Digital Signal processing & Its Applications
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Lecture 12 c
Parseval's Theorem and its Interpretation

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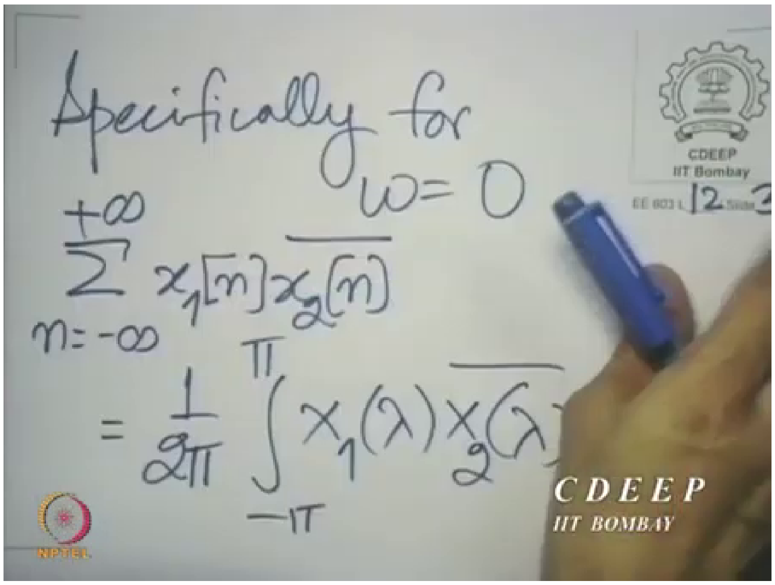

But this very important result has a very very central place in discrete time signal processing. This is called the Parseval's theorem.

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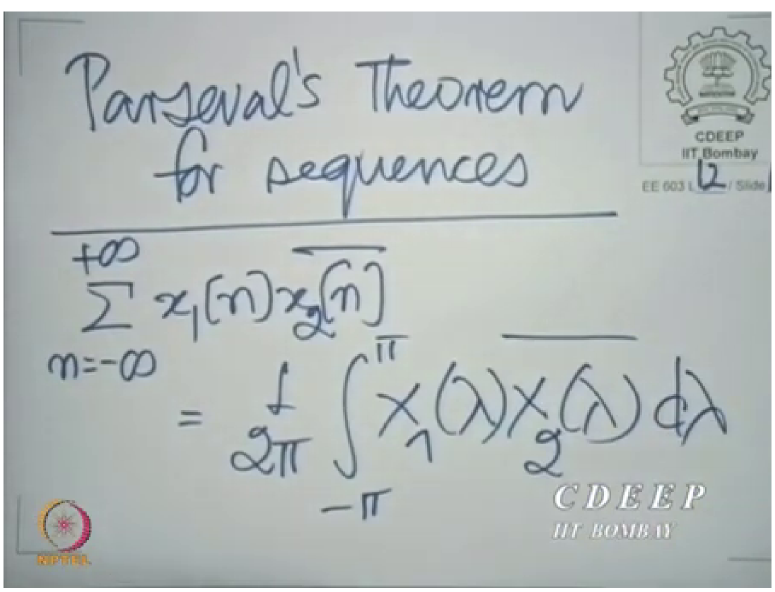

For sequences and we are saying in effect that the dot product of the sequences is unchanged whether you look at it in time or in frequency. And there is one specific consequence of this dot product which we shall see in a minute. That consequence follows if you take $x_1[n]$ equal to $x_2[n]$.

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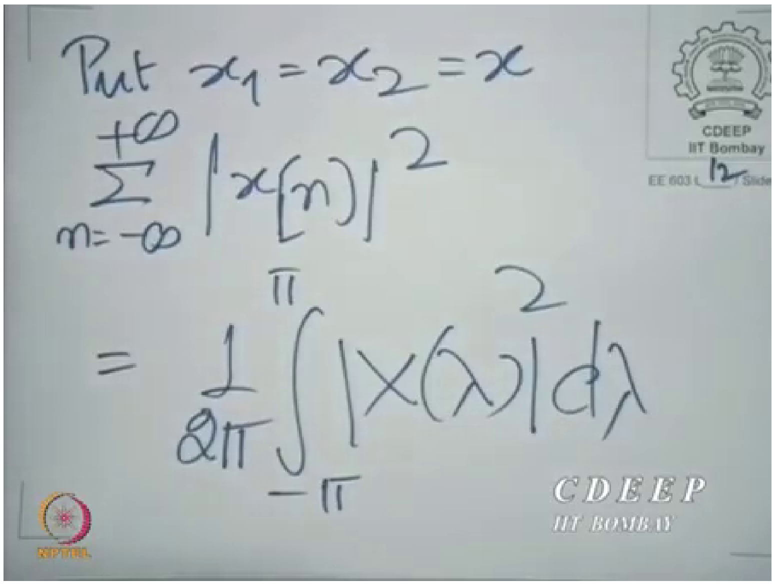

We get $\sum_{n=-\infty}^{\infty} |x[n]|^2$. Lets call this x so $|x[n]|^2$ is $\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\lambda)|^2 d\lambda$. And here we have the same specific interpretation as we did for the sequence. When we took both the sequences the same in the dot product. The dot product of a sequence with itself of course gives you the magnitude or the norm squared of that sequence. We said that when we talk about generalized vectors sequences are generalized vectors.

We do not like to use the word magnitude anymore as we do for finite dimensional vectors. We call it the norm. So, we say the norm and of course I also to be very careful said that we should call it the L2 norm. That was a technical point. Anyway, you know for the moment let us just call it the norm. So, this is the norm squared you know $\sum_{n=-\infty}^{\infty} |x[n]|^2$ is the norm square of the sequence. And that same norm squared can be calculated in the frequency domain.

Of course the physical interpretation is you have taken the dot product of the sequence with itself so you are getting the magnitude squared. So, magnitude squared is like the norm squared. And of course the same norm squared is calculable from the frequency domain. Now, this gives a very beautiful interpretation. You see what is the physical interpretation of this norm squared?

It is easier to see if we look in the frequency domain. You know what we have said in effect is if you had a band limited signal as you started off with when you sampled, then this is actually giving you the integral of the magnitude squared of the Fourier transform of that band limited signal.

Now, you see suppose that band limited signal were a voltage signal and you applied that voltage across a 1 Ω resistance. The total amount of energy that that signal would dissipate in the 1 Ω resistance is the quantity that you have on the right hand side here.

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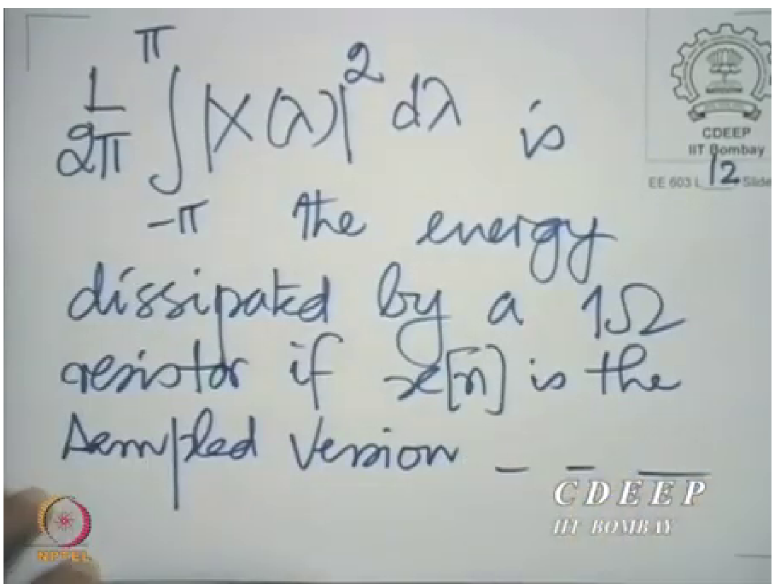

This quantity $\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\lambda)|^2 d\lambda$ is the energy that would be dissipated, is the energy dissipated by a 1 Ω resistance if x[n] is the sampled version let us continue.

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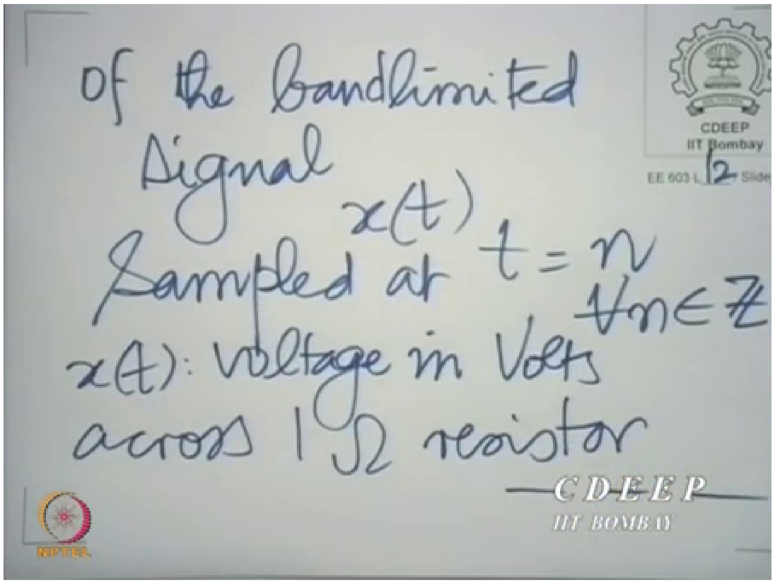

if x[n] is the sampled version of the band limited signal x(t) sampled at $t = n \; \forall \, n \in \mathbb{Z}$. x(t) is the voltage in volts across the 1 Ω resistor. And the Continuous Time Fourier Transform of x(t) is $X(\Omega)$ which is equal to you see.

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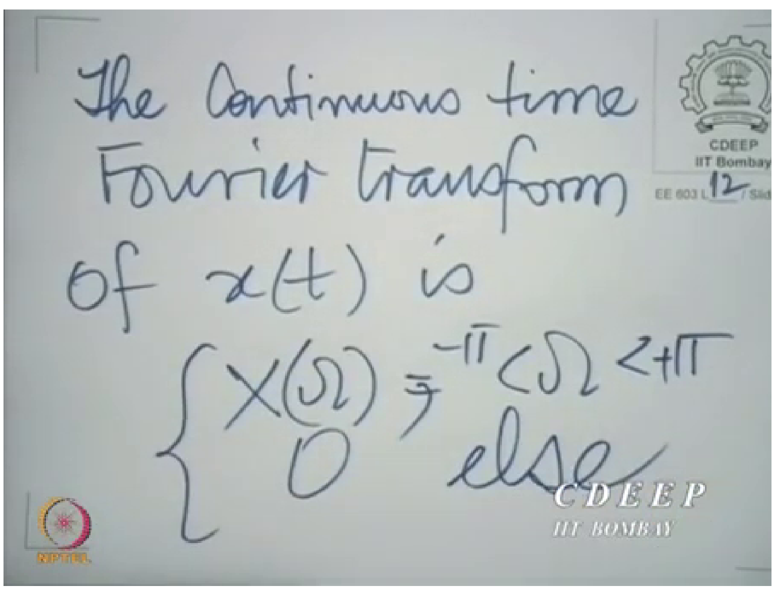

It is $X(\Omega)$ for $-\pi \leq \Omega \leq \pi$. And 0 else. So, what we are saying is restrict the Discrete Time Fourier Transform only to the interval $(-\pi, \pi)$. In fact here I introduce this idea of the underlying continuous time signal in general. Whenever you have a Discrete Time Fourier Transform, restrict that discrete time Fourier transform to the interval $(-\pi, \pi)$. And cut off all the rest, make it all 0. What do you get? You get the original band limited signal which was then sampled at the integer points to give you the samples x[n]. Is that correct?

If you take this restricted Fourier Transform band limited signal if you sampled it at the integer points you are sampling at a sampling rate of 1. What would happen in the Fourier domain? You would take the original spectrum translated by every multiple of 2π and add these translates. And therefore you see what let me let me try and explain graphically.

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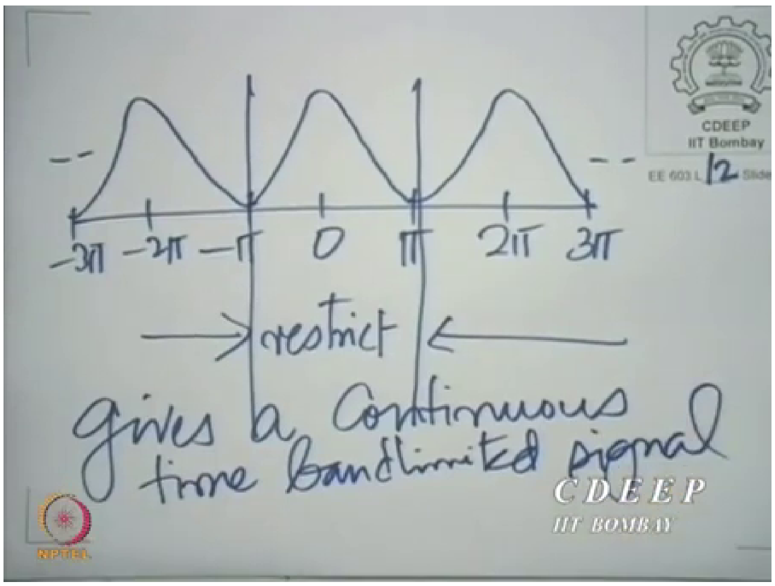

I am saying this is the DTFT. I am showing a couple of periods. Whatever it be. This is of course periodic with period 2π. Now, restrict it. This gives a continuous time band limited signal. Yes, of course.

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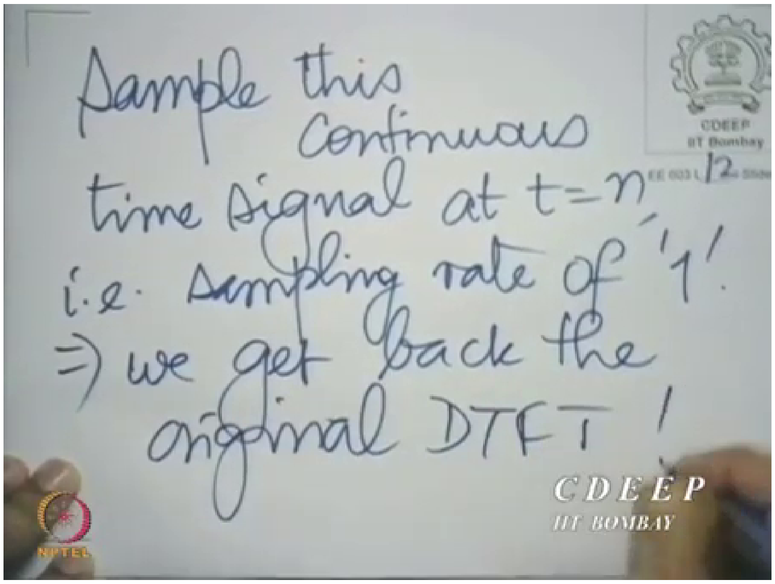

Now, sample this continuous time signal at $t = n$ that is a sampling rate of 1. So, we get back the original DTFT because, when you sample at a sampling rate of 1 the effect is to shift the original spectrum by every multiple of $2\pi/1$ which is 2π on the frequency axis.

And these translates are added. You get back the original DTFT. So, this is the notion of the underlying continuous time signal. And what I am saying is think of the underlying continuous time signal for the sequence x[n].

Let that continuous time signal be the voltage in volts applied across a $1\,\Omega$ resistance. The energy dissipated in that $1\,\Omega$ resistance is $\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\lambda)|^2 d\lambda$. And the same energy is of course obtained by taking the mod square of the samples and adding. So, both of them correspond to an energy. Let us go back to this. So, what we are saying is both of these quantities are now in energy.

This is an energy as seen in frequency and the same energy is seen in time. So, we also call this the energy of the sequence for the obvious reason that I have just explained. And now we also have an interpretation for just the integrand here. The integral of course we have interpreted it is the total energy. But, we also have an interpretation for the integrand $|X(\lambda)|^2$. The integrand is the way in which the energy is distributed over the frequency axis. So, it is called the energy spectral density.

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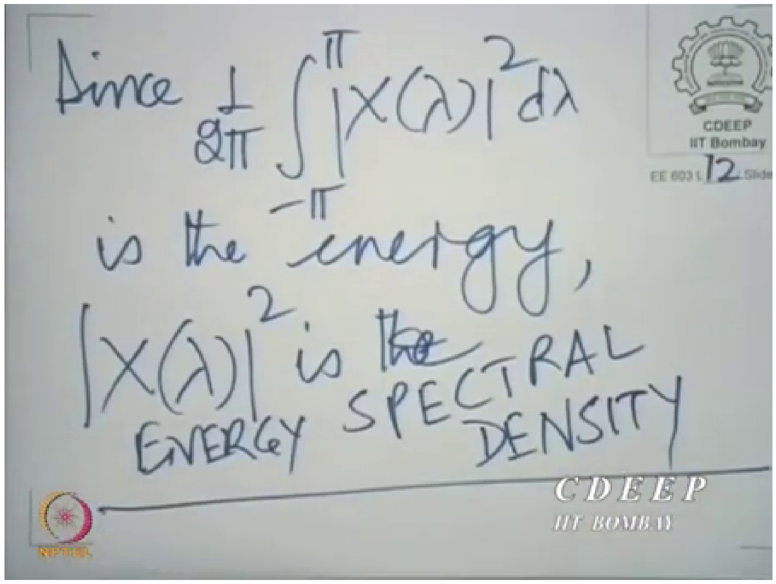

Since, $\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\lambda)|^2 d\lambda$ is the energy. $|X(\lambda)|^2$ squared is called the energy spectral density. Now, this is not very different from what we understand density to be, you say in the case of a mass, how would you calculate the mass of an object? If you knew its density at different points you would integrate the density over the volume on which the mass lies and that gives you back the mass.

And of course, the density does not have to be uniform. So, here too the density may not be uniform for all frequencies. And essentially, when you integrate the density you get back the mass or the energy as the case might be. So, this is what is this clear to everybody so we must this is a very important notion in signal processing. The notion of energy spectral density. now of course though this is not very commonly used, yes there is a question. Yes.

Student: (())(12:33)

Professor: So, the question is when you restrict so let me try and explain the question. The question is when you restrict the seek, the spectrum only to $(-\pi, \pi)$ we are going from discrete time to continuous time. How are we drawing the parallel between discrete time and continuous time?

Now, you see when you restrict the Discrete Time Fourier Transform to between $(-\pi, \pi)$ it cannot correspond to a sequence anymore. It cannot be the discrete time Fourier transform for sequence because, it is no longer periodic with period 2π.

So, it is in fact it is it is a band limited Fourier transform so it is about it corresponds to a band limited signal. Now, which band limited signal that band limited signal which you had sampled at a sampling rate of 1 to get the sequence that you have. Why, why are you saying that it is that that particular band limited signal? Because, if you took that band limited signal and sampled it at a sampling rate of 1, what would happen in the frequency domain?

You take the original spectrum translate by every multiple of $2\pi/1$ because, the sampling rate is 1, sampling time is 1. So, $2\pi/1$ and move these this original spectrum by every multiple of $2\pi/1$ and add these translates. That is how you get the DTFT. And of course, there is a 1 to 1 correspondence between the DTFT and the sequence. So, if you are getting back the DTFT by sampling this signal at a sampling rate of 1 then indeed you know there is a 1 to 1 correspondence between these samples and that underlying band limited signal.

So, for every for every DTFT there is an underlying band limited signal that you can think of and that is often useful to do. Is that clear?

Student: (())(14:45)

Professor: So, his question is then what is the filter that you would apply? Well, you see you would apply the same I assume you are asking, what is the filter that you would apply to reconstruct the original signal from its samples? Is that the question? So, its the same I mean you know when you have a band limited signal the filter that you would apply is a low pass filter with a cut off at π.

So, you know so it would I mean if you want to reconstruct the signal from its samples you could think of it as putting an ideal filter an ideal low pass filter with a cut off at π. That is what you would do. Any other questions?

Student: (())(15:27)

Professor: Well, so the question is would this analogy hold true only if the signal is band limited? Now, you see we are assuming there is no aliasing or the other way of thinking about it is the same samples can correspond to many signals. We know that that is what aliasing means. Out of them we are choosing the signal where there is no aliasing. There are of course many signals to which these samples could correspond.

But, we are choosing that signal for which there is no aliasing which must of course be the band limited signal band limited to pi. Any other questions? Is a good question. Yes, yes, yes there is a question. The question is this can be written only mathematically and there is no practical meaning? That is not correct. You see we just gave a practical interpretation. We said that you know if you took the underlying band limited signal and looked at the energy across a 1 Ω resistance you get the energy. Isn't it?

So, it is not just it is not just it has of course a mathematical meaning but it also has a very important significance. So, the word energy actually is very meaningful it is the energy in many situations. So, you know it is it is so here the in fact that is how the word energy is used it has energy has implications in terms of energy as we understand it in physics. So, if you have a signal whether it is a voltage signal current signal pressure signal any other kind of signal which is sensed this could actually correspond to the physical energy that is being delivered.

That is not correct. It has a practical meaning. You see one must remember and this is the comment that I must make in general. It is very important to see the congruence between what we do on paper in terms of algebra and math and what happens in practice. And that congruence is there. So, one must not assume that one is working out certain certain things on paper and it has no practical meaning it every everything that we do here has a practical implication. Yes. Any other questions? Yes.

Student: (())(17:45)

Professor: So, the question is every or any meaningful real signal is expected to have finite energy? Does it mean that every real signal must be band limited. Now, I will give you an analogy again to explain why the statement is not correct. You see we of course know that Indians would always live only for a finite time. We cannot live for of course there are legends that people live for thousand years and so on but I do not know of such big things happening in the last century or maybe last millennium.

So, let us assume that people have a lifetime for 200 years at most right 200 250 years let us say. So, Indians do live for a finite time isn't it? Now, you know that is also true of all people in the world we have not yet come across a person in the world who has lived for infinite time, isn't it. So, now you know if you say that now what are we saying what are we trying to say here we are saying see human being so when lets bring an analogy between band limitedness and human beings.

And again band limited so what what is the thing we are we are trying to we are seeing whether band limitedness and finite energy or reality or practicality is the same. Now, band limited it is like asking whether humanness and Indian-ness is the same. In both cases there is limited amount of life. Humanness and Indian-ness is not the same. Humanness is much bigger than Indian-ness. But, both of them have the property that they live only for a finite time.

Similarly, finite energy is broader than band limited. You can have a finite energy signal which is not band limited. You can have a you can have a signal whose spectrum extends all over the frequency axis. You see take for example

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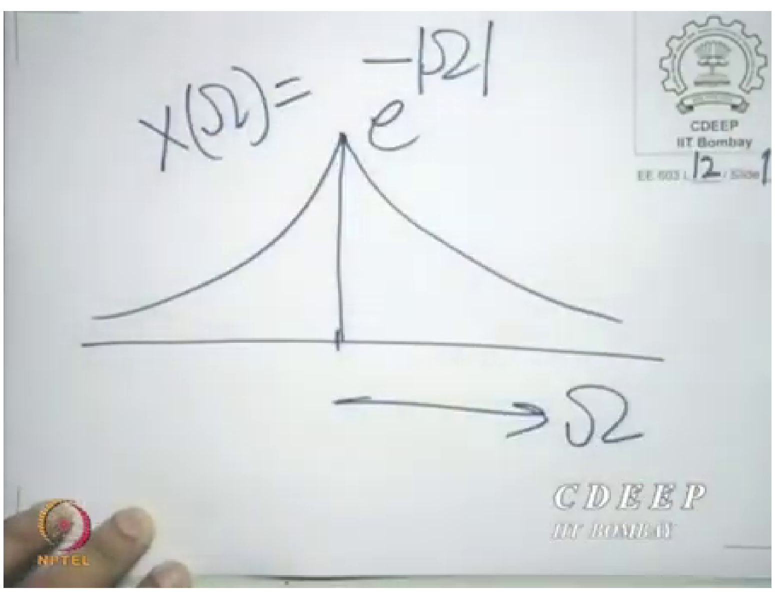

The signal which has a spectrum which decays like this. So, let this be the frequency domain and you have the spectrum e raised to the power minus mod omega. This is the spectrum of a signal a continuous time signal. Let us call this $X(\Omega)$ and of course the energy in this is easily seen to be finite.

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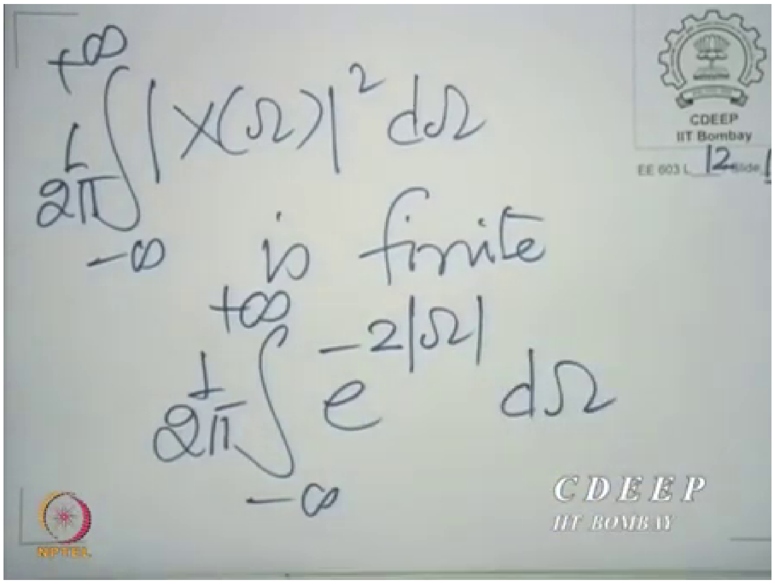

$\int_{-\infty}^{\infty}|X(\Omega)|^2 d\Omega$ is of course finite. Its integrated of indeed $\int_{-\infty}^{\infty}e^{-2|\Omega|}d\Omega$ and this is a very easy integral to evaluate. Isn't it? It is finite. However, this is not band limited, this is like being human. And band limited is like being Indian in that. Both of them of course live for a finite time but being human does not necessarily mean being Indian.

Being finite energy does not necessarily mean being band limited. Of course, being band limited and if the spectrum does not have what are called you know I mean you are not taking a casual you know an extreme example. You know what is called I mean you know you are not taking a pathological example of a band limited signal where for example you have an impulse a continuous time impulse or something in between then a band limited signal will of course be finite energy in most cases.

Except of course, if there are impulses sitting inside continuous time impulses. So, that is that is an important question one must be clear that band limitedness and finite energy is not the same thing. Finite energy is broader than land limitedness. Is an important question I am glad it was asked. Any other questions? Alright. So, then we come to this important conclusion that $|X(\Omega)|^2 d\Omega$ is the energy spectral density.

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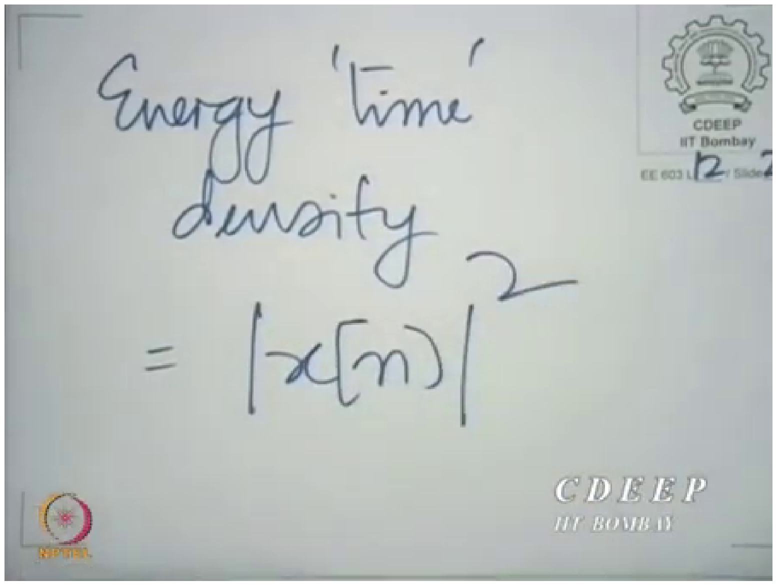

Now, one comment just as you can talk about the energy spectral density technically you should also be able to talk about the energy time density. So, of course we do not use this term too frequently and the energy time density or the way the energy is spread over time is $|x[n]|^2$. Notionally, that is of course correct but very rarely do we use this term.

The term energy spectral density is used more frequently than energy time density. Yes. There is a question.

Student: (())(22:51)

Professor: So, the question is when we calculate the energy time density we are averaging or we are taking the sampling rate to be 1, is that the question? Indeed, we have normalized is not it. We have normalized the sampling rate so we have normalized the sum we have normalized the time axis and we have also normalized the frequency axis.

So, of course if you when you go back to the actual times and frequencies you need to bring in that normalization factor. So, we are working with normalized time and normalized frequency. Alright. So, let us just take 1 example of a discrete time Fourier transform before we conclude.

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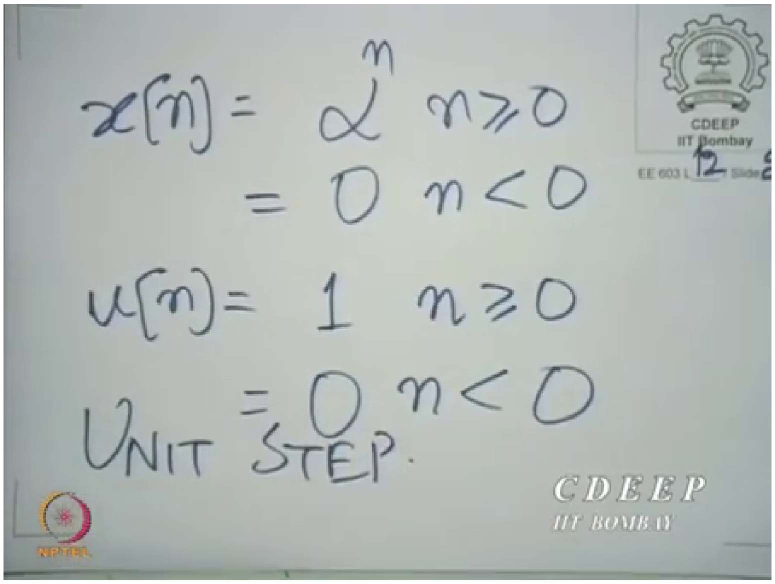

Take the sequence $x[n] = \alpha^{n}\ \forall\, n \geq 0$ and $0\ \forall\, n < 0$. Now, we define the sequence $u[n] = 1\ \forall\, n \geq 0$ and $0\ \forall\, n < 0$. This is a sequence which we call the unit step. The sequence unit step is going to use it is going to come frequently in our discussions. Interestingly, the unit step itself does not have a Discrete Time Fourier transform. The Discrete Time Fourier transform of the unit step would not converge.

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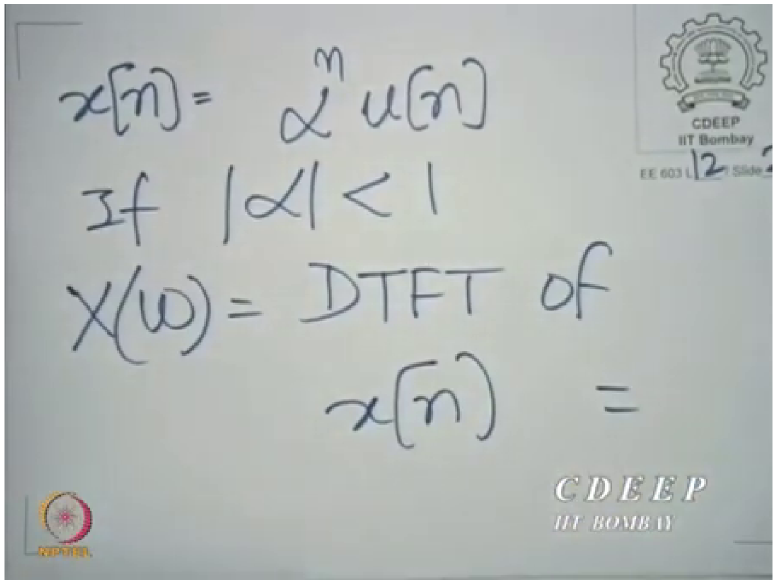

But, multiply the unit step by an exponential as we are doing here as we can see x[n] can be written $\alpha^{n} u[n]$. And if $|\alpha| < 1$ then we can calculate the Discrete Time Fourier transform of x[n]. And that is very easy to do. It turns out to be.

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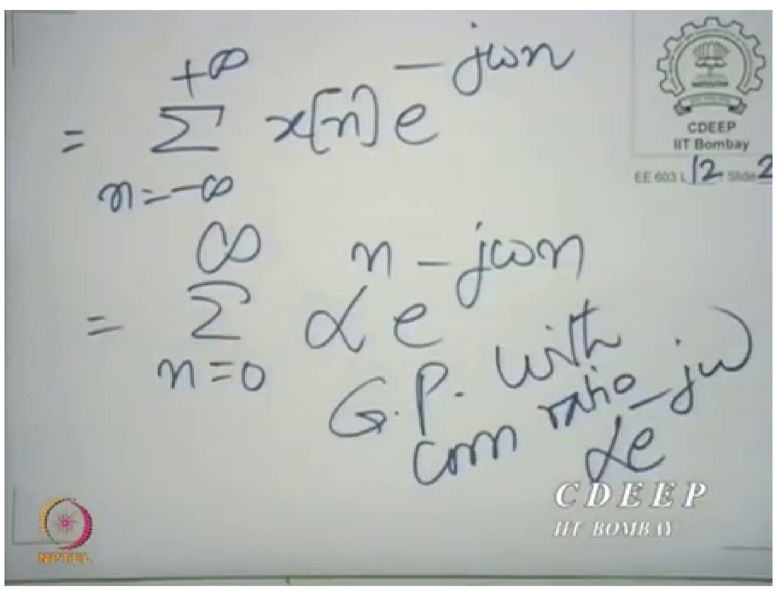

$\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$. Which is summation $\sum_{n=0}^{\infty} \alpha^{n} e^{-j\omega n}$. That is a very easy integral to evaluates a geometric progression with common ratio of $\alpha e^{-j\omega}$. And the modulus of this common ratio is the modulus of α. And since $|\alpha| < 1$ this is a convergent sum.

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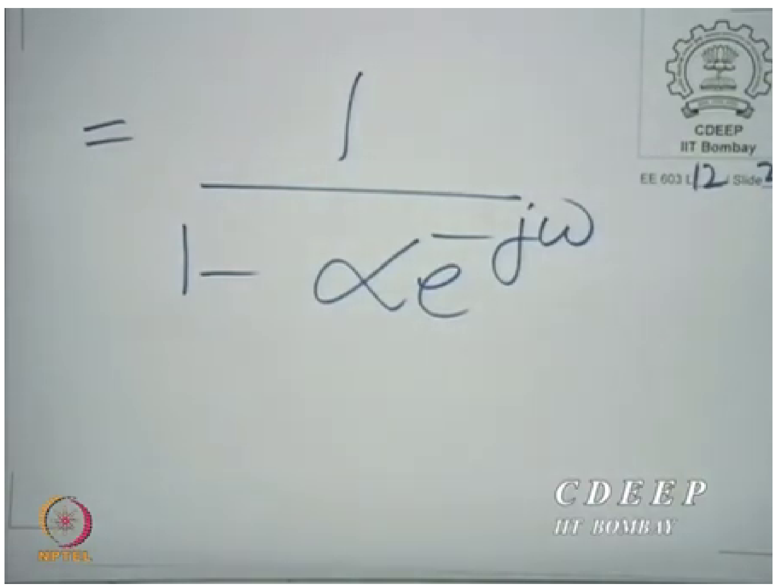

And it converges obviously to $\frac{1}{1-\alpha e^{-j\omega}}$. So, this is the Discrete Time Fourier transform of a 1 sided exponential. I may also mention that if you have a finite length sequence.

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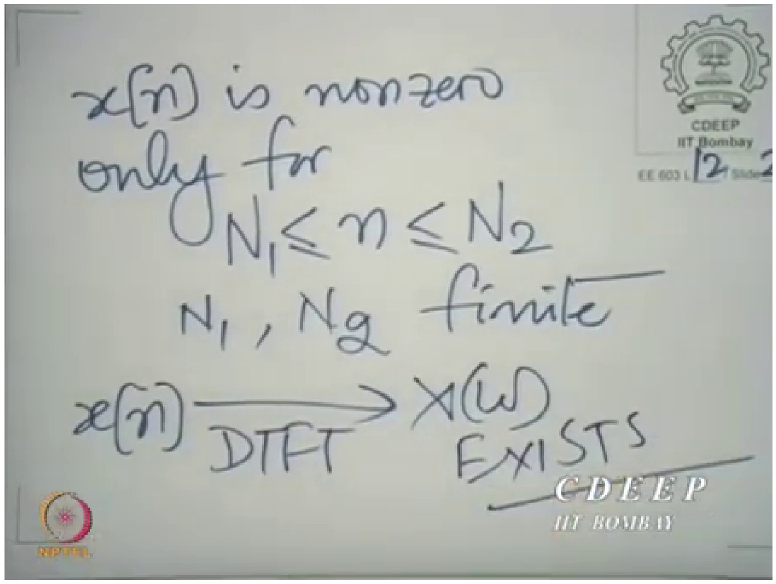

If x[n] is finite length x[n] is non zero only for $N_1 \leq n \leq N_2$. N_1 is strictly N_1 and N_2 are finite. They can be positive or negative it does not matter. Then of course the DTFT of x[n] always exists that is easy to see because, it is a finite summation. In fact, it is not very difficult also to see that if you took x[n] here to be the impulse response of a linear shift invariant system then that system is bound to be stable.

That is because the absolute sum of this impulse response is necessarily going to be finite can be infinite. There is only a finite number of samples and of course we assume each sample is finite. So, a finite length sequence always has a discrete time Fourier transform. We have given an example of an infinite length sequence which also has a discrete time Fourier transform. In the next lecture we shall take an example of an infinite length sequence which does not have a discrete time Fourier transform.

In fact we did it before but we will take it again and we will also use that to build a new transform called the z transform. With that then we conclude the lecture. Thank you.