\pi}^{\pi} Y(\omega)e^{j\omega n}d\omega$. And y[n] is also equal to this. So, y[n] must have this expression from our reasoning of Eigen sequences, and y[n] must have this expression from the inverse discrete time Fourier transform. And therefore, it is very clear that.

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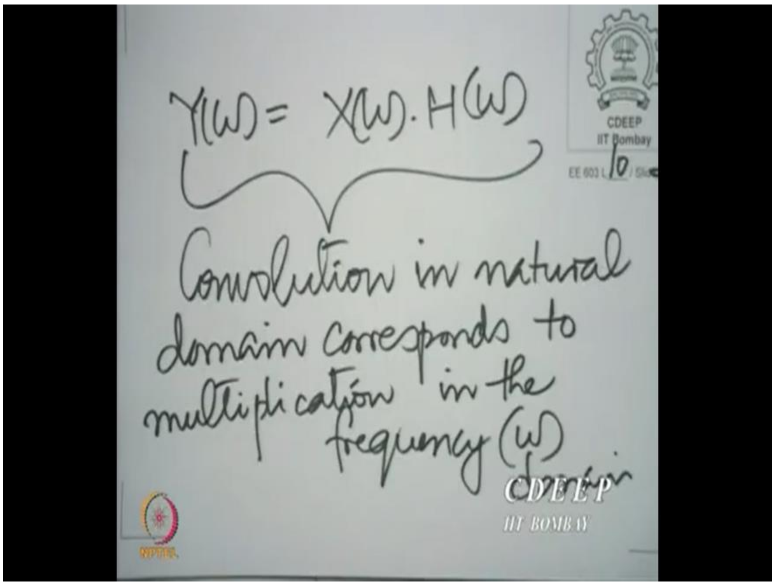

$Y(\omega) = X(\omega)H(\omega)$ very important results. So, here we have an interpretation of the result before we even derive it mathematically. The interpretation is that if we have thought of the input as a linear combination of several rotating complex numbers, rotating with different angular velocities normalized angular velocities ω. The output is going to be this a linear combination of the same complex exponentials, but the complex exponentials are going to get multiplied by their Eigen value H(ω).

And then you are going to integrate this over the ω. And here, we are using four things in the context of linear shift invariant systems additivity, homogeneity, shift invariance, and finally the eigen sequence property that $e^{j\omega n}$ when it goes into the LSI system comes out $H(\omega)e^{j\omega n}$. Now, here we have seen effectively that when you convolve x[n] and h[n]. And then take the discrete time fourier transform if it is equivalent to multiplying the discrete time Fourier transforms of the individual sequence.

Now this is in the context of an LSI system. And we now have a beautiful interpretation for it. But we want to prove this in general. So, what we are saying is convolution in time. Convolution in the natural domain, it could be time; corresponds to multiplication in the frequency domain. So, you see here we are now introducing this whole idea of domains. You know the same signal or the same sequence can be viewed in different domains, all this while we have been viewing it in what is called the natural domain. The domain in which that signal occurs, that could be time or it could be space or whatever.

But we can take the sequence of the signal to a different domain where you can equivalently view it. Now you see this is a reversible process, as we have seen. You can go from the natural domain to the frequency domain by using the discrete time Fourier transformation. And you can come back from the discrete time Fourier transform to the natural domain or the frequency domain, to the natural domain, by using the inverse discrete time Fourier transform.

So, it is invertible. Now we need to prove this, we have proved this or we have explained or we have justified this in the context of an LSI system. But let us prove it in general algebraically. So, let us prove this in general.

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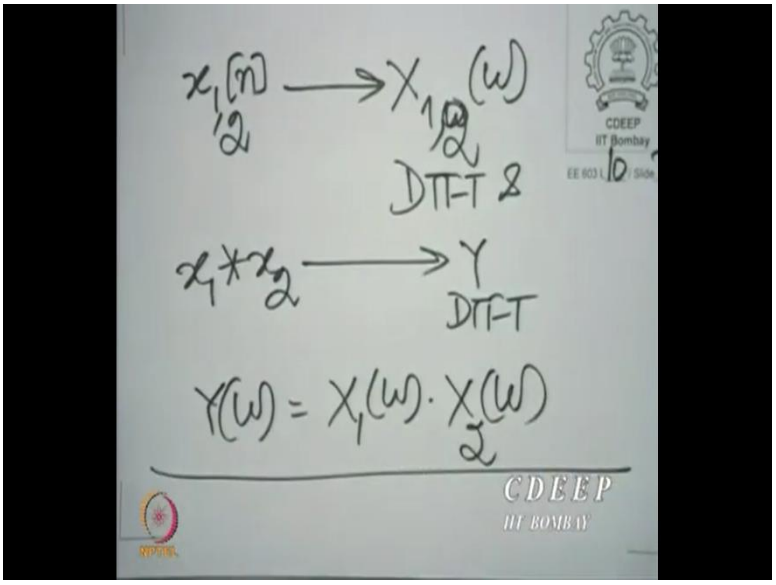

That is if I take $x_{1,2}[n] \rightarrow X_{1,2}(\omega)$ If $x_1 * x_2$ also has a DTFT. Let us call it y. Then Y(ω) is equal to $X_1(\omega)X_2(\omega)$. You want to prove this in general. You want to prove it algebraically. Now that is easy.

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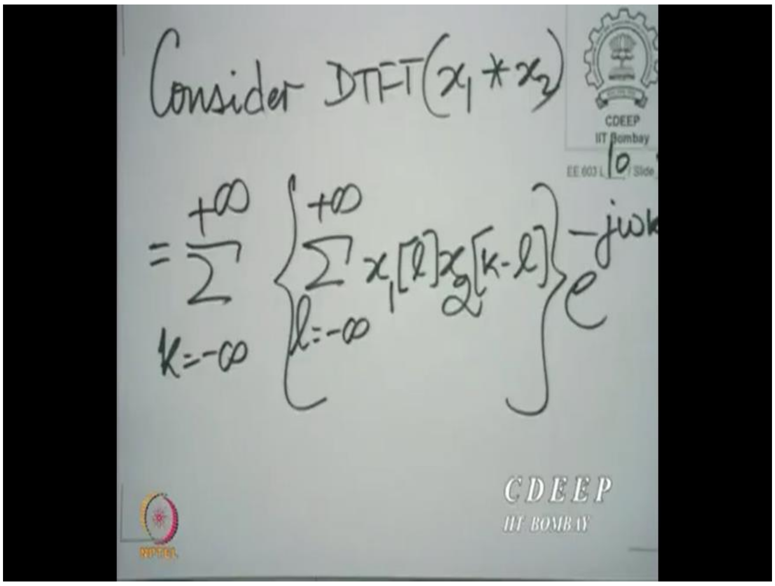

In fact, consider the DTFT of $x_1 * x_2$. Of course, it is $\sum_{k=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[l]x_2[k-l]e^{j\omega k}$. And I have assured myself that this double summation converges, that is why I said that the DTFT of the convolution exists. Now if it does let us collect all the summations into one.

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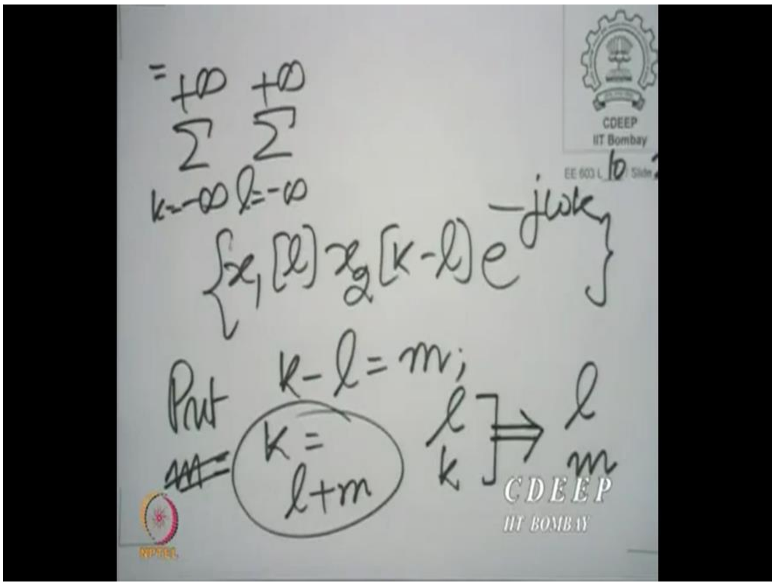

So, clearly this becomes $\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \{x_1[l]x_2[k-l]e^{j\omega k}\}$. And we use the standard trick, you see the troublesome term is $x_2[k-l]$, so let us put (k - l) = m.

Now, what we are going to do is consider instead of l and k we are going to move to l and m, we know that l and k independently run over all the integers. And, of course from this we also see that k = l + m. You see when l is of course as it, l runs independently over all the integers, but what about m? For a fixed l, when k runs over all the integers so does m. And therefore, a double summation where l and k independently run over all the integers is equivalent to a double summation on l and m both of them independently running over all the integers.

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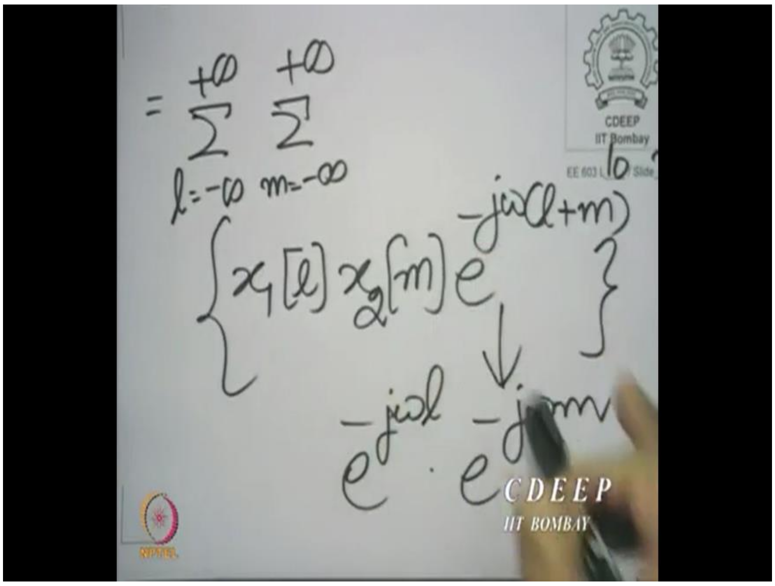

And therefore, this becomes $\sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \{x_1[l]x_2[m]e^{-j\omega(l+m)}\}$. And of course, it is very easy to decompose this, $\sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \{x_1[l]x_2[m]e^{-j\omega l}e^{-j\omega m}\}$. Now, we observe that it is only these two terms that depend on l. It is only these two terms that depend on m. I can bring the summation on m inside to act only on the terms that depend on m.

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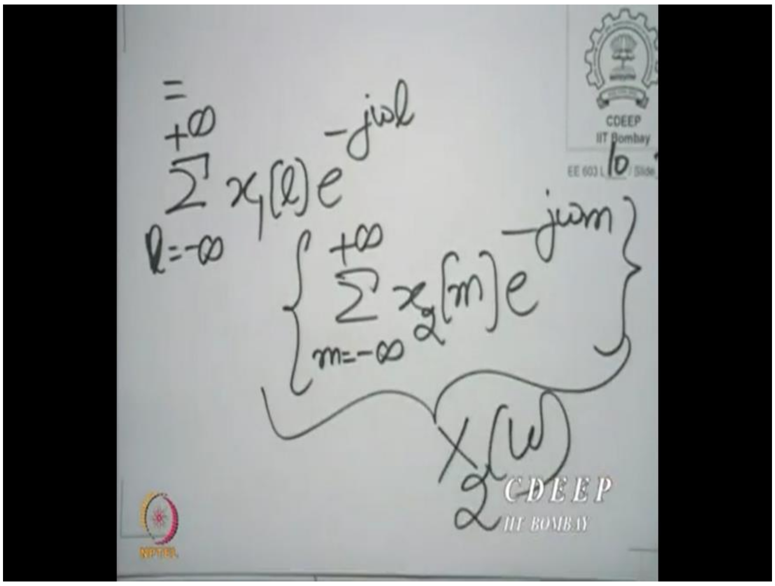

And I have $\sum_{l=-\infty}^{\infty} x_1[l]e^{-j\omega l}\sum_{m=-\infty}^{\infty} \{x_2[m]e^{-j\omega m}\}$. But this is very familiar. This is $X_2(\omega)$. So, this is $X_2(\omega)$ it is independent of l. And I can bring it outside. And once I bring it outside I see that this is nothing but $X_2(\omega)$.

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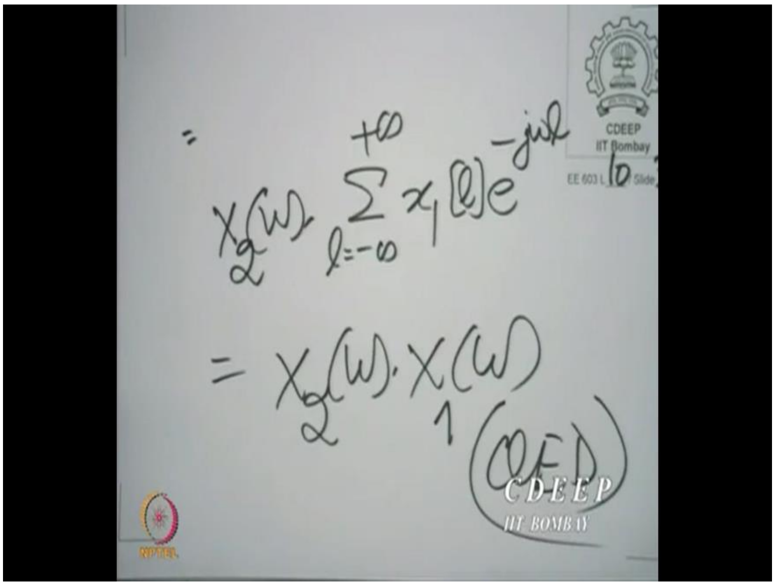

So, there we are so this becomes $X_1(\omega)X_2(\omega)$. So, we have proved this. So, we have a very significant result here. When you convolve two sequences if both of them have discrete time Fourier Transforms and of course, then assuming that the convolution does too the discrete Fourier Transforms of the convolution is the product of the discrete time Fourier Transforms of the individual sequences.