Digital Signal processing & Its Applications

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Lecture – 02 b

Sampling of sine wave and Associated Complications

So, in that sense now we understand how unique this property is for sine waves. When you add two sine waves at the same frequency you get a sine wave of same frequency. In fact now, let me give you a justification of that.

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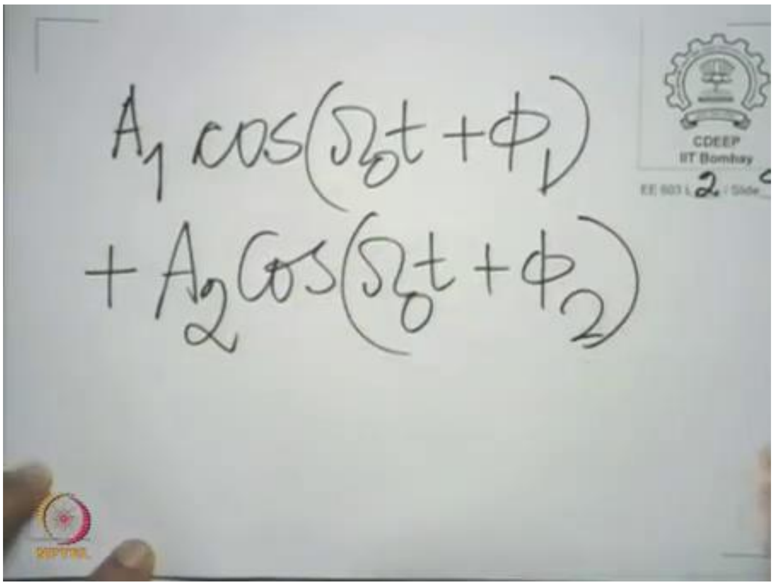

I will leave the details of the justification to you but if I have two sine waves of the same frequency A_1 (cos Ω_0t + $¢_1$) +A_2 now I am even giving them different amplitudes but the same frequency I am also giving them different phases. I can always decompose each of these is not it, so I can decompose them let me decompose them into their cosine and sin components.

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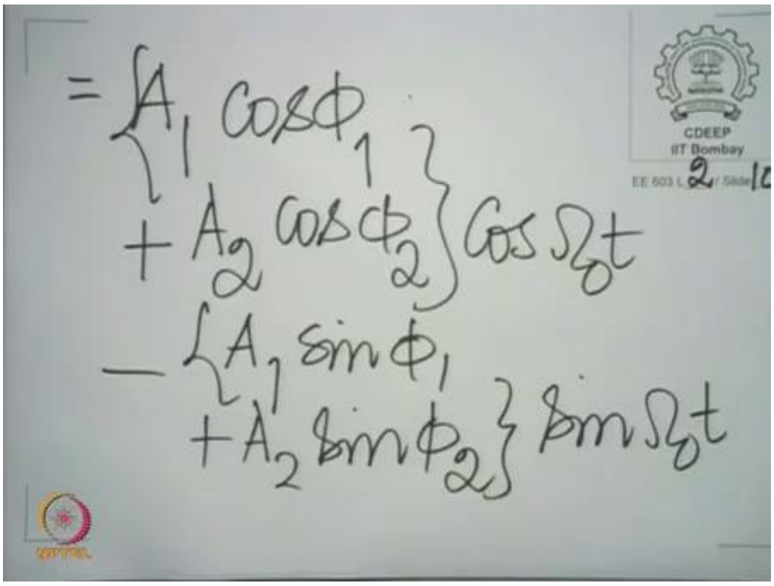

So, I have this is {A_1 cos ¢$_1$ +A_2 cos ¢$_2$} cos Ω_0t – { A_1 sin ¢$_1$+A_2 sin ¢1} sin Ω_0t. It is a simple trigonometric exercise to combine this back again to get a term of the form.

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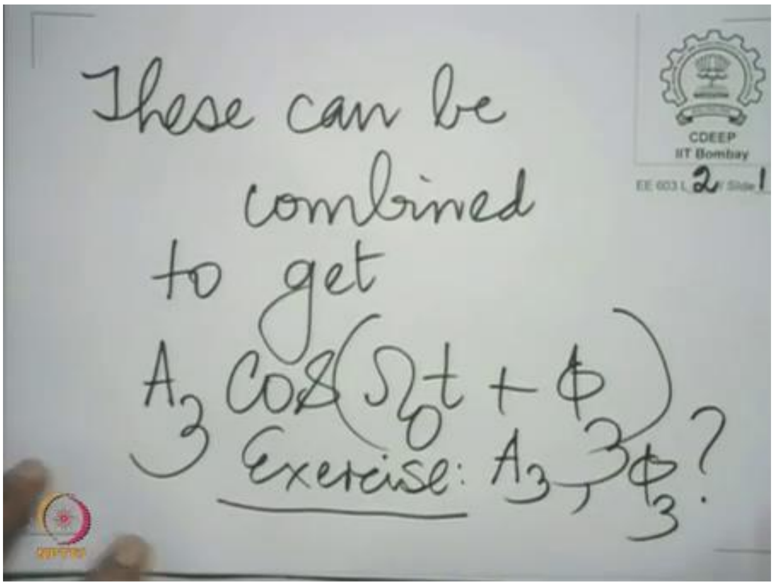

A_3 cos (Ω_0t + ¢$_3$). I leave it to you as an exercise to complete the details that means find A_3 and ¢$_3$ what are they? The simple exercise I mean it is a simple exercise and basic algebra trigonometry whatever you want to call it, anyway this proves this beautiful property of sine waves.

So, now there is one good reason why we like to build our thinking around sine waves. We very often think of as I said all reasonable signals as a combination of sine waves, we like to think of them like that.

In fact you may wonder if sine waves are so smooth, smooth means they have an infinite number of continuous derivatives, in fact there is a term used in the literature for such functions they are called analytic, they have an infinite number of continuous derivatives.

If sine waves are so smooth how is it that they come together and form rough functions? And we know that you know they do that to a certain extent and they do create trouble at discontinuities but we would not get into those details at the moment. What we do know and we have been exposed to this idea from our introductory courses on signals or signal theory or our introductory courses on transforms.

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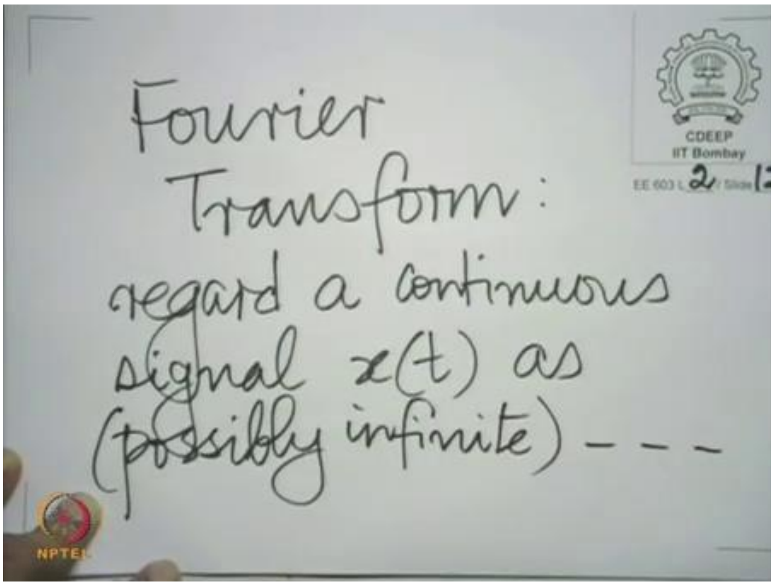

That, we can use what is called the Fourier transform to decompose or to regard a continuous signal x(t) as a possibly infinite sum or linear combination I will continue this.

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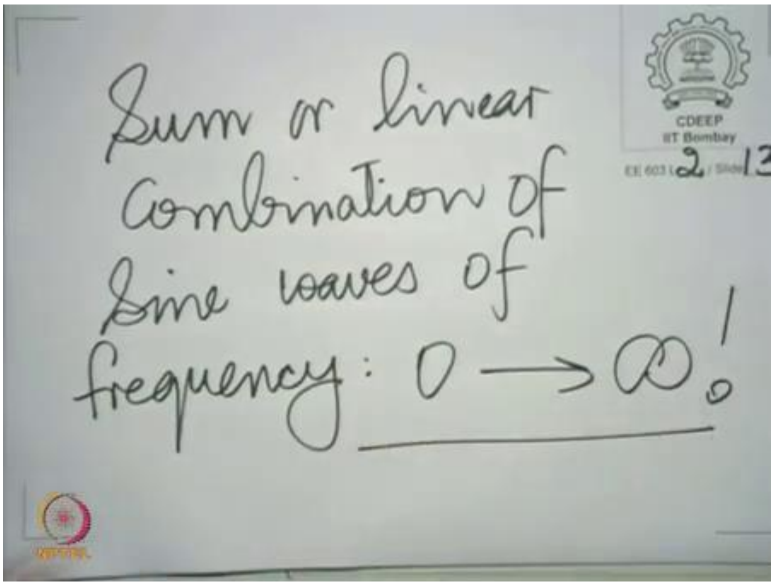

As a possibly infinite sum or linear combination of sine waves of frequency varying in principle from 0 to ∞. I repeat that because it is an important idea the Fourier transform regards a continuous signal x t as a possibly infinite sum or linear combination of sine waves of frequency ranging all the way from 0 to ∞.

Now, of course if the waveform is inherently periodic let us for the sake of argument take the period to be one millisecond then we have associated with that waveform what is called a fundamental frequency of the reciprocal of one millisecond, namely one kilohertz. And in that case you do not need all these frequencies from 0 to ∞ you need only discrete frequencies, you need the frequencies 0, 1 kilohertz, 2 kilohertz and all multiples of 1 kilo hertz.

On the other hand we could take this argument to its limit. So, suppose you have an periodic wave form, a waveform where the period is notionally not finite and that holds the key to generalization.

So, if you have a periodic wave form and if it is quote unquote “reasonable” I cannot qualify that at this time, let us assume and let us be satisfied with the explanation that we will only deal with reasonable signals most of the time, anyway the signal is reasonable and if it is a periodic then you could think of it as if it had a period of infinity.

And that means its fundamental frequency would tend towards 0 and in contrast to the case where you had one kilohertz fundamental frequency and you needed only frequencies which are multiples of that, now you need multiples of what is potentially a 0 fundamental which means you need essentially the continuous frequency axis.

So, for a periodic waveforms we need to move from a discrete frequency axis to a continuous frequency axis. Anyway, you see it makes it clear why we are in a position at least to deal with all reasonable signals if we can deal with one sine wave.

If we can see what happens to one sine wave when we sample we will be able to come to a conclusion on what happens to any reasonable signal when we sample. So, now let us answer the question, let me take just one sine wave.

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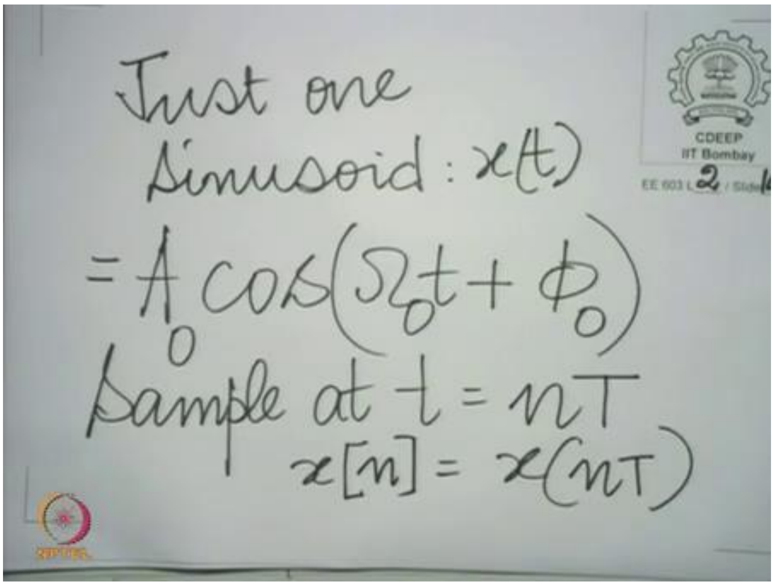

So, let that sine wave be A{cos (Ω_0t + ¢)} or if you like A_0{cos (Ω_0t + $¢_0$)}. . And let us sample this sine wave at t equal to all multiples of the period T. Let us give this signal a name let us call the sine wave x(t) and therefore we are essentially taking x (nt). And we shall again use a slightly compressed notation for this, we will say x(nt) is x [n].

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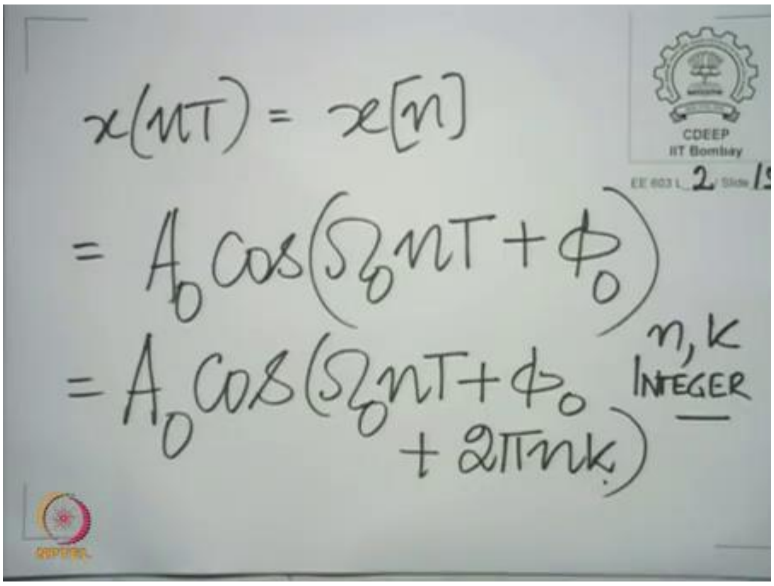

Now, let us make some observations. It is very easy to see that if you had x (nt) or x [n] b. As we have just called this which of course now would become $A_0\{\cos(\Omega_0 nT + ¢_0)\}$.

You could as well have got this from the expression $A_0\{\cos(\Omega_0 nT + ¢_0 + 2\pi nk)\}$ where n and k are integers any integer k here would give you the same value that is not at all difficult to see.

The same time instant or time instant index if you would like to call it that n and different integers k all of them are going to give you exactly the same value for every point n this is something rather striking.

In fact, what it tells you is that we seem not to be answering the right question, when we sample a sine wave what we landed up doing is creating too much of ambiguity or too much of confusion not loss of information as we thought initially.

We are asked the question how do we ascertain that we are not losing information by sampling, perhaps we have to rephrase the question by sampling what we have done is to create a lot of confusion.

There is too much of information now, these samples could have come from so many different sine waves there was just one sine wave initially you took samples of sine waves spaced at intervals of T. Now, when you do that your problem is not that you cannot reconstruct a sine wave, you can reconstruct a sine wave there are just too many sine waves that you can reconstruct.

In fact now let us put down explicitly what those sine waves are and in fact let us increase the confusion in a minute. So, let me go back to this expression here, I have $A_0\{\cos(\Omega_0 nT + \phi_0 + 2\pi nk)\}$, now you know I can invert the sin.

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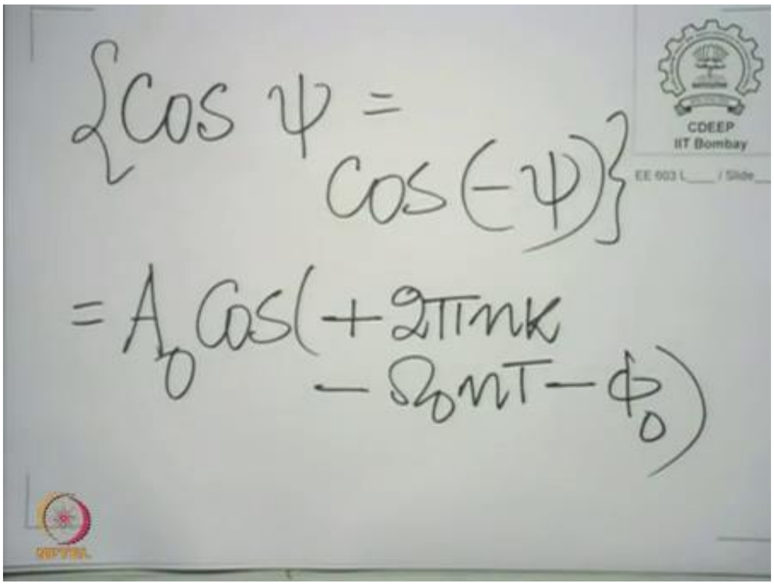

So, I know that cos Ψ = cos (-Ψ). So, I have trouble there, I can introduce a minus sign here and get the same values so I can put a minus sign in this expression here. So, therefore because of this I can also equate this to $A_0\{\cos(2\pi nk - \Omega_0 nT - \phi_0)\}$

But you see k took all integer values so since I am free to choose all integer values there, I do not really have to write (–k) I can also write just +k there so I will do that, I will write +k here.

So, now I have a very serious problem to deal with, I have $A_0\{\cos(\Omega_0 nT + \phi_0)\}$ is equal to $A_0\{\cos(2\pi nk + \Omega_0 nT + \phi_0)\}$ and it is also equal to $A_0\{\cos(2\pi nk - \Omega_0 nT - \phi_0)\}$

. In other words if I were to aggregate terms together, collect terms as they say.

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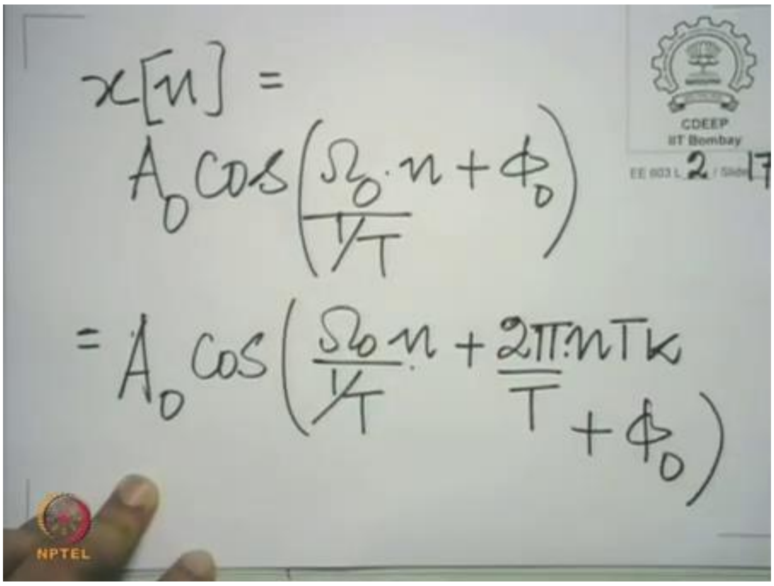

We are saying essentially that x [n] is equal to $A_0\{\cos(\Omega_0 nT)\}$ A naught cos omega naught n T. So, I will write it as Ω_0 divided by 1/T times n you will see why a little later. It is also equals to now I need to take the n term together so I need to write it like that I need to write A_0 cos $(\Omega_0/(1/T))n$ plus I will write 2π I need to write nT here so I will try multiply and divide by T. And of course $¢_0$.

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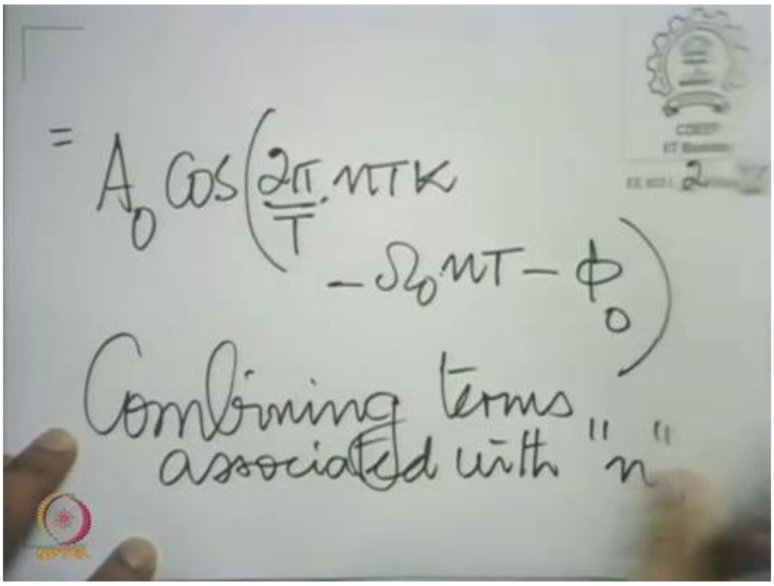

And that is also equal to A_0 cos(- Ω_0), so in fact let us take this term now, from this term I am multiplying and divide by T here so I have{ $(2\pi/T)nTk$ - Ω_0- $¢_0$}.and all of them give me the same value. So, essentially if I were to combine let me combine a few terms. So, for example let me combine these two terms on the same place, I will combine these two terms, now you can take n common from here, so let us combine terms.

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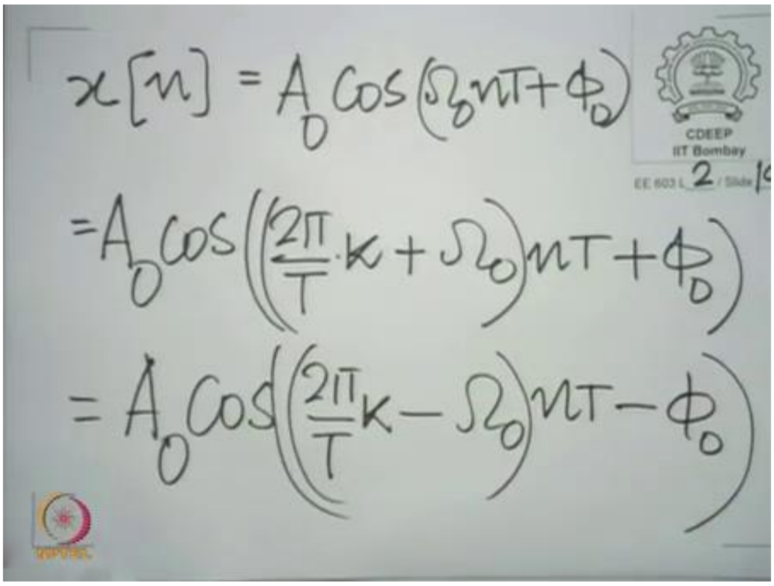

I get x [n] is of course the obvious $A_0\{\cos(\Omega_0 nT + ¢_0)\}$. Which is also equal to $A_0\{\cos((2\pi/T)k + \Omega_0)nT + ¢_0\}$ and that also equal to $A_0\{\cos((2\pi/T)k - \Omega_0)nT - ¢_0\}$.

All three of them and now we have a beautiful interpretation that we can work with. Let us look at this expression this series of equations a little more carefully here. What are we saying in this series of equations? We are saying that this stream of samples x [n] could have as well come from the original sine wave whose frequency is capital omega naught.

It could have as well come from another sine wave whose frequency is $\Omega_0+(2\pi /T)k$ and k can be any integer. So, k could be 1, 2, 3, -1, - 2, anything that you like.

And finally of course it could also be the frequency $(2\pi/T - \Omega_0)$. And notice one thing when you associate Ω_0 with $\{(2\pi/T)k\}$ you have the same phase as the original sine wave.

On the other hand when you associate $- \Omega_0$ with $\{(2\pi/T)k\}$ you have the opposite phase. So, omega naught becomes minus omega naught here, you have a change of phase, you have a reversal of phase. It will be much easier for us to appreciate this if we take some numbers.

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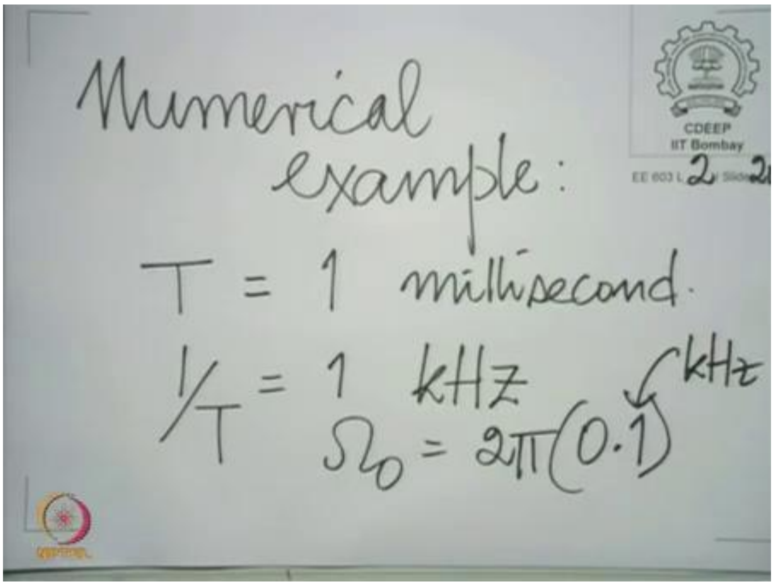

So, let us take some numerical values. Let us again go back to our example of T is 1 millisecond. And we are saying essentially that 1/T is 1 kilohertz then. And let us take Ω_0 to be 2π.

Now, remember omega naught is an angular frequency so it has to be 2π times a hertz frequency. So, let us take it to be essentially corresponding to the hertz frequency of 0.1 kilohertz in kilohertz.

Now, just for the sake of convenience we would now work only with the hertz or kilohertz frequencies rather than a mixture of angular frequency and hertz frequency. So, let us use the understanding of 0.1 kilohertz for the original sine wave and one kilohertz for the sampling frequency or the rate of sampling.

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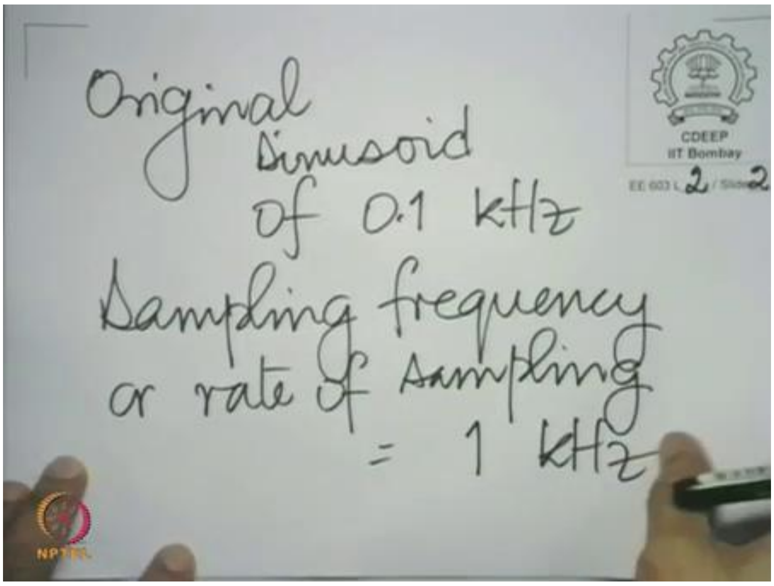

So, we are saying original sinusoid of 0.1 kilohertz and sampling frequency or the rate of sampling is 1 kilohertz. And now we sit down to analyze what we have just concluded.

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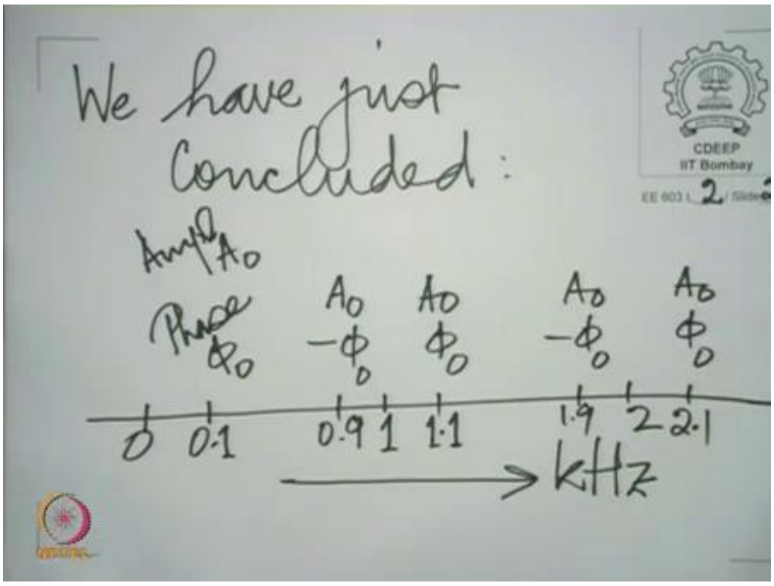

We have just concluded that, if you take the frequency axis now or the kilohertz frequency axis initially you had only 0.1 kilohertz. So, here you have 0 frequency here you have 0.1 kilohertz, let us mark 1 kilohertz there, let us mark 2 kilohertz somewhere here. Perhaps, not quite to scale but anyway representative.

What we are saying is that initially you had a 0.1 kilohertz frequency with a phase of ¢ of or $¢_0$ and amplitude A_0. After sampling what you have created is many many copies of this. Or in other words you have created so much of confusion that all of us class of sine waves could have generated those samples which are those sine waves, let us mark them on the frequency axis with their amplitudes and phases.

So, the same samples could have come from a sine wave of frequency 1.1 kilohertz amplitude A_0 phase $¢_0$ or it could have come from a sine wave of (1 - 0.1) that is 0.9 kilohertz with an amplitude of A_0 but a phase of - $¢_0$. Please note.

It could equally well have come from a sinusoid of frequency 2.1 with an amplitude of A_0 and a phase of $¢_0$ or it could have come from a sine wave of frequency (2 - 0.1) which is 1.9 kilohertz with an amplitude of A_0 but a phase again mind you of - $¢_0$ and this pattern is repeated.

So, around every multiple of the sampling frequency 3 kilohertz, 4 kilohertz, 5 kilohertz, 100 kilohertz, 1000 kilohertz and what have you. You have two adjacent frequencies, you have the sampling frequency plus 0.1 kilohertz with an amplitude of A_0 and a phase of $¢_0$.

And you have this other one with the frequency of sampling frequency minus the signal frequency or 0.1 kilohertz with an amplitude of A_0 and a phase of- $¢_0$. All these sine waves could have created these samples. So, this makes it very clear what we said a minute ago. The problem is not that we have lost some information in sampling, the problem is we have created too much of confusion in sampling.

Now, there is just too much to deal with, there are just too many sine waves that could have given me the same samples. In fact we need to spend a minute or two in understanding graphically what has happened, graphically what has happened is the following. You see let us take the original sinusoid.

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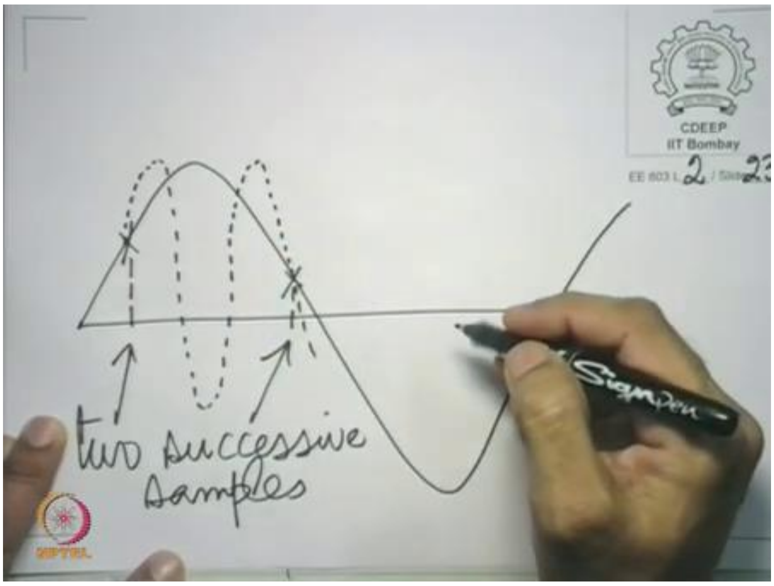

And let me expand just one period. Let us assume that we have taken one sample here and the subsequent sample somewhere let us say here. Now, the confusion has arisen because of the following reason. The same samples could have come from another sine wave where you have lost a complete period in between. So, you could have lost a whole period and you could be going back or you could have also lost a period and you remember there is a correspondence between this and this point on the sine wave.

So, now you could also draw another sine wave where this sample appears on the upward edge and this sample also appears on the upward edge. So, you could compress it even further, here of course you are drawing the sample you are putting the sample as it is originally on the downward edge.

But, you could even complete this and draw it on the upper edge. And then of course you could be skipping two whole periods, you could be skipping three whole periods and again whenever you are skipping one period you could either be putting the second sample on the upward edge or the downward edge.

That is another way to understand why for each possibility of multiple of the sampling frequency. So, like you had 1 kilohertz, 2 kilohertz, your original sine wave was of 0.1 kilohertz. The next one is 1 kilohertz and 1 plus 0.1 1 minus 0.1. Now, what is the ambiguity between plus 0.1 and minus 0.1? It is essentially in a way the ambiguity of phase whether you have plus phi naught or minus phi naught that is a mathematical way to describe it.

A graphical way to understand it is that the same point could have come from an upward or a downward edge as you see here. I encourage you to take the exercise of drawing a few of these possibilities and let me conclude this lecture today by asking you to do so.

Draw a few of these possibilities so that you get a complete feel of the ambiguity that is created by sampling. Find out many different sine waves graphically that can generate the same stream of samples and you know how to do it.

Essentially, you get one pair by skipping 1 cycle, you get another pair by skipping 2 cycles, 3 cycles, f cycles and if you do it a few times for a few cases the idea would be very clear to you.

So, there we are we have created so much of ambiguity by the process of sampling. What we now need to do is to resolve that ambiguity. And the ambiguity can be resolved if we go back to the original drawing here.

You know the ambiguity has come because of these false sine waves that have been created. We need to understand when it is possible for us to remove these false sine waves and retain the original true one and when there could be some trouble in doing so. We shall discuss this in the next lecture. Thank you.