**Digital Signal processing & Its Applications**
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**Lecture No. 11 b**
**Duality between Multiplication and Convolution**

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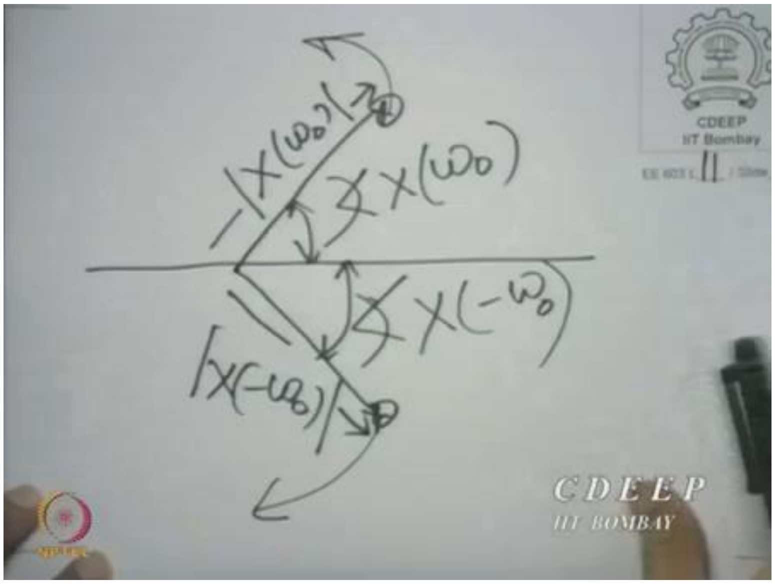


Now, let us look at a few more properties of the discrete time Fourier transform.

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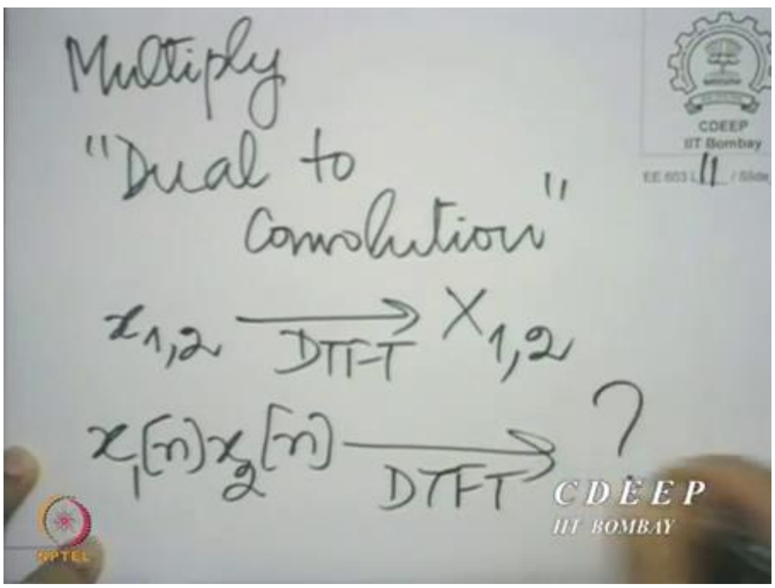

One important property is what happens when we multiply. That is in some sense dual to convolution. Let me explain the meaning of the word dual. So, in other words if I have $x_{1,2}$ with DTFT is capital X at is the DTFT of $x_1[n]\ , x_2[n]$? That is the question that we ask.

Now, I will spend a minute in explaining the idea of dual here. We are kind of for you know having a foregone conclusion we are trying to use our intuition to come to a conclusion before we actually arrive there.

There is one very common principle that is useful in many contexts in the Fourier transform. And that is that very often the roles of time and frequency can be reversed. Now, in continuous time and in analog frequency this is exactly true.

So, one can show that, one can more or less exactly reverse the roles of time and hertz frequency in properties. However, in discrete sequences and the discrete time Fourier transform this is true in a slightly more extended sense. It is not true obviously.

For example, what we saw was that when we multiply, when we convolve two sequences, we are multiplying the discrete time Fourier transforms. Now, duality would tell us that you could reverse the roles of time and frequency.

That means that when you multiply two sequences you expect some kind of convolution to take place in the frequency domain. But it is not obvious what kind of convolution that we will need to do with a little bit of algebra.

We broadly understand that we expect some kind of convolution that is what duality says. But we need to put down exactly what kind of convolution. To arrive at that answer let us actually consider a product so here we are.

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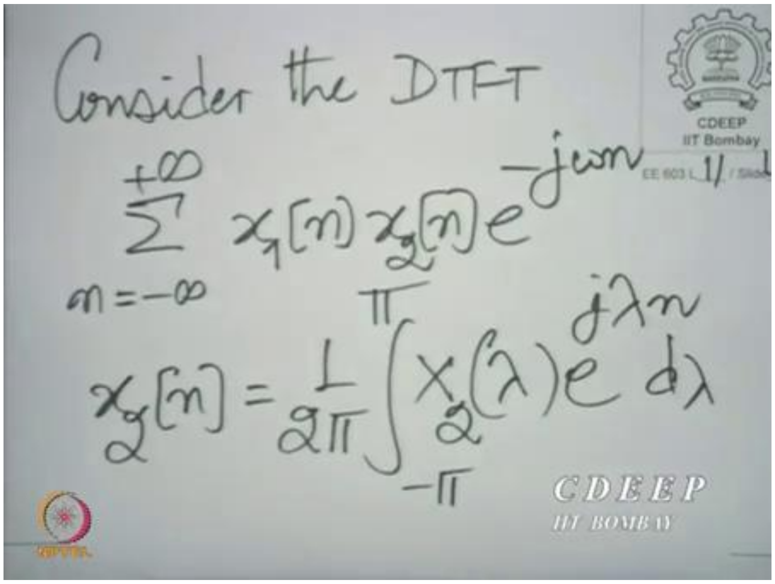


So, consider the DTFT. So, now of course we have assumed that both $x_1$and $x_2$have their own DTFTs and that this converges. We are assuming otherwise it has no meaning to discuss this. All that we do is cleverly replace one of them by their inverse DTFTs.

So, we write down $x_2[n]$ for example you could do it the other way too but we will write down $x_2[n]$ as. Now, here we cannot write ω. Because, you already have an ω there so you must use another variable of integration.

Let us use the variable λ. Is that right? So, we simply replace one of the sequences by the in by essentially write the sequence as an inverse DTFT of its own DTFT. Now, we substitute that.

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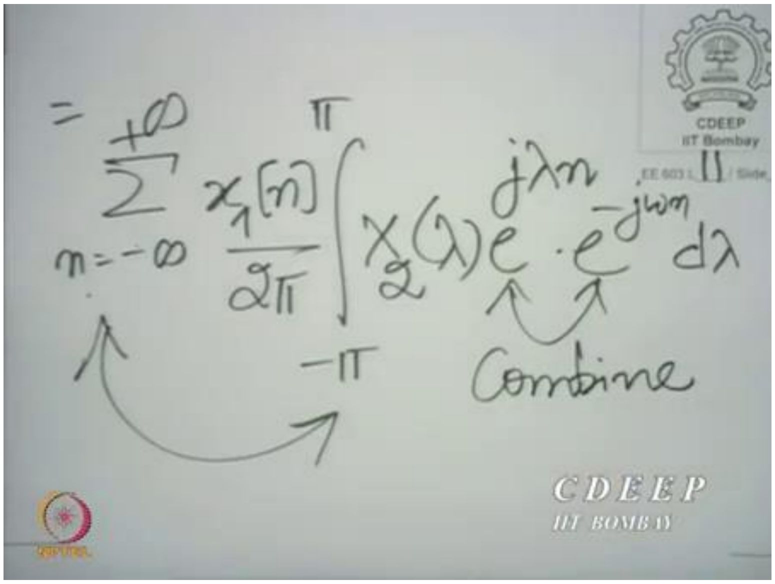


So, here we are. That becomes summation n going from minus to plus infinity $x_l[n]/2\pi$ integral minus π to π. And you see we realize, we realize that we could as well take the $e^{-j\omega n}$inside here it does not do any harm at all.

It is a constant anyway with respect to this. It is a constant with respect to λ so I can push it inside. And now I can interchange the summation and the integration. And I can combine these two terms and there I am.

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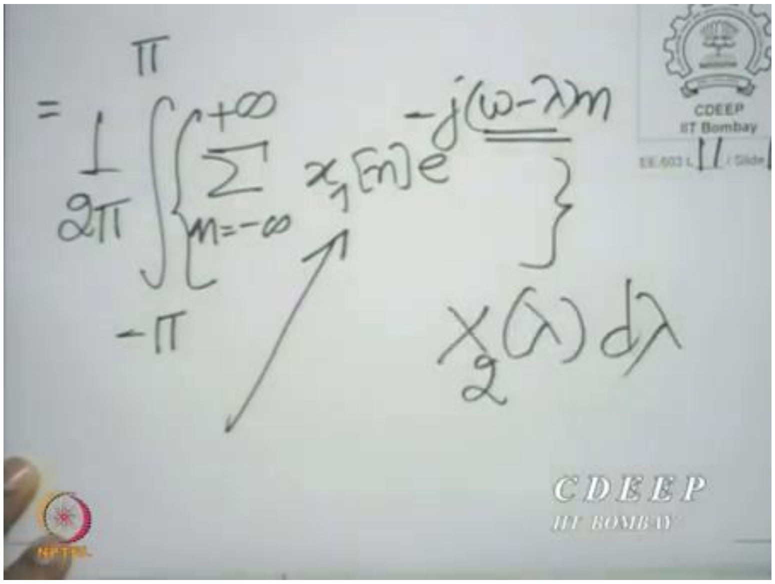


This becomes $1/2\pi \int_{-\pi}^{\pi} \{\sum_{n=-\infty}^{+\infty} x_1[n]e^{-j(\omega-\lambda)n}\}X_2(\lambda)d\lambda$. Now, here we have something very interesting.

What we have inside the curly bracket here is essentially the discrete time Fourier transform of $x_1[n]$ but evaluated at ω minus λ instead of at λ or at ω. You have just evaluated that discrete time Fourier transform at a different value. So, there we are.

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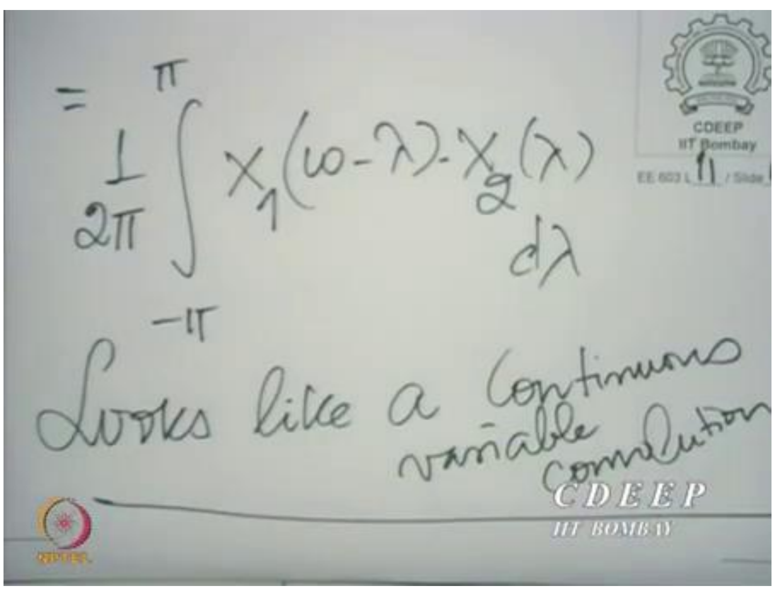

We are saying this is $1/2\pi \int_{-\pi}^{\pi} X_1(\omega - \lambda)X_2(\lambda)d\lambda$. And recognize that this is very similar to what you expect a continuous time convolution to be.

So, it looks like a continuous variable convolution. There is only one little catch. A continuous variable convolution should have run all over the independent variable axis. So, here λ is a continuous variable.

So, it should have taken you all over the λ axis if you were to convolve the continuous functions capital $X_1$and capital $X_2$. But, you are restricting the convolution to one period. You have restricted the convolution to the period minus π to plus π.

In fact, we do not really have to insist upon the interval minus π to plus π even in the inverse discrete time Fourier transform. In the inverse discrete time Fourier transform we can take any continuous interval of 2 π.

So, if you are really very fond of it we might take the interval from zero to 2 π instead of from -π to π. That is because from π to 2 π you have just the same thing that you have from -π to π.

If you are even more insistent on doing something unusual you could start from π and go up to 3 π whatever you please. Any contiguous interval of 2 π is π. And that is true here as well.

So, one must in general say in this expression that you need to calculate this integral over any interval any contiguous interval of 2 π. It could be from 0 to 2 π it could be from minus π/4 to 3π/4 if you like you know or I mean you know whatever you prefer.

So, or -π by two to 3π / 2. So, three I mean I it should be actually π + π by 3π / 4 so it could be from -π / 2 to 3π / 2. It could be from 0 to 2π or it could be from minus 2π to 0 whatever you please. Any contiguous interval of 2π is π. Anyway, this is what is called a periodic convolution.

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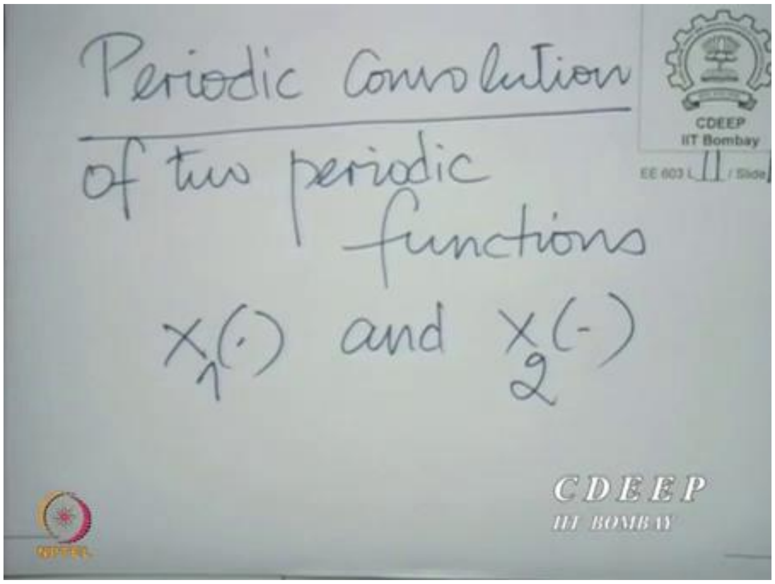


It is a periodic convolution of two periodic functions capital $X_1$ and capital $X_2$. Remember, both these DTFTs capital $X_1$ and capital $X_2$are periodic functions. Now, if you did actually write down the expression of convolution as you have done here.

But, you try to evaluate this integral going from minus to plus infinity of course that interval would diverge, that integral would diverge. Because, both of the functions are periodic this would be a sum of the integral over every period and if any of them is non zero that sum would obviously diverge.

So, it does not make any sense when you have two periodic functions to calculate the convolution by integrating over all the independent variable axis. It only makes sense to calculate over one period and that is exactly what we are doing here.

Remember, that in this expression both $X_2(\lambda)$and $X_1(\omega - \lambda)$for any fixed ω are both they are both periodic. Of course $X_2(\lambda)$is periodic with period 2π $X_1(\omega - \lambda)$ is also periodic with period 2π. And how do you get $X_1(\omega - \lambda)$ from $X_1(\lambda)$. Let us just spend a minute in doing that.

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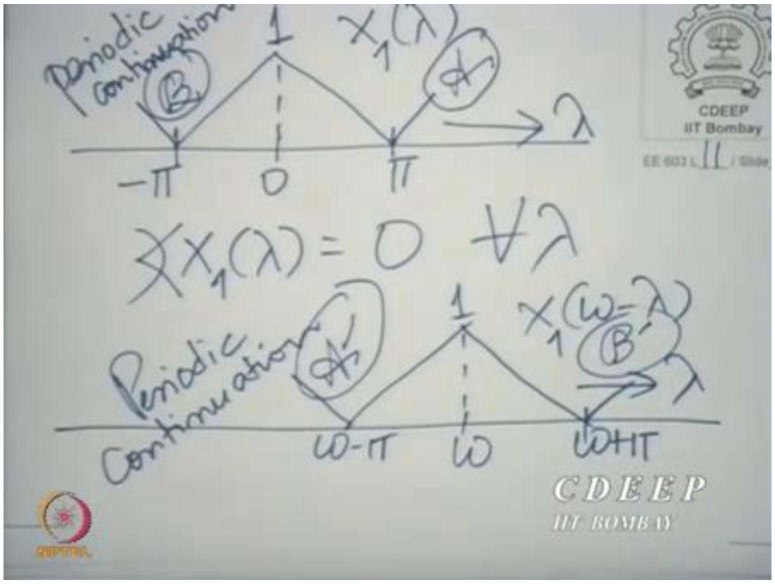


So, just to take an example. Suppose, you happen to have this from -π to π for $X_1(\lambda)$. I mean let me assume for the moment that $X_1(\lambda)$ has zero phase so we will say that the angle of $X_1(\lambda)$ is equal to 0 for all λ.

And we have shown the magnitude here let us assume that this is the kind of X1 λ we are dealing with. Just for simplicity. How would $X_1(\omega - \lambda)$ look? Essentially it would look with this occurring at ω.

And of course, always remember that you have periodic continuation here. There is periodic continuation there and there is also periodic continuation here. Also, whatever you see here let us mark it as script A is going to be seen as its own reflected version here let us call it A prime.

And what you see here let us call it script B is going to be seen here script B prime is also reflection. You see this can you can come to this conclusion in two steps. When you replace λ by ω + λ instead of ω - λ.

When you replace it by ω + λ you are going backward by ω. And then when you replace λ by ω - λ you are making a reflection. So, what was at 0 would have gone to -ω when you replace λ by λ + ω.

And then when you replace λ by ω - λ you are switching the λ sign of λ. And therefore, what is that minusω now comes to plusω what is after minusω goes before plusω and what is before minusω goes after +ω.

And that is how we come to the conclusion. That whatever was after this would now appear before this in reflected form, whatever is before this would appear after this again in reflected form. This is the relation between $X_1(\lambda)$and $X_1(\omega - \lambda)$. Now, let us take an example to illustrate this idea.

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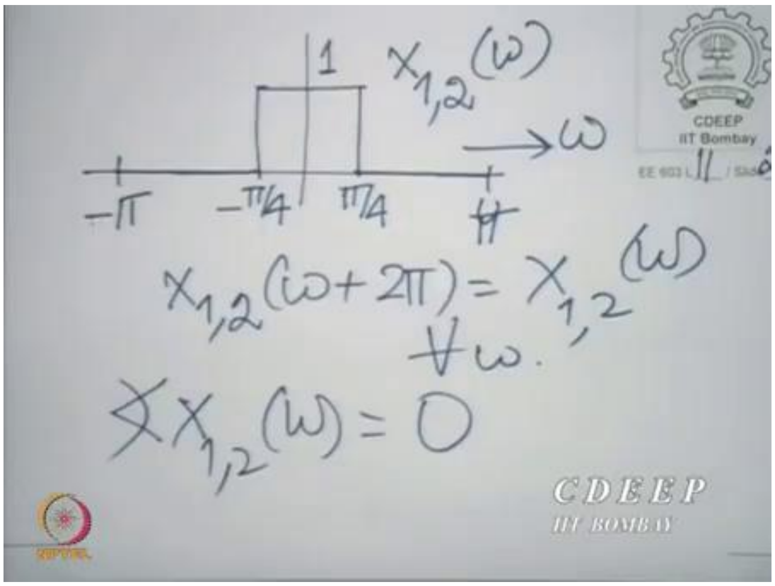


Let us take a very simple discrete time Fourier transform which is 1 between -π / 4 and +π / 4 and 0 else. And we will assume that the discrete time Fourier transform is 0 phase. So, in fact let us take this to be both $X_{1,2}(\omega)$.

Of course $X_{1,2}(\omega + 2\,\pi)$ is equal to $X_{1,2}(\omega)$ for all ω. And we will assume that angle of $X_{1,2}(\omega)$ is equal to 0. So essentially, its 0 phase and the magnitude is 1 between -π / 4 and +π / 4, 0 outside.

Let us find out the inverse discrete time Fourier transform of this. Now, of course how do we find out the inverse discrete time Fourier transform we multiply this $e^{j\omega n}$and integrate over ω from -π to π simple.

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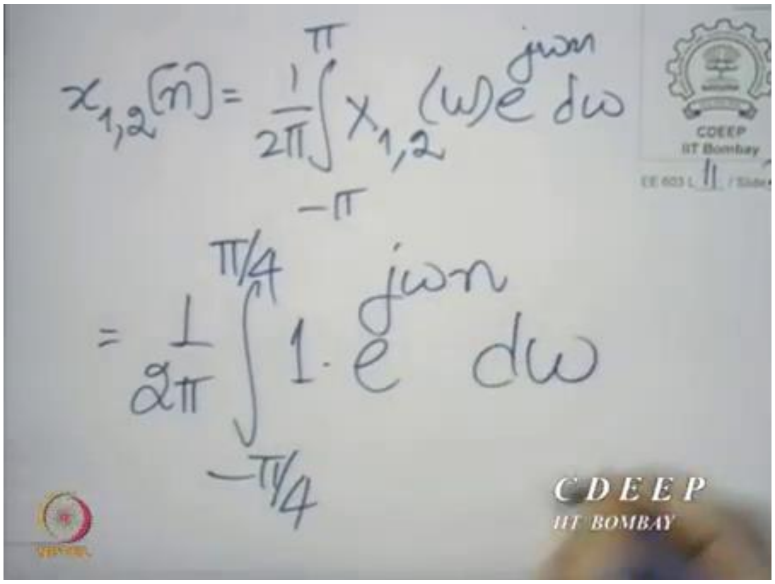


So, we have $x_{1,2}[n] = 1/2\pi \int_{-\pi}^{\pi} X_{1,2}(\omega)e^{j\omega n}d\omega$ And this of course boils down to $1/2\pi e^{j\omega n}d\omega$. This is a very easy integral to evaluate. In fact it takes us only a minute to evaluate it this simply becomes.

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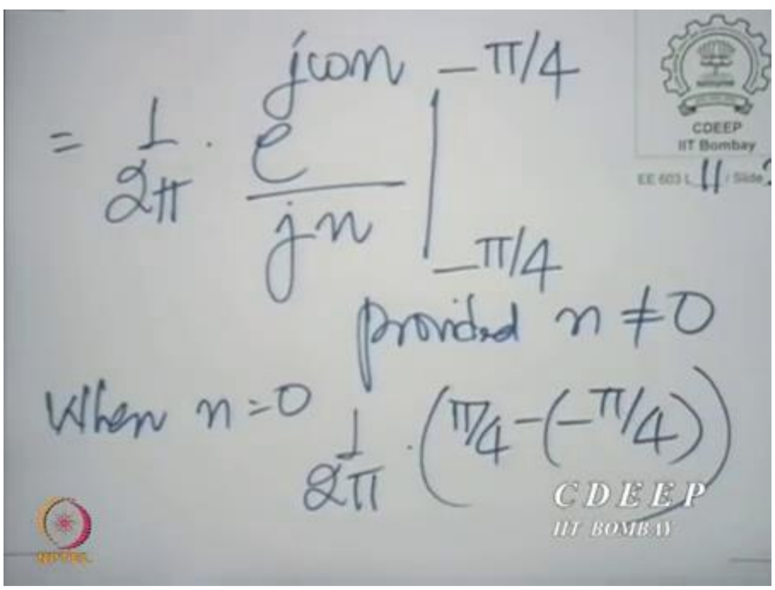


Now, please note I have written this do we need to put a condition here? Yes. Indeed what is the condition, the condition is that n must not be equal to 0 provided n not 0. Now, when n is 0 what does it become?

When n = 0 this becomes 1/2 π the integral is simply 1 dω so essentially π / 4 - (-π / 4). Which is π / 2. So therefore, this is valid only when n is not zero. So, there we are.

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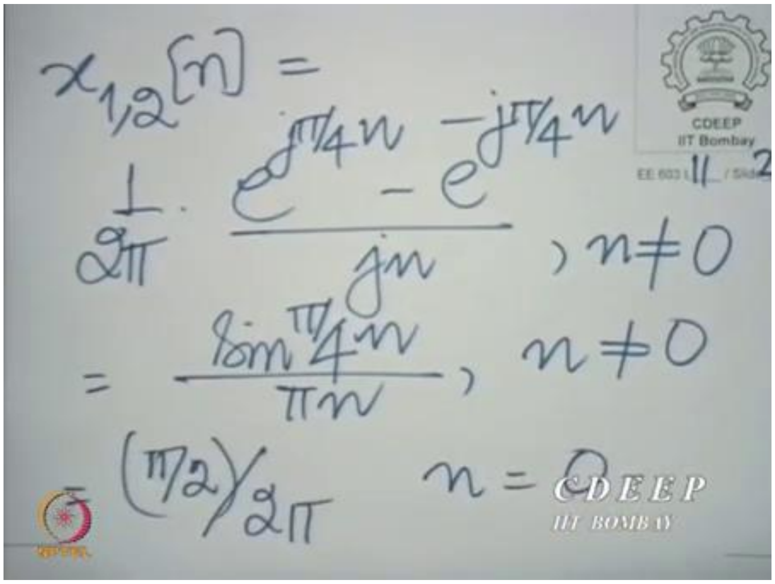


So, $x_{1,2}[n] = 1/2\pi.(e^{j\pi/4n} - e^{-j\pi/4n})/jn$ for n not equal to 0. And it is very easy to see that this becomes sin (π / 4n) / π n for n not equal to 0.

And sequel to π/2 divided by 2π for n = zero. Now, we ask what is the discrete time Fourier transform of $x_1[n]x_2[n]$.

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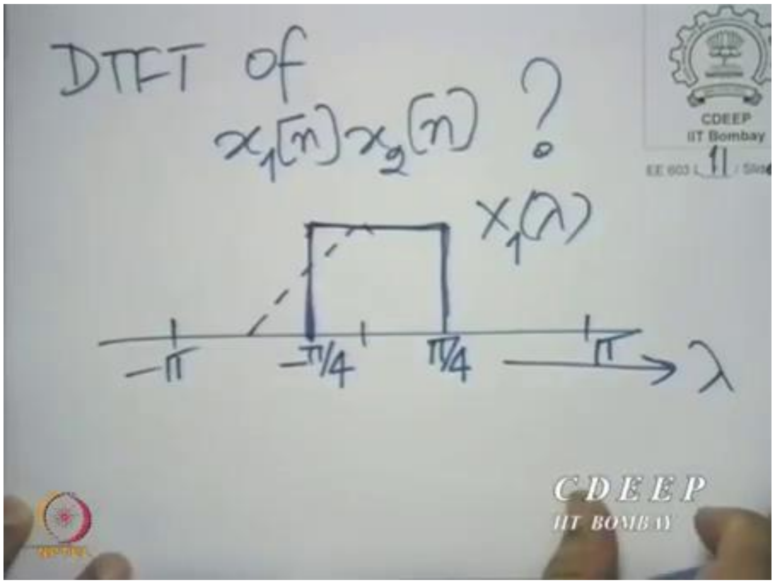

What do we expect it to be we need to convolve the discrete time Fourier transform of any one of them with itself so we take the λ axis we put here. Of course we you know lets first just draw one of them so this is just one of them. And we can visualize the other so let us visualize the other.

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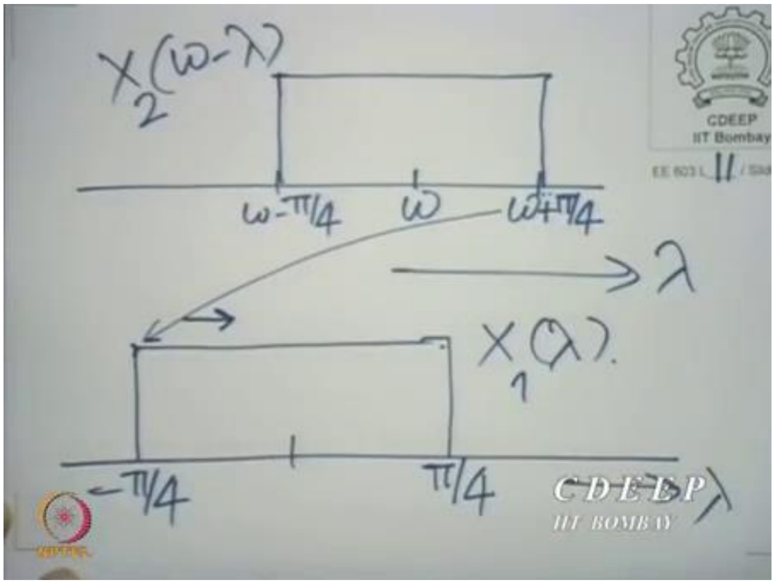


I am kind of expanding it a little bit. Let me draw them on the same scale so to speak. Now, what do we need to do? We need to multiply these two and to integrate over one period and we can choose that period to be between -π and +π.

How much is this interval? This interval is π / 2? So, let us assume that this tip has reached you see now you must you see take the energy of the train and the platform. So, you have this the only catch here is now you know instead of the train having people standing at discrete locations people stand all over the train.

That is the continuous time situation. So, here we are here this is these are the people on the platform these are the people in the trainω moves. Now suppose, this point has reached here what is the situation, of course this is going to be repeated at every multiple of 2π.

But, luckily when this point reaches here we the trouble would have come from the next such, the next such square is not it? How far is the next such square? 2π away from here that is very far from so it is even if even if this point is here the next of these square is not going to clash with this.

So, what we are sure is that one time only one square clashes with one square here that I leave it to you to verify. You will never have a situation where two such squares clash with this one square here.

I leave it to you to verify that. That follows from the fact that you have only a $\pi / 2$ interval of spread. Anyway, so what would happen as $\omega$ moves from a point where $\omega + \pi / 4$ is equal to $-\pi / 4$.

You see when $\omega$ when this point is here $\omega + \pi / 4$ is equal to $-\pi / 4$. In other words, $\omega = -\pi / 2$. And you know at that point this you know so you can visualize this comes here and it moves as $\omega$ moves in this way.

So, you have more and more of this square overlapping with the square here or this rectangle. And as $\omega$ moves along this the area increases linearly. The area is obviously equal to this height into the base which overlaps.

And that base increases linearly with time. So, right from the point where this is here up to the point where this has reached here there is a linear increase. Afterwards as this point you see so I mean well no slight correction.

Right from the point where this has reached here up to the point where this, so let us go back to this discussion here, we begin from the point where this edge has reached here. When this edge reaches here up to the point where this edge reaches here there is an increase of area.

And the area goes to a maximum when this whole rectangle overlaps with this rectangle here. Afterwards this edge begins to move away and this edge starts to approach this one. When there is a linear decrease. What is the maximum area? The maximum area is when these two rectangles overlap entirely and that area is of course $\pi / 2$ multiplied by 1. So, we have the following shape for the convolution.

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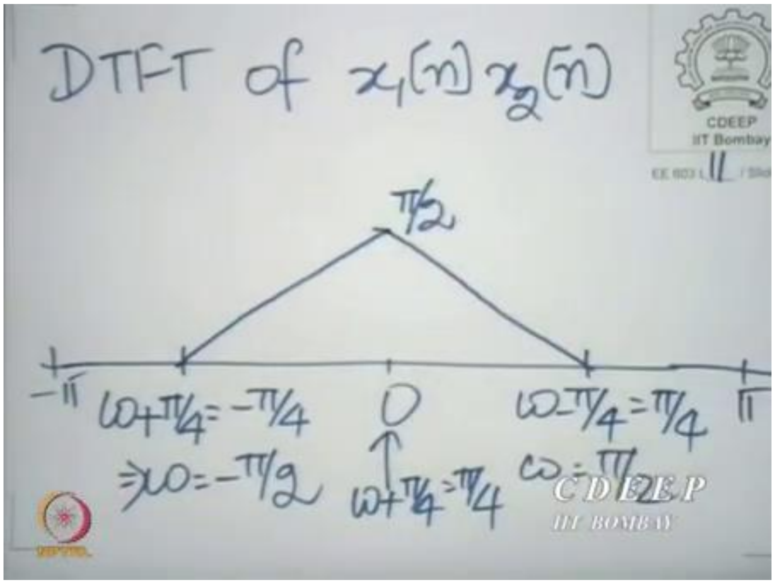


So, the DTFT of $x_1[n].x_2[n]$ would look like this. It would start when ω + π / 4 is equal to -π / 4 or ω = -π / 2. I could end when ω - π / 4 is = π/4 or ω = π / 2.

At 0 it would take the maximum that is when ω + π / 4 is equal to π / 4. There would be a linear increase here and a linear drop there and this height reached would be π / 2. This is how the DTFT would look.

Now, you know this so called dual result that we have here is very important later when we talk about the effect of windowing on sequences or when we try to design finite impulse response filters with windows.

We shall gain a lot of insight into what happens when we truncate an impulse response by using this idea of multiplication of sequences. So, it is not without application or without reason that we are discussing this property.

Anyway, so much so for that property but this property at the moment gives us something equally valuable and interesting. Let us write it down. Yes. There is a question. That is right. And that is a very good question.

So, the question is here we did not run into any trouble or we you know the whole convolution became easier because two rectangles did not overlap at one. You know each of those DTFTs is periodic.

So, what we said is I left it to you as an exercise to show that when one of the rectangles was interacting with the basic rectangle between -π and π no other rectangle interfered. Because the other rectangle was too far away.

What would happen if this were not the case? Well, if this were not the case you have to account for rectangles that come at once carefully. So, in fact that is a very good question and let me leave it to you as an exercise to do the following.

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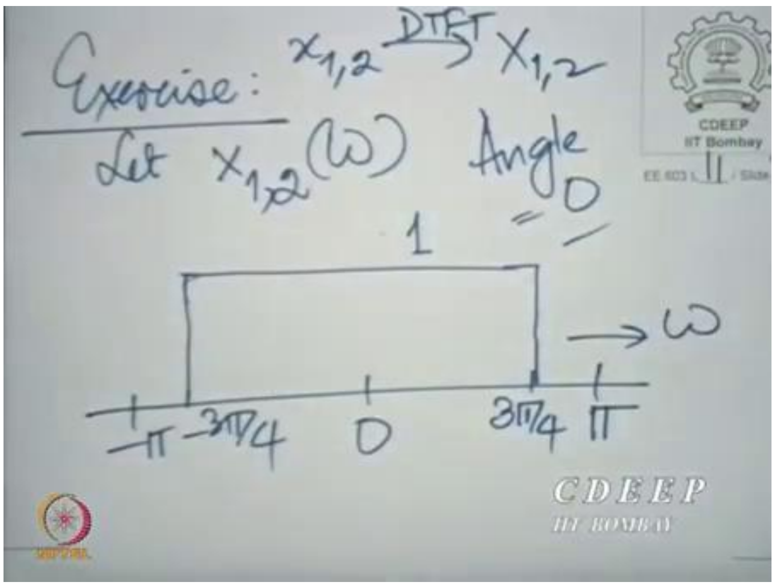


Exercise. Let the two DTFTs $X_{1,2}(\omega)$take the following form. Let them be one between -3π /4 and 3π / 4. And of course angle is 0. Say, 0 phase. And of course you know that $x_{1,2}$ have the DTFT $X_{1,2}$ as before.

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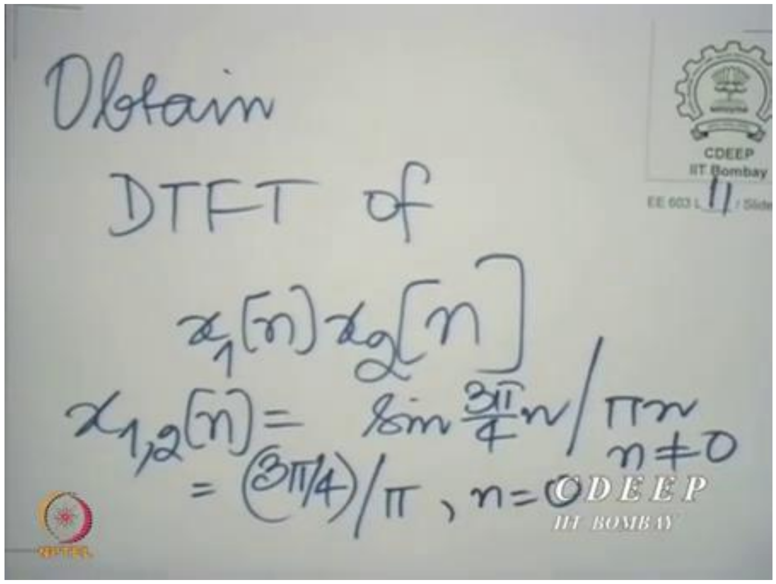


The exercise is obtain the DTFT of $x_1[n], x_2[n]$. In fact, I will also leave it to you as an exercise to show that $x_{1,2}[n]$ comes out to be sin(3π / 4 n) / πn for n not equal to 0. And (3π / 4) /π for n = 0. Anyway, here when we do this exercise, we will have to worry about more than one rectangle overlapping at once. So, you have to carefully account for the rectangles that would overlap with the basic rectangle each time you move.