Digital Signal Processing & Its Applications
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Lecture 13 e
Properties of Z Transform Its Applications

Now, you see, let us look at the properties of Z-transform as we did for the discrete time fourier transform.

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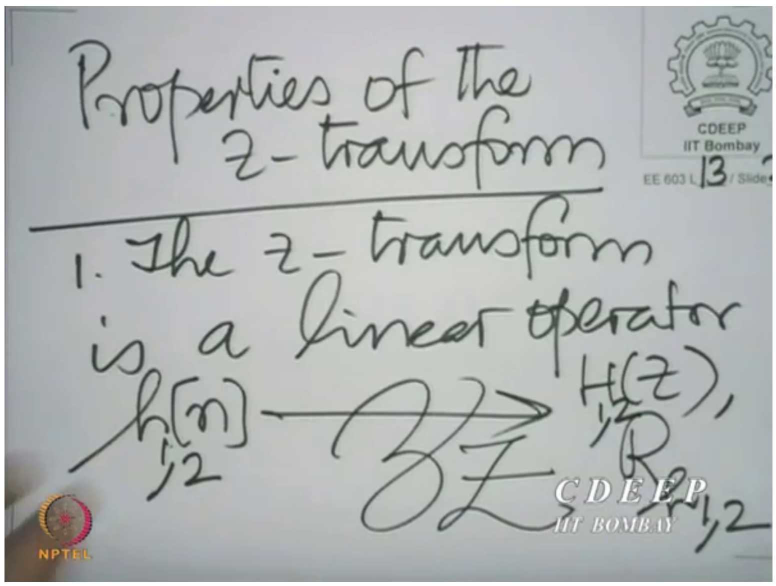

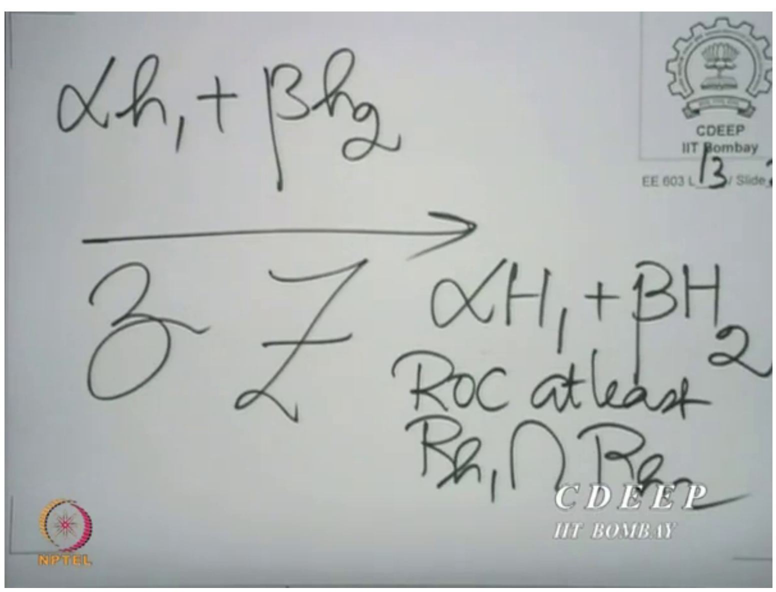

Let us, first look at the linearity or otherwise now it is in fact, it is almost trivial to show that the Z-transform is linear or it is a linear operator, what I mean is if you think of the Z-transform as an operator, which takes you from the set of sequences and you know, the Z-transform is denoted this way a $\underset{z}{\rightarrow}$ often. It is also sometimes denoted like this Ƶ, both of these are used as symbols for saying take the Z-transform.

If $h[n]$ leads you to $H(z)$ with an ROC R_h. So, you know, he was always write it like this an expression $H(z)$with an ROC or region of convergence R_h, $R_h$is a region between two concentric circles, is that right? So, if $h[n]$ and in fact now we use the same shorthand notation if $h_{1,2}[n]$ respectively have the Z-transforms $H_{1,2}(z)$ with regions of convergence $R_{h_{1,2}}$ then $\alpha h_1 + \beta h_2$ would have the Z-transform, you can write Z-transform $\alpha H_1 + \beta H_2$. Now, comes the question of region of convergence. What do you think we can say about the region of convergence?

Students: ...

Professor: Vikram M. Gadre: The region of convergence is at least the intersection of R_{h_1} and R_{h_2} it could be bigger ROC at least R_{h_1} intersection R_{h_2} could be bigger. This is often the case with the regions of convergence. The region of convergence is of course at least the intersection. It could expand beyond the intersection and the expansion takes place because the trouble created by one sequence can be outdone by the other.

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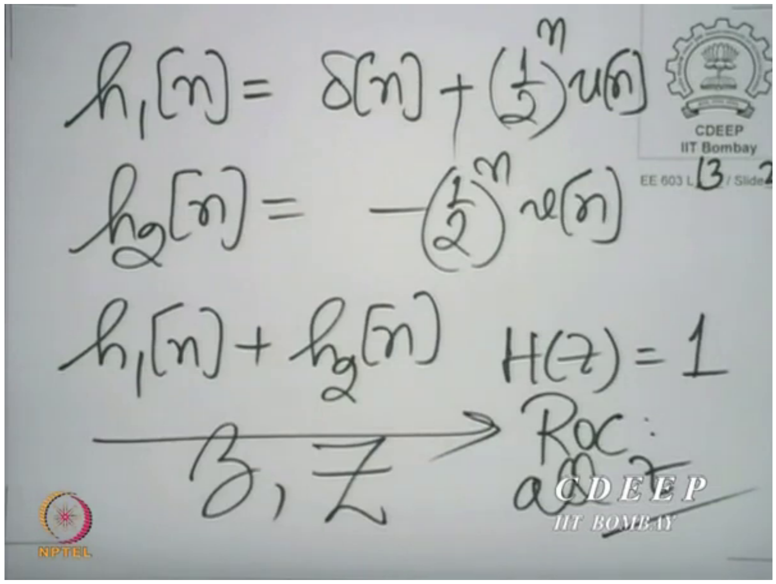

Now, let us take an example. You see let us take the sequence $h_1[n]$ is $\delta[n] + \left(\frac{1}{2}\right)^n u[n]$ and let $h_2[n]$ be $-\left(\frac{1}{2}\right)^n u[n]$, it is very easy to see that of course, $h_1[n] + h_2[n]$has the region of convergence is of course the Z-transform is essentially the Z-transform of $\delta[n]$ and that is *1* for $\forall$ $z$and the region of convergence is all z.

On the other hand, the region of convergence of either $h_1$or $h_2$is $|z| > \frac{1}{2}$. So, here the intersection is $|z| > \frac{1}{2}$ but the region of convergence has expanded beyond the intersection because the trouble created by one sequence has been outdone by the other, is that right? So much, so, for the linearity of the Z-transform. Now, we look at some other properties.

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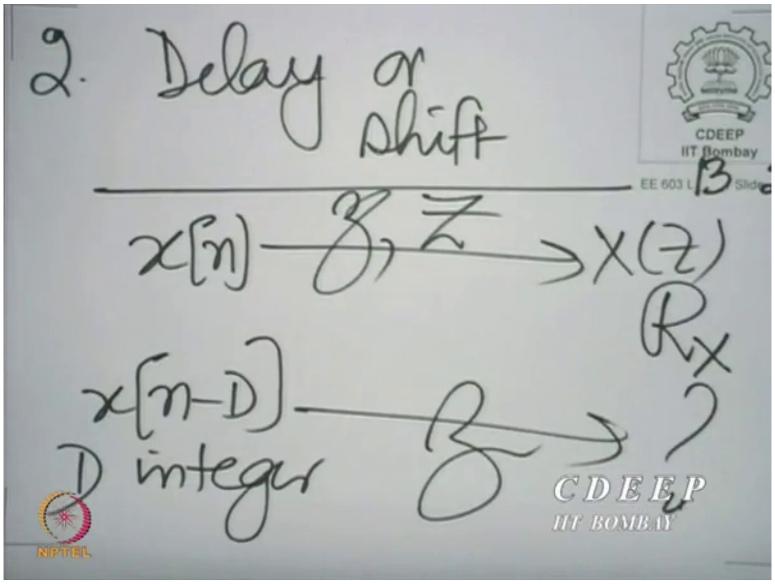

What happens when we delay? Delay or shifts thats very easy. So, if $x[n]$ has the Z-transform therefore variety let us use $x[n]$ will use one of them in future the Z-transform $X(z)$with a region of convergence R_X. What is the Z-transform of $x[n - D]$? D is an integer. It is very easy to answer this question. We only need to write down the summation.

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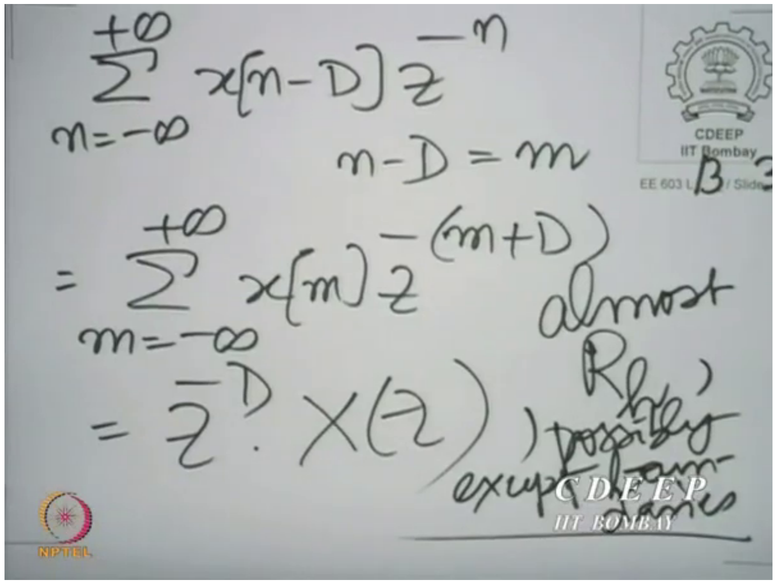

The $\sum_{n=-\infty}^{+\infty} x[n - D]$ is z^{-D} you see can be evaluated by putting $n - D = m$. Now, when *n* runs over all the integers so does *m* for a fixed integer *D*. And therefore this can re-written as $\sum_{m=-\infty}^{+\infty} x[m]z^{-(m+D)}$and therefore that is $z^{-D}X(z)$. The only difficulty is what happens the region of convergence.

Now, the region of convergence is going to be affected by the fact as z^{-D}. So, you know, the only effect that the factors z^{-D}can have is to affect what happens at the boundaries. So, essentially it is almost $R_h$as a region of convergence, possibly except boundaries region of convergence could almost be the same but the boundaries have to be carefully seen.

Let me give you an example, again a very simple example take $\delta[n - 2]$, $\delta[n - 2]$had the Z-transform z^{-2}, we have seen that before the region of convergence is the entire Z-plane excluding $z = 0$, advance this by *2*. So, take $n + 2$ in place of *n* there and you get $\delta[n]$.

Now, the boundary also gets included. So, because of the multiplication by z^{-D} you need to worry about what is happening at the boundaries. Otherwise, the rest of the region of convergence is unaffected. The interior is unaffected. The boundaries could be affected so much so far delaying or shifting.

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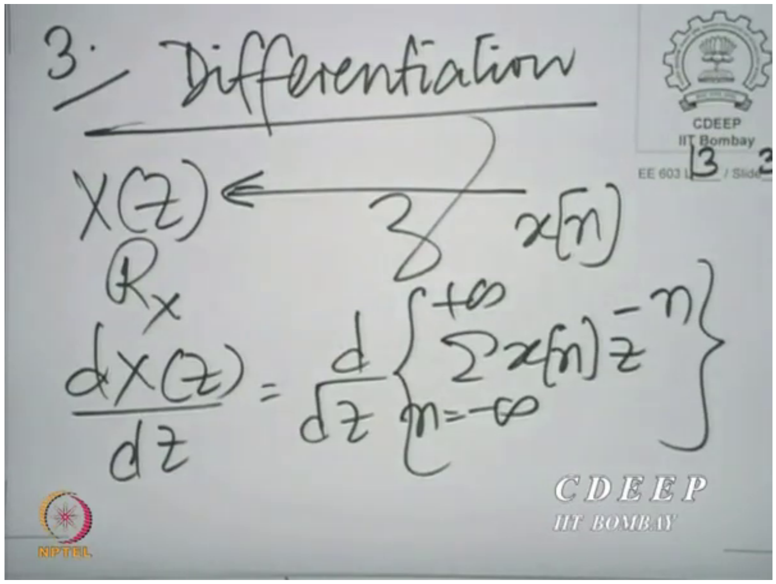

Now, we see a very interesting variant, will see one variant of a property today and then we will look at more properties in the next lecture, the apparent variant we will see is that of differentiation and here we differentiate the Z-transform. So, here we go the other way let $X(z)$be the Z-transform of $x[n]$with region of convergence R_X we asked the question, what is it that has the Z-transform $\frac{d\,X(z)}{dz}$ or can we do something to figure out what happens to $x[n]$ when we take the derivative with respect to *z* if it exists.

Let us, write down the expression so $\frac{d\,X(z)}{dz} = \frac{d}{dz}\{\sum_{n=0}^{\infty} x[n]z^{-n}\}$. Now, let us assume that this derivative exists. So, if it exists, then we can evaluate the derivative term by term if it exists and is analytic. So, we see normally we expect now we are going to consider this certain point functions in the complex plane can be analytic or not analytic in the region of analytic means they have an infinite number of continuous derivatives.

Now, we will initialize deal only with Z-transforms which are in fact all through this course will deal largely with Z-transform which are analytic in the region of convergence. So, in the region of convergence, they have an infinite number of derivatives of course one can always conceived a functions that do not have this property, but we shall not do it in this course. That is more of mathematical interest than practical.

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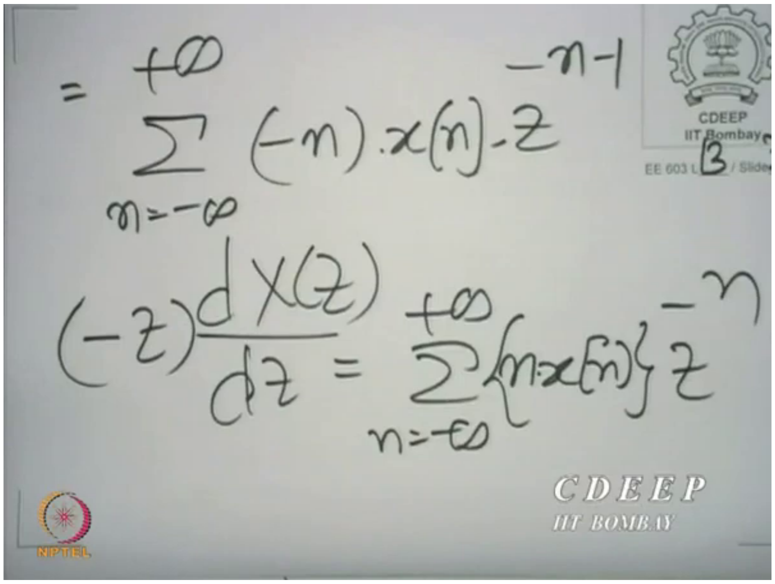

So, anyway, we can then take the derivative term by term and that gives us $\sum_{n=-\infty}^{\infty}(-n)x[n]z^{-n-1}$ Remember $x[n]$ is the constant with respect to z, and therefore if you take $(-z)\frac{d\,X(z)}{dz} = \sum_{n=-\infty}^{\infty}\{n \cdot x[n]\}z^{-n}$, this is very interesting. So, what you are saying is that the Z-transform of $n \cdot x[n]$ is essentially the derivative of the original Z-transform of $x[n]$ multiplied by $-z$.

Now, we have assume the Z-transform to be analytic in its region of convergence. So, all over the region of convergence interior to the region of convergence the derivative is valid so the region of convergence continues to exist at the derivative, you know, I mean, so the derivative can spread all over the region of convergence so the region of convergence R_h is definitely included here. Again, because of a multiplication by $-z$ you need to worry about the boundary. So, region of convergence is at least R_h if not more at least the interior.

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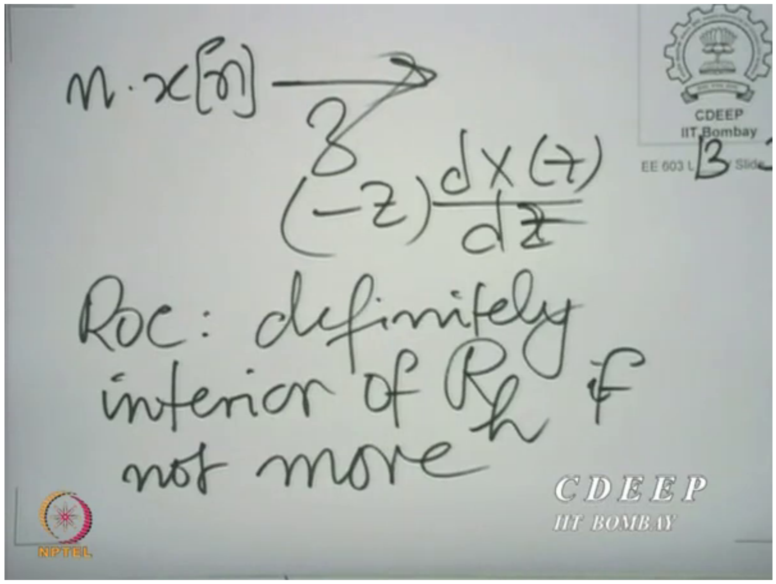

So, what we have said effectively is $n\,x[n]$has the Z-transform $(-\ z)\frac{d\,X(z)}{dz}$and the region of convergence is definitely the interior of $R_h$if not more. And the more essentially refers the boundaries, this brings us to a very interesting point in fact, we intentionally took this property before others because this property can also be taken to the discrete time fourier transform. If $x[n]$has a DTFT we can ask what is the discrete time fourier transform of $n\,x[n]$.

And in a way that it can be answered by this property. If you can evaluate the Z-transform on the unit circle unit circle means were $r\ =\ 1$then you can evaluate essentially you are saying the Z-transform evaluated on the unit circle, unit circle is a circle with a radius of *1* or $r\ =\ 1$ then so if you can evaluate the Z-transform on the unit circle, then you can find out what is the Z-transform or what is the sequence whose Z-transform is the derivative multiplied by $-\ z$ and that gives you the Z-transform of $n\,x[n]$when you multiply a sequence by the time index and you multiply a sequence by *n* the effect in the Z-domain is to take the derivative with respect to *z* and then $x\ -\ z$. These operations cannot be interchanged. You must first take the derivative with respect to z and then multiply $-\ z$.

And in particular if you do this on the unit circle you get what happens to the discrete time fourier transform that is an interesting property. Now, we have to answer several other questions about the Z-transform. The first question is what happens when you convolve two sequences the million-dollar question when you talk about linear shift invariant system. So, when I want to see what happens to unstable systems when I give them inputs I need to know what happens to Z-transform under convolution. I also keep to answer similar questions about the Z-transform with respect to inversion, can I invert the Z-transform.

Let me mention that it is most common to invert the Z-transform by experience that means we associate inverses for certain typical forms, and we use those inverses to calculate the inverse for more complicated Z-transform. Anyway in the next lecture, let us see little more about some of these questions. Thank you.