Digital Signal processing & Its Applications
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Lecture – 13 d
Region of Convergence

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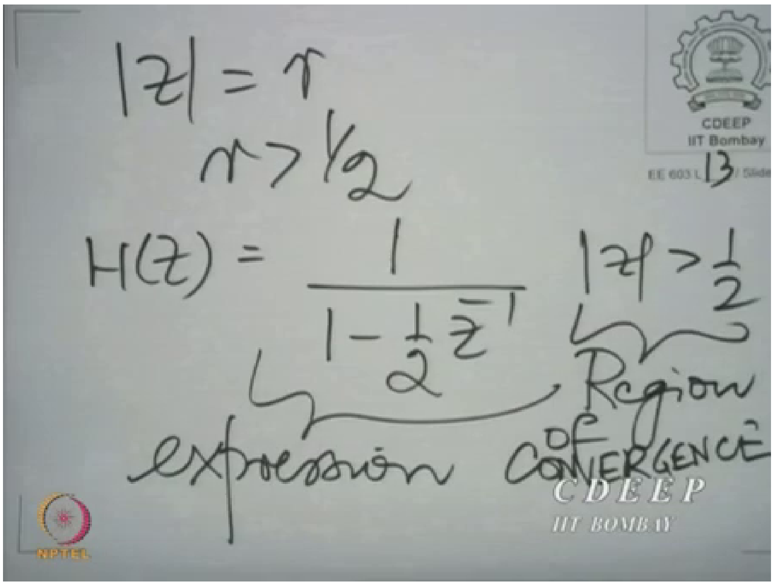

Now, what can we say about the region of convergence. The region of convergence has nothing to do with ω. It has only to do with *r*. So, the region of convergence has always to do with circles centred at the origin. What is *r*, *r* is the radial distance of the complex number from the origin.

So, the convergence or non-convergence of the Z-transform has only to do with the radial distance, with the radial distance of the points that you are considering from the origin of the Z-plane. And therefore the regions of convergence of the Z-transform are always essentially between concentric circles with the centers located at the origin that is a very important observation.

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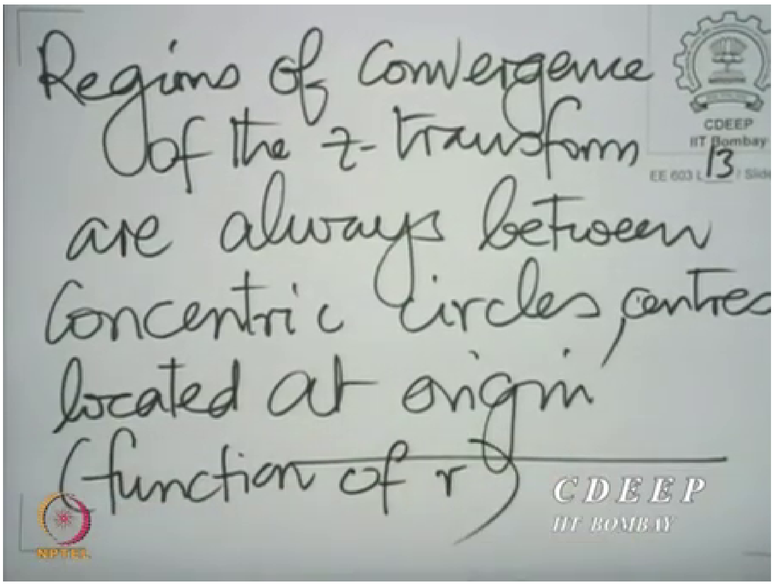

Regions of convergence of the Z-transform are always between concentric circles located with centers located at the origin. So, essentially a functions, function of r not of ω. Now, it should be noted that one of these circles could go all the way up to ∞ radius. And one of the circles can go all the way down to *0* radius.

So, circles do not need to have non-zero radius or finite radius. Also it is a moot point whether the circles themselves are a part of the region of convergence. The circles that bound the region of convergence may or may not be a part of the region of convergence. That needs to be seen case by case. This is particularly a tricky point when you are considering ∞ radius and *0* radius.

There is an issue there. Let us take an example. Let us take the sequence $h[n] = \delta[n]$.

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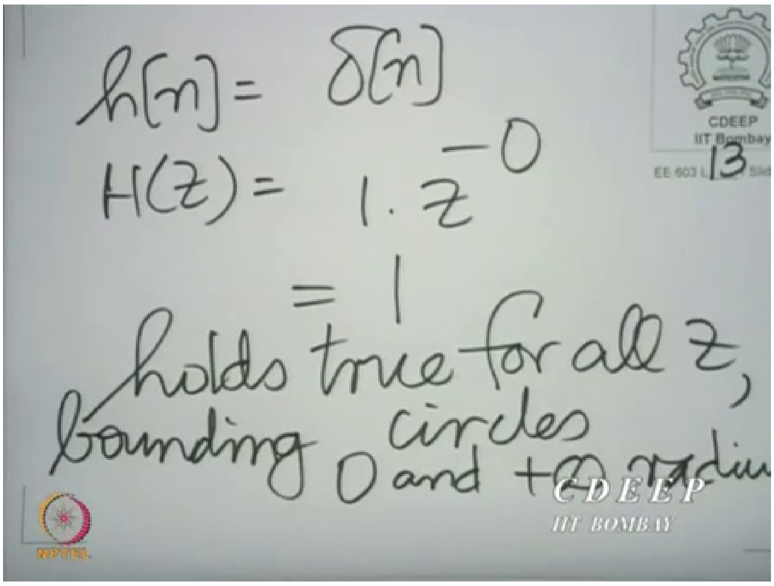

The simplest possible sequence that we can conceive of the unit impulse sequence. The Z-transform of the sequence is very easy to see. It is essentially $1 \cdot z^{-0}$which is *1* and this holds true for $\forall\ z$. And therefore the, the bounding circles have *0* and ∞ radius.

Essentially, the region of convergence lies between the concentric circles of radius *0* and radius $+ \infty$. The entire Z-plane is the region of convergence here. However, in this case the two boundaries are also part of the region of convergence. The circle of radius *0* and the circle of radius ∞ are parts of the region of convergence.

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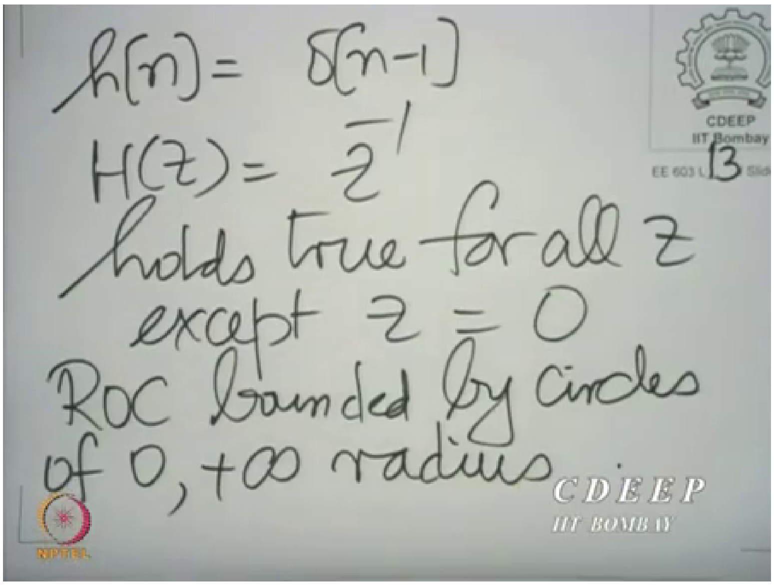

Now, let us take the sequence $h[n] = \delta[n - 1]$. It is very easy to see that the Z-transform is z^{-1} and it holds true for $\forall\ z$ but $z = 0$. Again the region of convergence is bounded by circles of *0* and ∞ radius. However, the circle of radius *0* is not included in the region of convergence. Whereas, a circle with radius ∞ is.

So, you see it's a very, it is a very tricky point. The bounding circles may or may not be in the region of convergence that must be carefully seen. Again let us take another such example.

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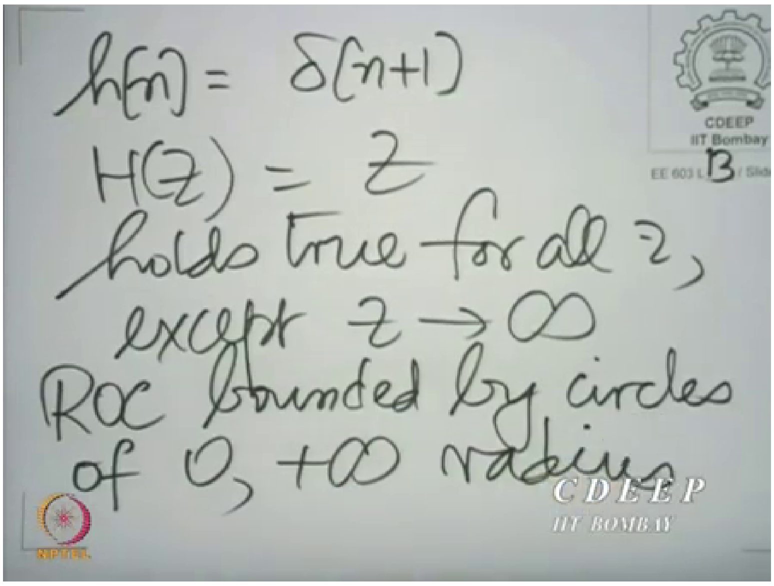

Let us take $h[n] = \delta[n + 1]$. And it is very easy to see that the Z-transform is simply z holds true for $\forall\, z$ except $z \rightarrow \infty$. So, the ROC is again bounded by circles with 0 and $+\ \infty$ radius, very interesting. So, here we have all of the Z-plane included the same boundaries. The boundary $z = 0$or $r = 0$ is included in the region of convergence.

But, the boundary, the circle with ∞ radius is not included. I must also draw the attention of the class to one difference between the notion of ∞ for reals and ∞ for complex numbers. The ∞ in complex numbers is not a point or a set of points it is a contour in fact it is several contours. Essentially, an ever enlarging contour takes you to the ∞ on the complex plane.

So, ∞'s on the complex plane are several ever enlarging contours. The circle is a possibility. As the circle enlarges and continues to enlarge it takes you to the ∞ on the complex plane. Any other contour to any other closed contour as it expands indefinitely takes you to an ∞ in the complex plane. So, the ∞, the notion of ∞ in the complex plane is a little different, little more general, little more you know intricate than the notion of ∞ on the reals.

Anyway, so this is essentially about the reach so you see the Z-transform now this of course illustrates that you have boundaries which are concentric circles. The boundaries need to be checked for inclusion or exclusion. But now we need to explain why it is important to specify the region of convergence along with an expression. Although, sometimes we take that for granted it is not correct to do so.

One must always specify a region of convergence along with an expression. Let us take the very same sequence that we had. Well, let us take the very same Z-transform that we had a minute ago.

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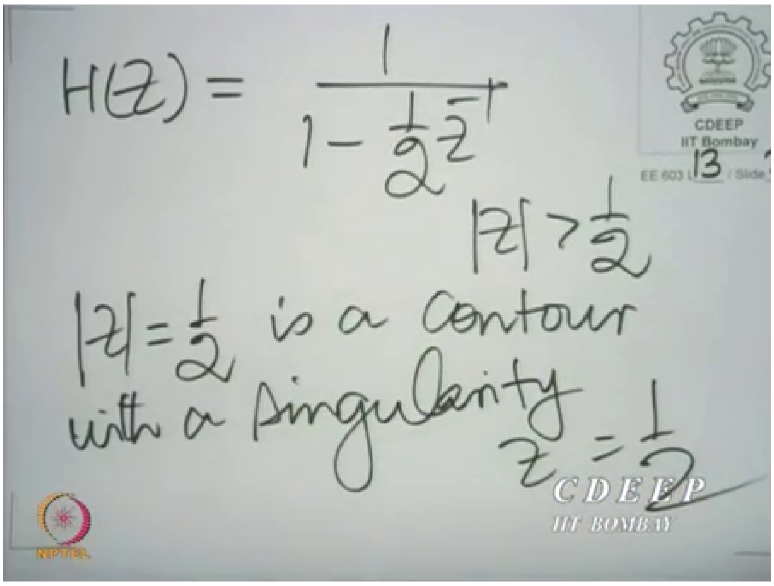

We had the Z-transform $\left(\frac{1}{1-\left(\frac{1}{2}\right)z^{-1}}\right)$, $|z| \ > \left(\frac{1}{2}\right)$. Now, here of course it is very clear that $z = \left(\frac{1}{2}\right)$and that means $|z| \ = \left(\frac{1}{2}\right)$ can never be in the region of convergence here. That is because $|z| \ = \left(\frac{1}{2}\right)$ is a singularity is a contour with a singularity. You see at $|z| \ = \left(\frac{1}{2}\right)$ this expression diverges this expression has no meaning the denominator becomes *0*.

That is called the point of singularity where the expression is unevaluable. Now, you see when a contour holds a singularity on it then that contour cannot be in the region of convergence. A contour which hosts a singularity cannot be in the region of convergence. And therefore, we have only two possible regions. You see the contour must be excluded.

When there is a singularity on a contour in the expression that contour cannot be in the region of convergence. So, the only choice is as far as the region of convergence must be you see it must be what is called a simply connected region. You cannot have parts of it you know you cannot have a region of convergence separate with some you know part which is in no man's land. No, it must be a simply connected region.

That means if you take any two points in that region any contour joining the two points must all lie within the region. So, for example in this case it is very clear that we cannot allow the, the circle with radius half to be in the region of convergence. So, either you have the region of convergence within $|z| = \left(\frac{1}{2}\right)$ or outside of $|z| = \left(\frac{1}{2}\right)$. And we have chosen the region of convergence outside $|z| = \left(\frac{1}{2}\right)$. But we could choose it within $|z| = \left(\frac{1}{2}\right)$ too. So, let us see what happens if you take the region of convergence to be $|z| < \left(\frac{1}{2}\right)$.

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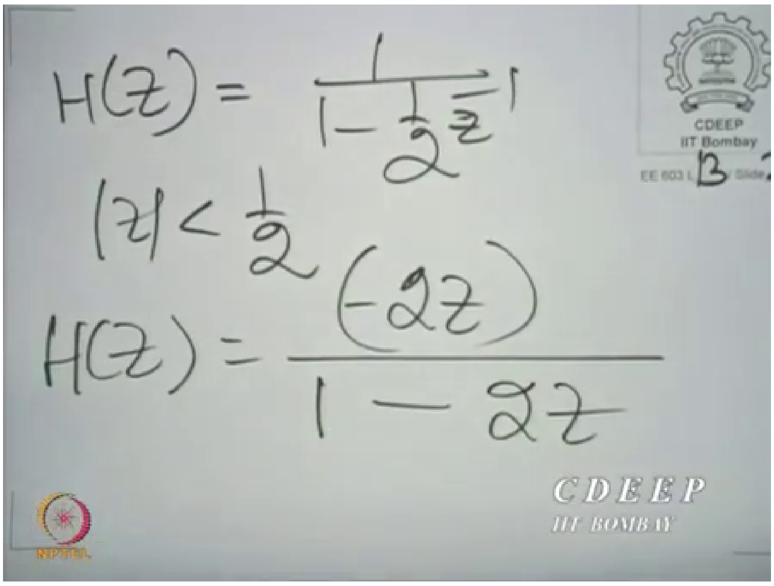

In that case what we are essentially saying is we multiply the numerator and denominator by $2z$. In fact, $-\ 2z$ if you please. And in fact, it is very clear that $|2\ z| < l$. Since, $|2\ z| < l$, $\left(\frac{1}{1-2z}\right)$ can be expanded using the idea of a geometric progression.

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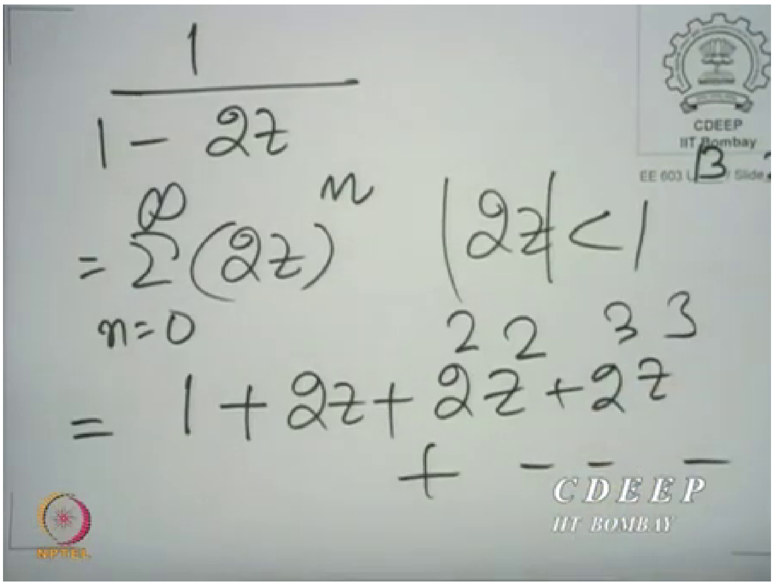

$\left(\frac{1}{1-2z}\right)$ can be expanded as $\sum_{n=0}^{\infty}(2z)^n$ and$|2z| < 1$. By using the simple idea of a geometric progression. And we can write this out explicitly. So, we have 2^0z^0 so $1 + 2z + 2^2z^2 + \ldots$. And of course we can now see $H(z)$.

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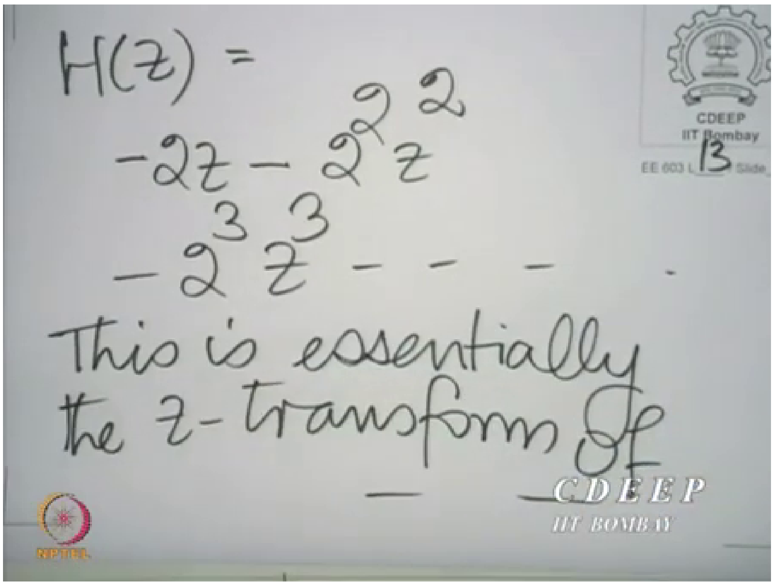

$H(z)$ is very clearly $-2z-2^2z^2-2^3z^3....$ And it is very easy to see that this, this is essentially the Z-transform of the sequence.

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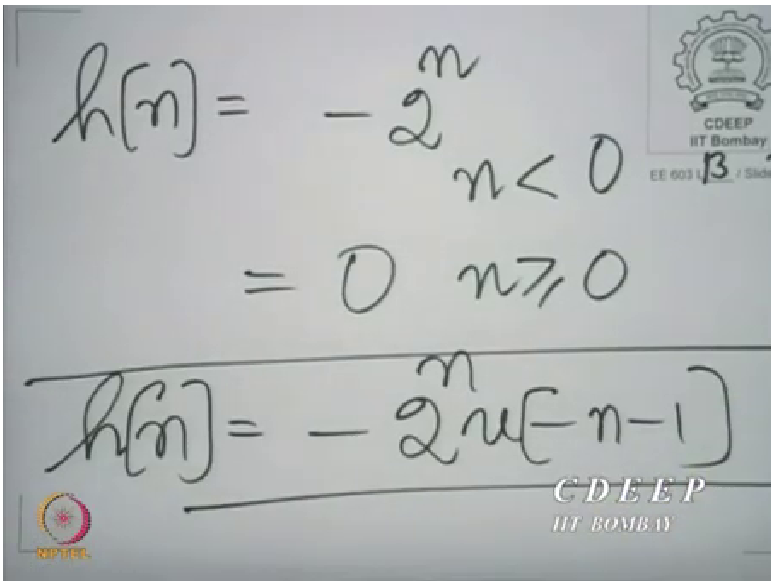

$h[n] = \ - 2n$ for $n \ < \ 0$strictly and *0* for $n >= \ 0$. In fact, I leave it to you to verify that this can be written as $h[n] = \ - 2^n u[- \ n \ - \ 1]$. I mean the way we arrive to that conclusion is that you have a summation here you can identify. *z* is the term or *z* is contributed by the sample at $n = \ - \ 1$.

z^2 is contributed by the sample at $n = \ - \ 2$and $- \ 3$and so on so forth. And of course, you can see there are no samples from *0* onwards. And of course this is a matter in point if you look at $u[- \ n \ - \ 1]$.

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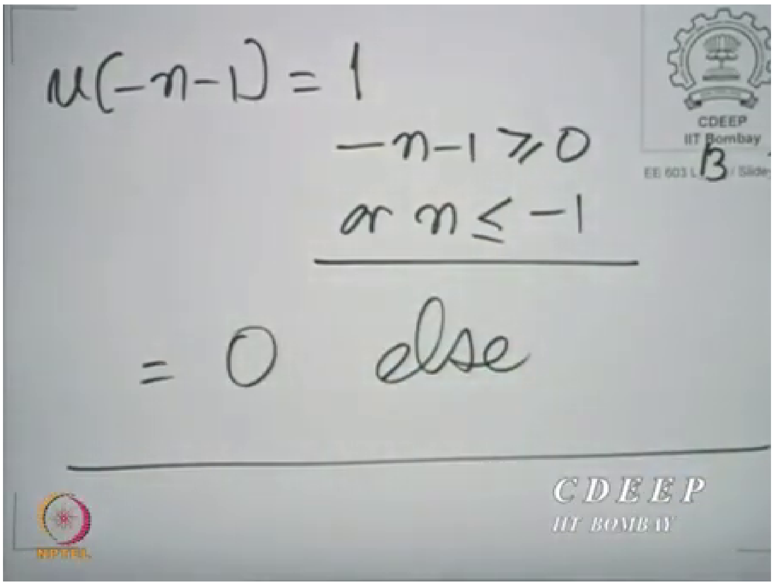

Is expected to be *1*. Whenever $-\ n\ -\ 1\ \geq\ 0$ or $n\ \leq\ 1$. Is not it. And *0* else as we expect. Is that correct? So, the same expression $\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)$ when associated with the region of convergence $|z|\ > \frac{1}{2}$ leads to one sequence and when associated with the region of convergence $|z|\ < \frac{1}{2}$ leads to another.

Now, in fact we can have even more complicated situation we can have more than two possibilities for the region of convergence that we see later. And then we have more than two possible sequences if we do not specify the region of convergence properly. But, we will come to that later we will come to that or we will see examples of that later when we look at a few more properties of the Z-transform.