Digital Signal Processing & Its Applications
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Lecture 14 a
Convolution Property (Z-Transform)

So, warm welcome to the 14th lecture on the subject of Discrete Time Signal Processing its Applications. This lecture will continue with where we left in the previous lecture, namely, on the properties of a Z-transform, we are introduced the Z-transform in the previous lecture and we had started looking at some of its properties, I would like to recall a couple of points from the previous lecture before we proceed to say more about the Z-transform and its properties. We had looked at the expression for the Z-transform, let us recapitulate that.

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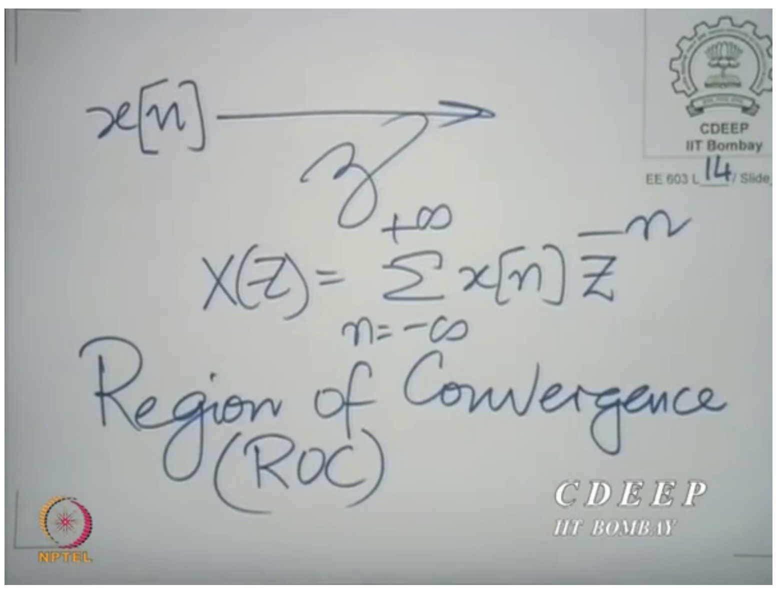

So, if you have a sequence $x[n]$ then we said it is Z-transform is given by $\sum_{n=-\infty}^{+\infty} x[n]z^{-n}$. And this always has a region of convergence or ROC as it is often called for short. Now, the region of convergence is always a region between two concentric circles with the circles centre at the origin.

And we said that one of the concentric circles could possibly have radius *0* and the other one could possibly have radius infinity. So, *0* and infinity are also possible radii. It was also a point that needed to be addressed to look at whether the boundaries are included in the region of convergence, the boundaries may or may not be included in that needs to be checked in each case individually.

One guideline is if there is a singularity on the boundary point where that quantity diverges, then of course that boundary cannot be included in the region of convergence. So, these are about the Z-transform in general. And its definition and its contents, Z-transform all always has an expression with a region of convergence. Yes, as a question.

Student 1: But the centred $z = re^{i\omega}$.

Professor Vikram M. Gadre: Yes.

Student 1: Does R have any physical significance?

Professor Vikram M. Gadre: So, that is a good question. It says we had said, so let us come to that.

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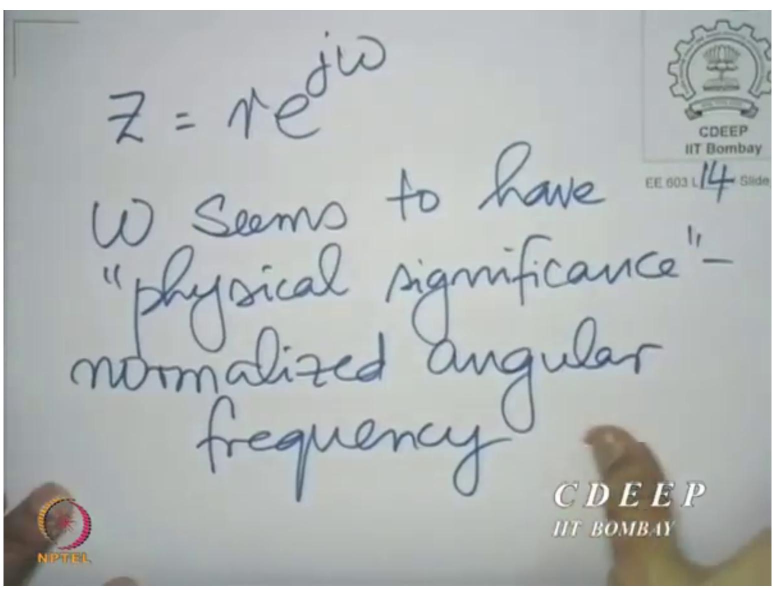

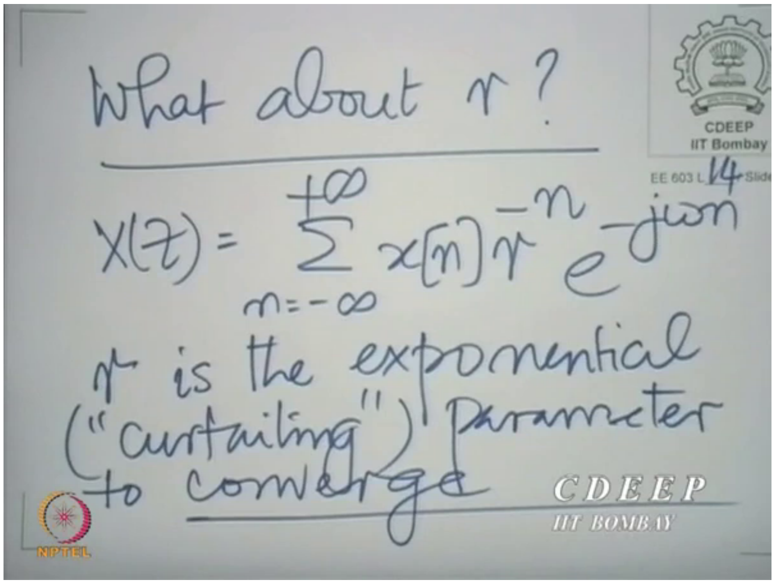

We had said that z is of the form $re^{i\omega}$. So, the question is, ω seems to have a physical significance. And that physical significance is the angular, normalised angular frequency, is not it? However, the question is does r have a physical significance, r does indeed have a physical significance. There are two ways of understanding r, one way is a formal way that is, it is the rate at which the exponential should multiply that sequence so as to bring the Z-transform into convergence that is a formal way of understanding r.

So, if you look at it, $X(z) = \sum_{n=-\infty}^{+\infty} x[n]r^{-n}e^{-i\omega n}$. Now, let us write this in terms of r and ω, so $e^{-i\omega n}$ and r^{-n}. So, r is the exponential parameter in a way exponential or curtailing parameter, if you may call it that. Essentially it, in, it ensures that the sequence converges on multiplication by this exponential.

So, r is in some sense, the rate at which the exponential must grow to capture the sequence or another way of looking at it is r^{-n} is the exponential that must capture the sequence to bring it into convergence that is the way to understand it. If we speak informal language, if we recall the analogy of the tiger that I gave last time, r is the size or the strength of the cage that you need to encapsulate the tiger.

So, you have the sequence which refuses to converge, which you cannot train for whom you cannot find a frequency response. So, you need r to capture it to tame it so that a frequency response exists, that is the physical significance of r that is a good question. Any other questions from the previous lecture? Yes, there is a question, yes.

Student 2: ...

Professor Vikram M. Gadre: So, the question is, can we view the Z-transform as the discrete time fourier transform of a sequence, can you think of the Z-transform as indicating the frequency content of some other sequence. Yes, indeed, of course.

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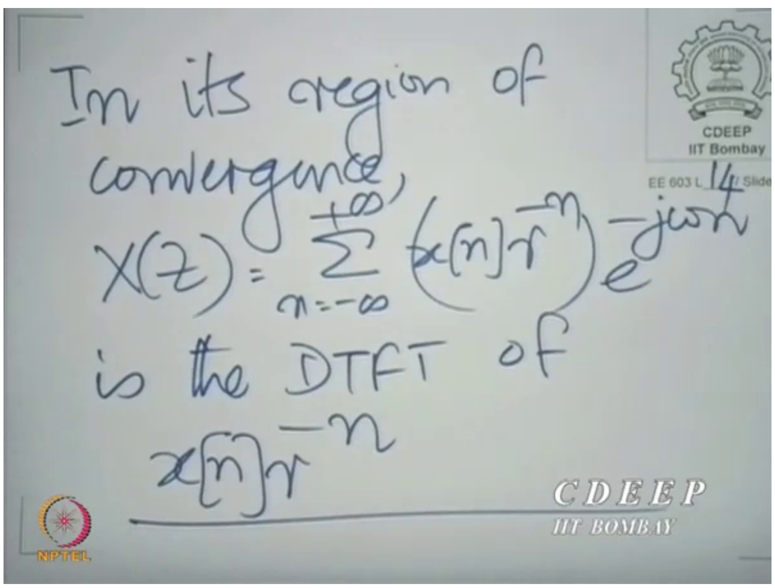

If you look at it, in its region of convergence $X(z) = \sum_{n=-\infty}^{+\infty} x[n]r^{-n}e^{-i\omega n}$ is the DTFT of $x[n]r^{-n}$. So, in fact, it is the discrete time fourier transform, or in other words, it indicates the frequency content in a suitably weighted sequence $x[n]$ or $x[n]$ weighted by an exponential.

The exponential should strongly or it should be strong enough the exponential should be quick enough to capture the growth of $x[n]$. So, that this discrete time fourier transform converges, so in a way that Z-transform on any particular circle in the region of convergence take any circle centred at the origin, lying entirely in the region of convergence.

On that circle, the Z-transform is essentially the sequence, is the discrete time fourier transform of the sequence given by $x[n]r^{-n}$, where r is the radius of that circle that is the way to understand. Yes, any other questions? Any other question before we proceed, yes, no other questions, none at all. So, then we proceed to look at a few more properties of the Z-transform.

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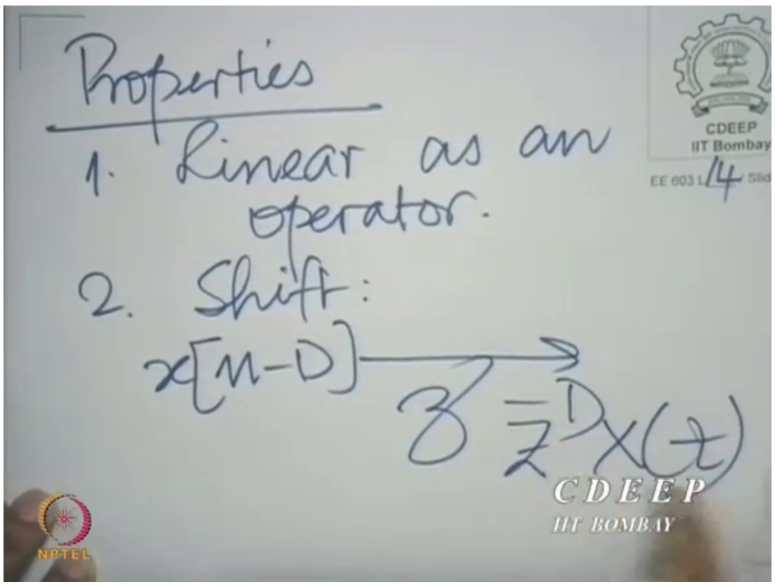

Now, we had seen some properties, we had seen the Z-transform is linear. We had seen that it is linear, linear as an operator. Secondly, we had seen the Z-transform has a property of delay, the delay property or the shift property, is not it? That is, if you shift the sequence by D, then the Z-transform gets multiplied by z^{-D}, we have talked about the region of convergence. We also saw what happens when we differentiate this was what we were looking at last time. So, we saw that if you differentiated the Z-transform.

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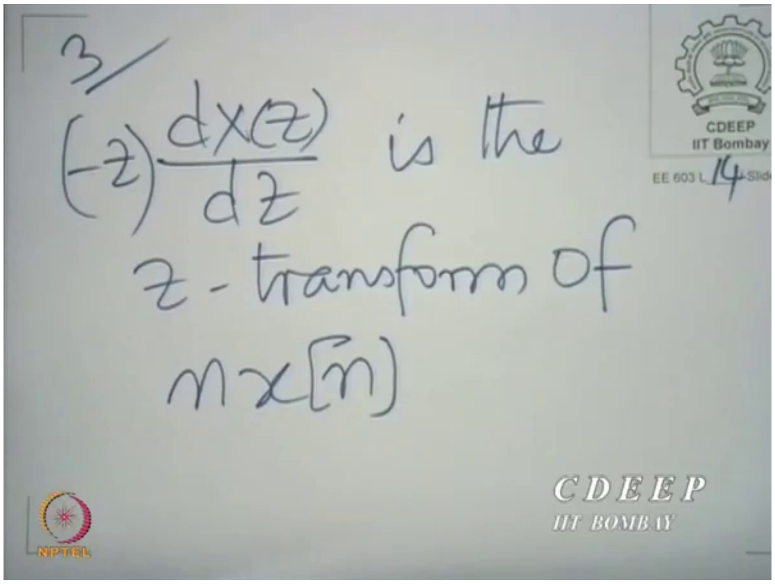

So, if you have $\frac{d\,X(z)}{dz}$ and multiply this by $-\,z$, it is the Z-transform of $nx[n]$ that is interesting. And we also looked at the region of convergence, each case we had seen the region of convergence, we saw that normally a Z-transform is analytic in its region of convergence in fact we are going to restrict ourselves to that class of Z-transforms. So, the same region of convergence would hold for this derivative, save for the boundaries, these boundaries need to be checked.

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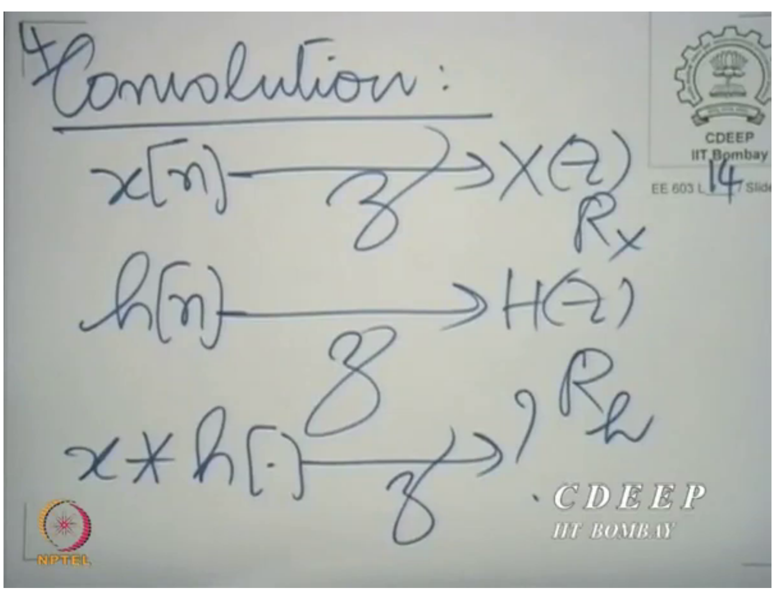

Now, we look at a very important property of the Z-transform namely the property of convolution that is the real value of the Z-transform. So, what happens when we convolve two sequences, so let $x[n]$have a Z-transform, $X[z]$ with the region of convergence R_x , and let $h[n]$ have a Z-transform, $H[z]$ with a region of convergence R_h , we ask, what is the Z-transform of the convolution of x with h?

Now, you see the principle that we use to arrive at the answer is very similar to what we did in the case of the discrete time fourier transform, there is no fundamental difference. The only thing is that we had to remember that z must lie, you see, in principle here let us take a z to lie in the region of convergence of both, namely the intersection of the regions of convergence.

Now, there are tricky issues here. What happens if the intersection is null? Can you still convolve? We will not answer these questions. They are tricky ones. But for the moment, let us take a situation where there is a non-null intersection of the region of convergence. And let us pick a z from that intersection of the regions of convergence of $R$and $R_X$and R_h.

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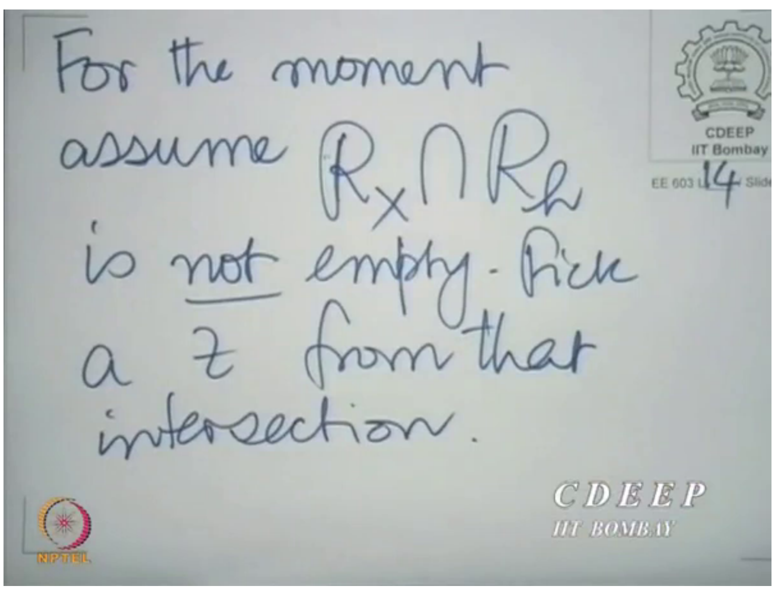

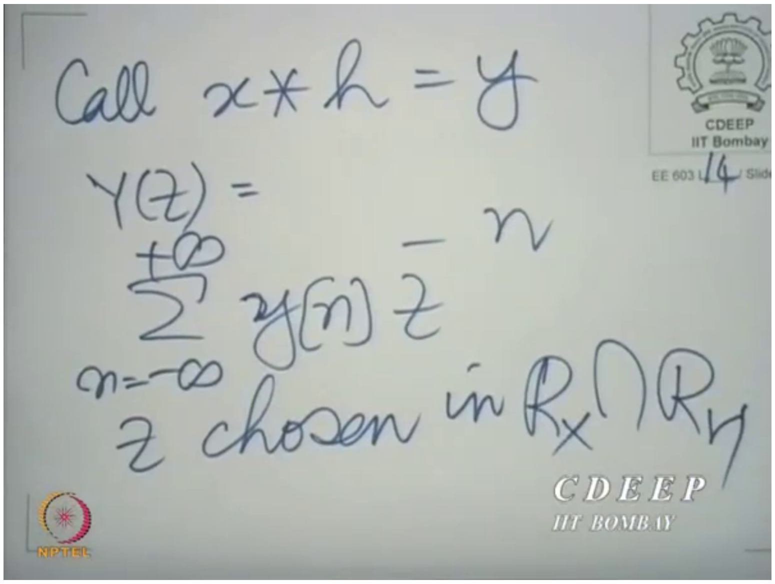

So, let us see for the moment I am saying for the moment. Assume that $R_X$intersection R is not null, is not empty, pick a z from that intersection and use that z in the discussion that follows, or use any of those z in the discussion that follows. Now, let us look at the Z-transform, let us call x convolved with h, let us call it y. So, of course, we need to find out the Z-transform of y, the $Y(z)$ is of course, $\sum_{n=-\infty}^{+\infty} y[n]z^{-n}$and we have picked a z, z chosen suitably chosen in R_X intersection R_h as we have said.

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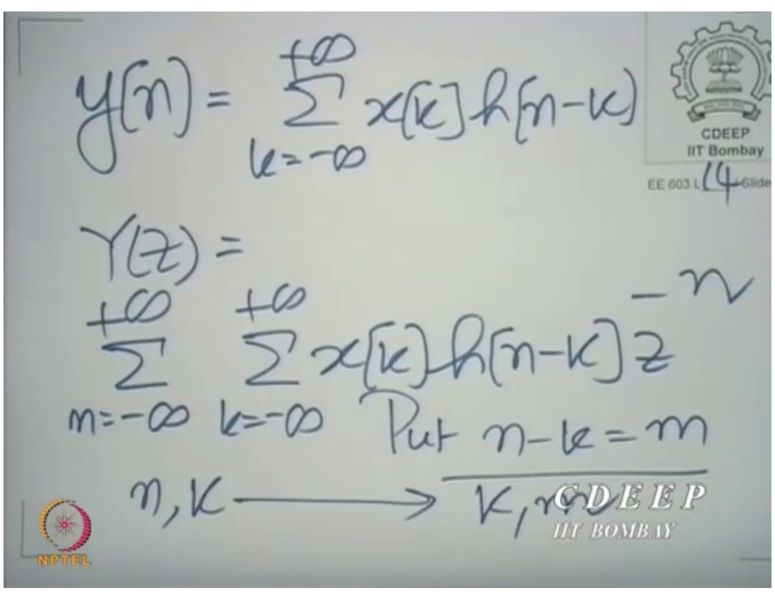

Now, expand $y[n]$. So, you have $y[n]$ is $\sum_{n=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$, whereupon we have $Y(z)$ is given by $\sum_{n=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} x[k]h[n-k]z^{-n}$. And of course, we use the same strategy as before, we put $n - k = m$.

And now instead of *n* and *k*, we go to *k* and *m* of course, so *k* is as it is, *k* runs, you see *k* and *m* independently run over all the integers. So, for a fixed *k*, *m* runs over all the integers. Now, if *n* and *k* independently run over all the integers, then for a fixed *k*, *m* would also run over all the integers independent of *k*.

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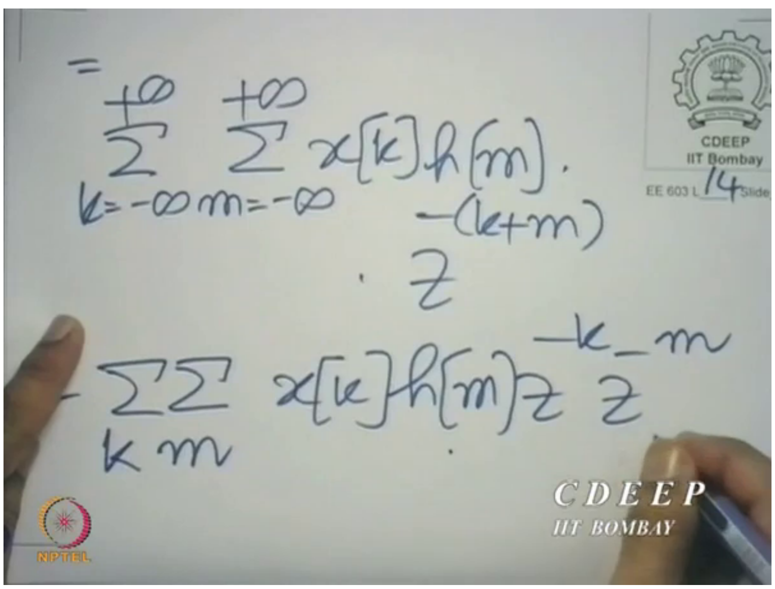

And therefore, we could rewrite this summation here as $\sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty}$. And we have $x[k]h[m-k]$, and m is of course, $k + m$ that becomes a summation, I am not writing the limits every time they are understood $h[m]z^{-k}z^{-m}$.

And now we notice something very interesting. There are terms that depend only on *k*, there are terms that depend only on *m*. So, we can act $\sum_{m}$ on the terms that depend only on *m*. And that leaves us with so you see, I mean, what I am saying is, we could take this term and this term and operate the $\sum_{m}$ on them that gives us a quantity independent of *k*.

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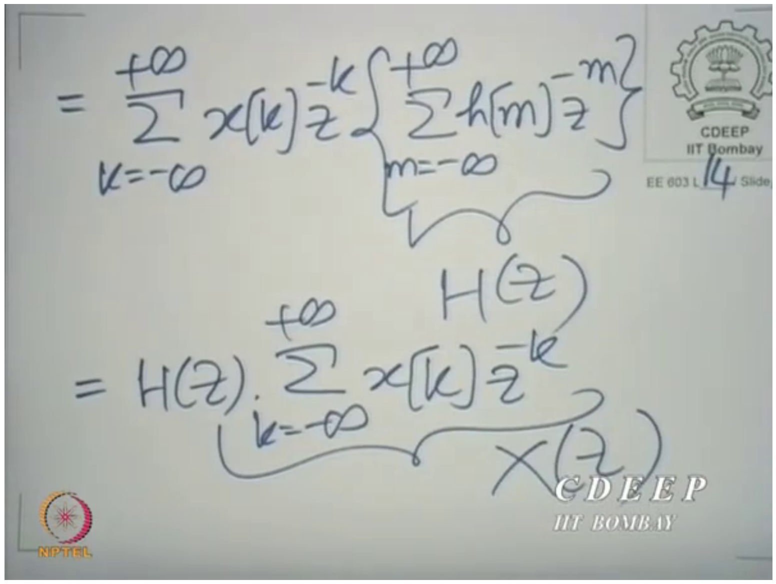

So, that leads us to $\sum_{k=-\infty}^{+\infty} x[k] z^{-k} \left\{ \sum_{m=-\infty}^{+\infty} h[m] z^{-m} \right\}$ and note that this is essentially $H(z)$ and $H(z)$ is of course independent of k, so I can draw it, $H(z)$ outside the summation, because it is independent of k and that leaves me with this, but then this happens to be $X(z)$ and therefore, we have a product of $H(z)$ and $X(z)$ that is very interesting.

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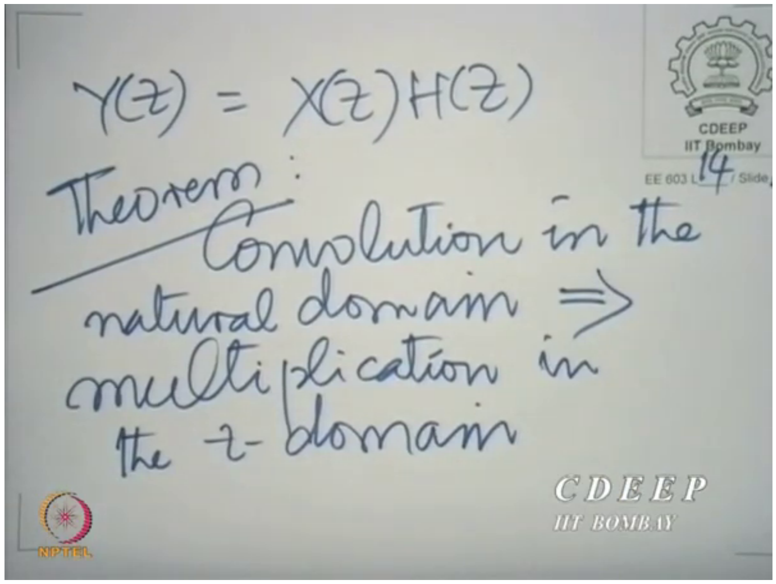

So, $Y(z)$is clearly $H(z)X(z)$. So, in other words, this is a very important theorem. It says that convolution in the natural domain leads to multiplication in the Z-domain, this is not surprising, because in particular, if the unit circle or $|z| = 1$or $r = 1$ is included in the regions of convergence of both or at least definitely of $Y(z)$ then you would find that this spoils down to the specific property of the discrete time fourier transform, namely when I convolve two sequences and each of them has a discrete time fourier transform, their convolution would also have a discrete time fourier transform given by the product of these.

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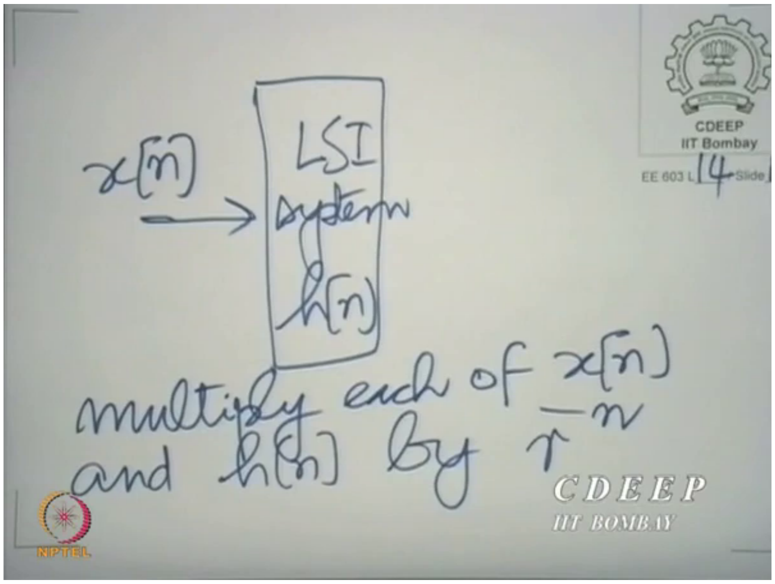

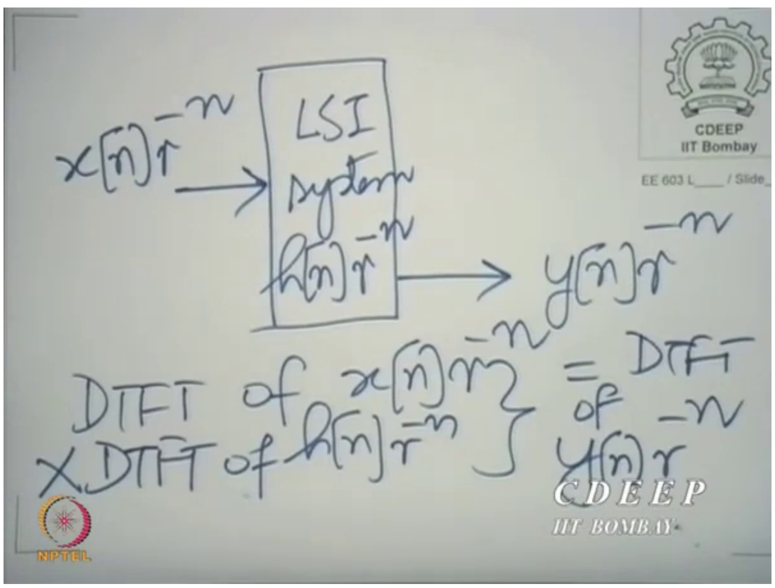

Now, you see, we can also give this an interpretation. What we are saying is that if you have a linear shift invariant system with impulse response $h[n]$ and if you have a sequence $x[n]$. We are saying that if you see as such $x[n]$ and $h[n]$ may not had discrete time fourier transforms. But, again, you could encapsulate these tigers in a cage multiply them by suitable exponential.

So, you could multiply $x[n]$multiply each of $x[n]$and $h[n]$ by a suitable r^{-n}, and now you can use the property of the discrete time fourier transform. So, you could now treat x, you see the beauty is that now the same LSI system, but now the impulse response should be viewed as $h[n]r^{-n}$.

And here we have $x[n]r^{-n}$ given us the input. The beauty is that you would get y instead of $y[n]$, you would get $y[n]r^{-n}$or in other words, if you took the DTFT of this DTFT of $x[n]r^{-n}$ multiplied by DTFT of $h[n]r^{-n}$ is equal to DTFT of $y[n]r^{-n}$.

So, you use the same property what we are saying is you use the same property of the discrete time fourier transform, but on a suitably treated or in a suitably tamed sequence or suitably tamed input and impulse response tamed by r^{-n}. That is what we are saying effectively, of course, in some cases you may in fact view this not as a taming but also as a licence.

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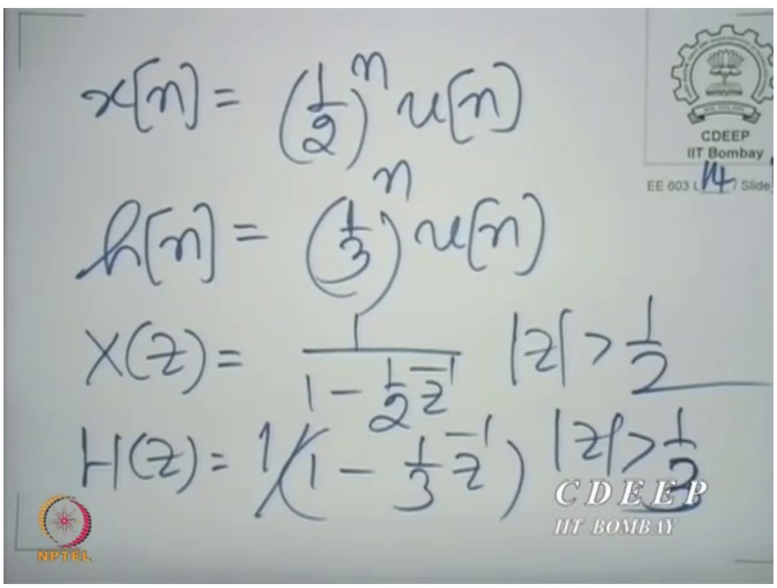

So, for example, if you look at it, suppose we take $x[n] = \left(\frac{1}{2}\right)^n u[n]$and $h[n] = \left(\frac{1}{3}\right)^n u[n]$. We can easily obtain the Z-transforms, it is very easy to see that $X(z)$is $\frac{1}{1-\frac{1}{2}\times z^{-1}}$with $|z| > \frac{1}{2}$. And similarly, $H(z)$is $\frac{1}{1-\frac{1}{3}\times z^{-1}}$ with $|z| > \frac{1}{3}$ very easy to see.

So, of course, if you take the convolution if you happen if this $h[n]$ happens to be the impulse response and of course, you note that an LSI system with this impulse response $h[n]$ would be stable. Because this sequences easily seem to be absolutely sum, in fact you can find its absolute sum.

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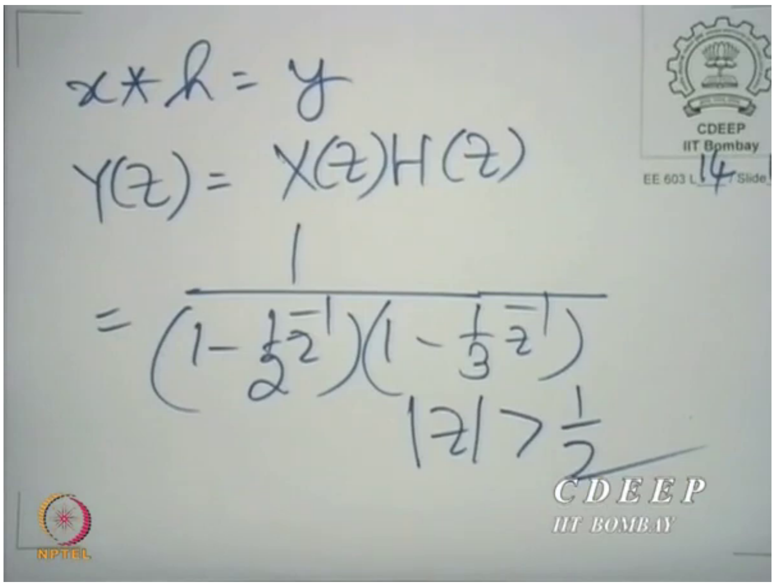

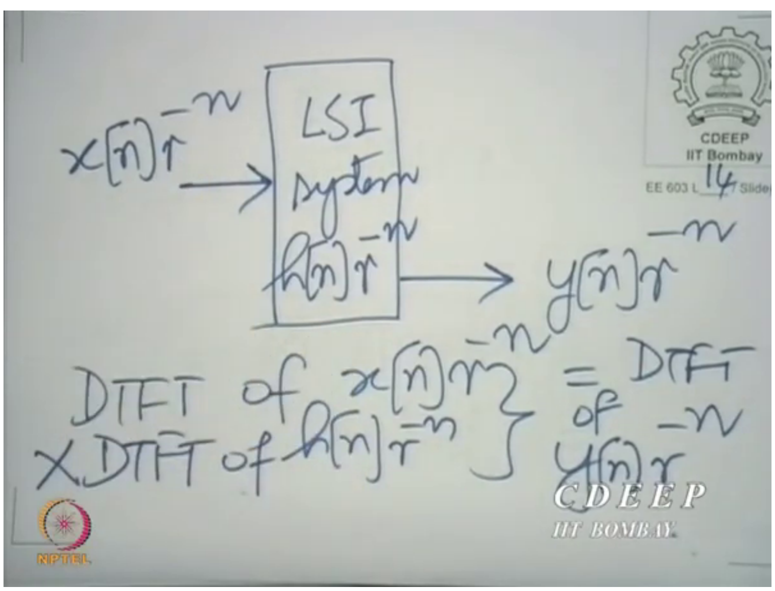

But that apart the sequence gives us a stable LSI system for the impulse response. And x convolved with h which is why has the Z-transform $Y(z)$ is it given by $X(z)H(z)$, which is

$\frac{1}{(1-\frac{1}{2}\times z^{-1})\left(1-\frac{1}{3}\times z^{-1}\right)}$. And of course, now the region of convergence is indeed the intersection of the two regions of convergence.

So, $|z| > \frac{1}{2}$, $|z| > \frac{1}{3}$. So, in total $|z| > \frac{1}{2}$. Because that is the intersection. Now, $|z| > \frac{1}{2}$ is like a licence, I told you we have been talking about encapsulating or taming the sequence. But here $|z|$ needs only to be greater than $\frac{1}{2}$, you can take for example, $|z| = \frac{3}{4}$.

In fact, you can take $|z| = \frac{3}{4}$ if you like. Now, $\left(\frac{3}{4}\right)^{-n}$ is actually an exponentially growing sequence. It is $\left(\frac{4}{3}\right)^{n}$. So in a way, you are saying, even if you multiply $x[n]$and $h[n]$ by an exponentially growing sequence, and take their discrete time fourier transforms, you could apply the property that we did a couple of points a couple of minutes ago, what I meant was, here we had $x[n]r^{-n}$ applied to $h[n]r^{-n}$ giving you $y[n]r^{-n}$.

So, here *r* could be a quantity less than *1*, *r* could be $\frac{3}{4}$in this example, which means it is actually an exponentially growing sequence. So, there is a licence also, is a bit of a licence to allow the sequence to grow, but still you can encapsulate, you can allow it to apply the discrete time fourier transform property of course, this is because the sequence itself is exponentially, both of the sequences are exponentially decaying.

So, if the exponential growth is slower than this decay, then on and on this leads to a decay and therefore you can have some licence that is what we are saying effect.