Digital Signal Processing & Its Application
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Lecture 14b
Rational Systems

Anyways, so much so for license and taming.

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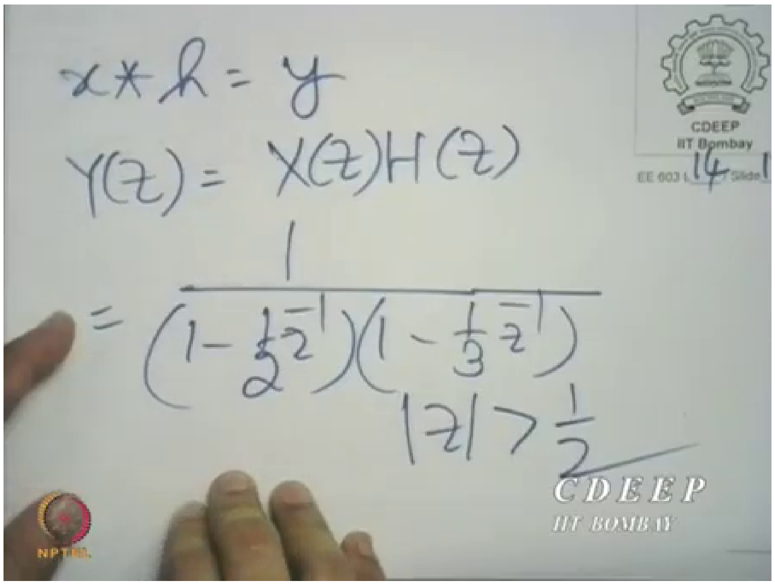

Now, let us get back to this example of why $Y(z)$ here. You see here you have a product of two Z-transforms. And the beauty is, this product of the two set transforms can also be expressed as a linear combination of the two Z-transforms. So, suppose I wish to find the sequence $y[n]$from here, how would I go about doing? Now, one would use what is called a partial fraction expansion.

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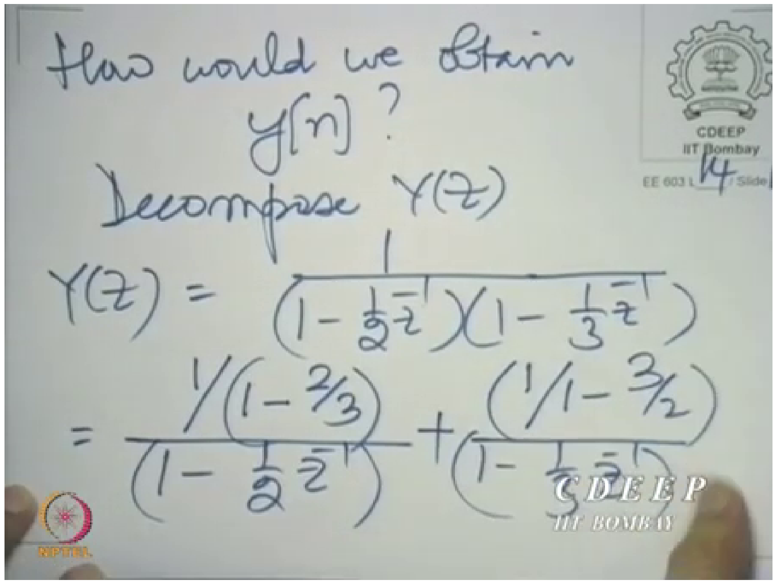

How would we obtain $y[n]$? Decompose $Y(z)$. Now, $Y(z)$ which is $\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$can be decomposed as something on $(1 - \frac{1}{2}z^{-1})$ and something upon $(1 - \frac{1}{3}z^{-1})$. What upon each, and the idea is very simple. When I want to find this factor, I multiply $Y(z)$by $(1 - \frac{1}{2}z^{-1})$and put $z = \frac{1}{2}$. Essentially, I am multiplying by $(1 - \frac{1}{2}z^{-1})$and making this factor *0*.

So, multiply by $(1 - \frac{1}{2}z^{-1})$, put $z = \frac{1}{2}$. So, I get $(1 - \frac{1}{3} \times \left(\frac{1}{2}\right)^{-1}),\left(\frac{1}{2}\right)^{-1}$that is *2*. So, I have $(1 - \frac{2}{3})$. And here I would use the same idea; I would multiply this by $(1 - \frac{1}{3}z^{-1})$ and put $z = \frac{1}{3}$. So, in other words, z^{-1}, so $z = 1$; z^{-1} is 3, so there we are, $\frac{1}{\left(1-\frac{3}{2}\right)}$. Now let us verifying the, let us verify.

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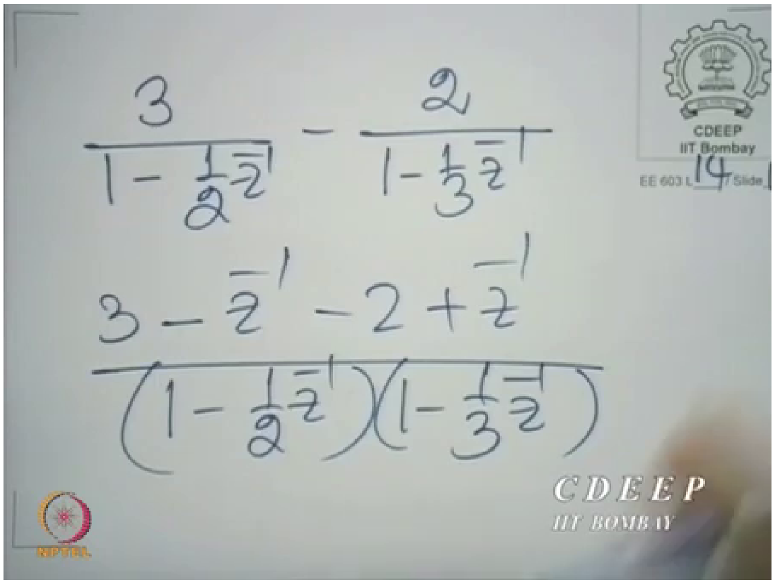

What we are saying is $\frac{1}{(1-\frac{1}{3})}$, that is $\frac{1}{(\frac{1}{3})}$, so $\frac{3}{(1-\frac{1}{2}z^{-1})}$ and $(1-\frac{3}{2})$ that is $(-\frac{1}{2})$. So, $\frac{-2}{(1-\frac{1}{3}z^{-1})}$, this is very easy to combine. $3-z^{-1}-2+z^{-1}$ and that indeed gives you back *1* by exactly the $Y(z)$ that we had.

Now, what I just illustrated is what is called a method of partial fraction expansion. It is a method of expanding a rational $Y(z)$, in terms of factors where there are, where there are denominator terms all acquiring the value *0* only at one point in the Z-plane.

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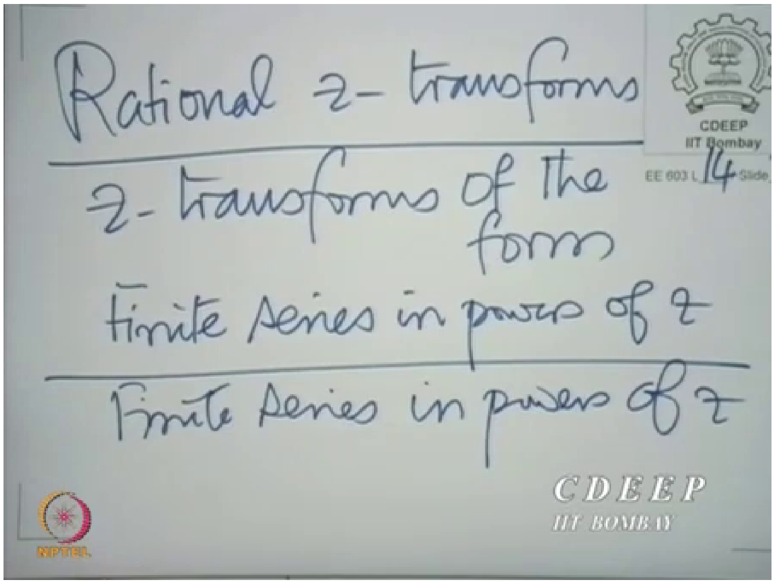

Now, we often deal with this class of Z-transforms. Let us introduce this class of Z-transforms, we call them rational Z-transforms. Rational Z-transforms are Z-transforms of the form, finite series in powers of z divided by finite series in powers of z. There is a numerator and denominator, each of which are a finite series in powers of z, not infinite. And of course, the word rational comes actually from the idea of rationality in the integers.

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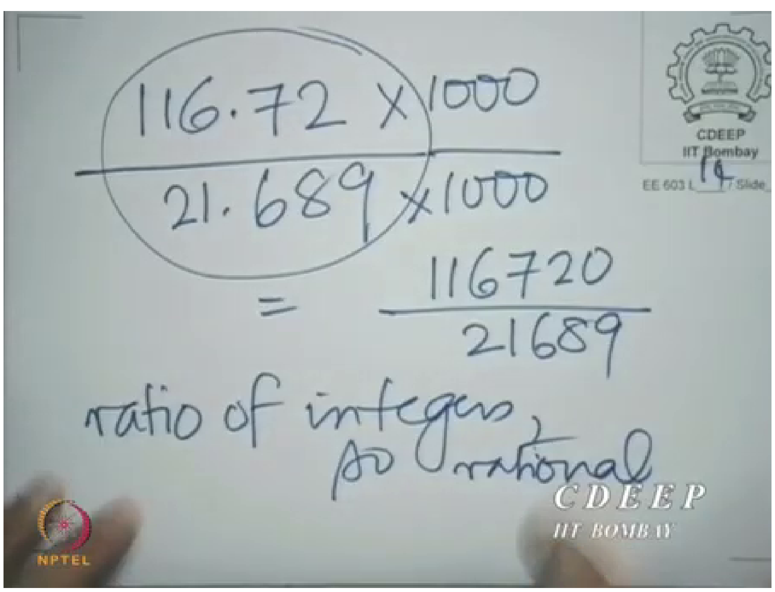

Let us take rational numbers we understand for integers. $1160.72 \div 21.68$ is a rational number. In fact, even if I put *689*, it will be a rational number. Why is this a rational number, because I can multiply the numerator and denominator by *1000*. And that leads me to $\frac{116720}{21689}$, which is a ratio of integers, so it is rational. Now let us expand this quantity $\frac{116.72}{21.689}$, in terms of powers of *10*.

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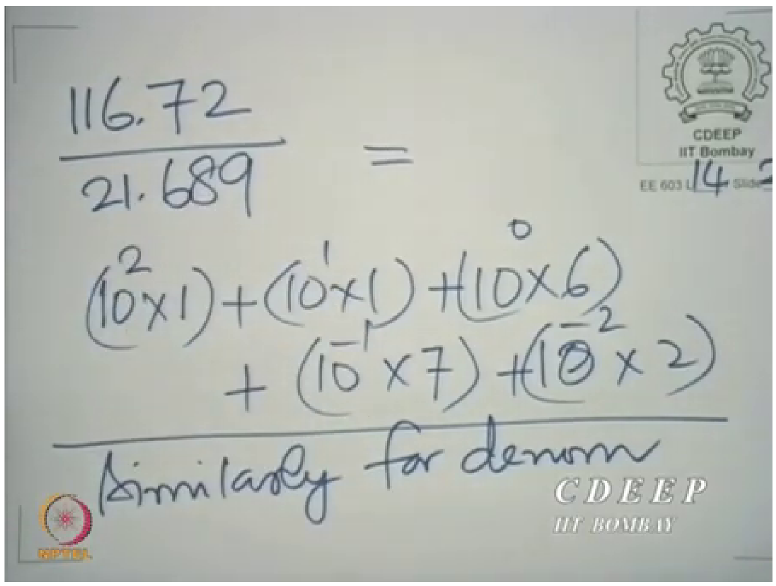

So in fact, we can see that $\frac{116.72}{21.689}$ can be rewritten as, $\left(10^2 \times 1\right) + \left(10^1 \times 1\right) + \left(10^0 \times 6\right) + \left(10^{-1} \times 7\right) + \left(10^{-2} \times 2\right)$ divided by similarly for the denominator. You can easily write the denominator as $\left(2 \times 10^1\right) + \left(1 \times 10^0\right) + \left(6 \times 10^{-1}\right) + \left(8 \times 10^{-2}\right) + \left(9 \times 10^{-3}\right)$.

So essentially, a rational number is a finite series in the base, here the base is *10*. Of course, you could write it, you could write the numbers, the base *8* or you could write the numbers the base *3*, or you could write it in binary, whatever you like. So, what is a rational number, rational number is a ratio of two series, each of which is finite in the base with respect to which those numbers are written.

The same idea is being used here except the base is an indeterminate z. It is a generalization of the idea of rationality as understood from the context of integers or numbers. Now, this generalization does not stop there. There are certain important properties that rationals have, and systems whose impulse response has rational Z-transform are called rational systems. So now let is define rational systems.

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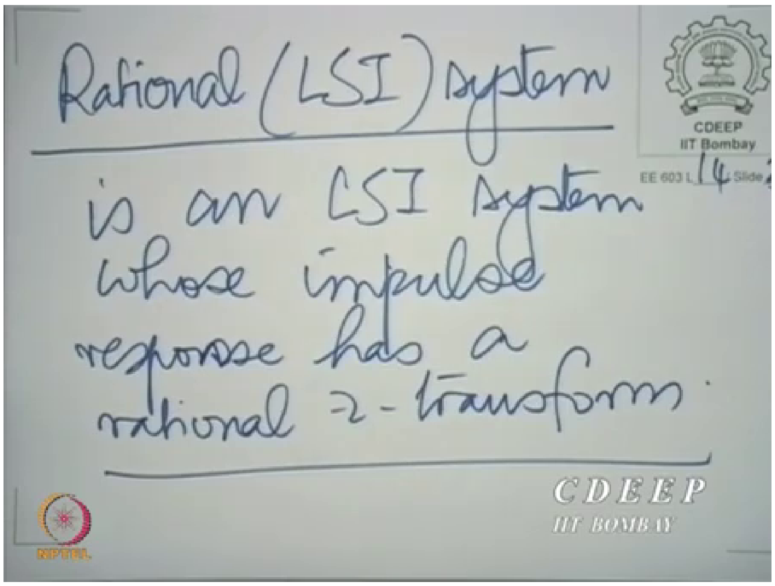

Rational and of course the word, rational system immediately refers to an LSI system otherwise it has no meaning. So, rational LSI system is an LSI system whose impulse response has a rational Z-transform. And of course, the LSI system that we saw a few minutes ago is an example. Rational systems are very important. In fact, almost all through this course in the sequel, we shall be talking about rational systems.

We shall of course, take examples of irrational systems, but we should actually design only rational systems. There is a good reason for that. Today, it is only rational systems that have a meaningful realization. What is mean, what is meant by realizing a system, realizing a system means translating a system into a hardware and, or software structure. Implementing a system with concrete components and operations. And of course, they must be finite in number.

You can always conceive of a system that requires an infinite number of operations to implement. That is of course of no practical significance. Unless a system can be implemented with finite resources, the system is not very useful in practice. It is only rational systems that can be implemented with finite resources. As far as LSI systems, that is why rational systems are of such great importance and relevance to us.

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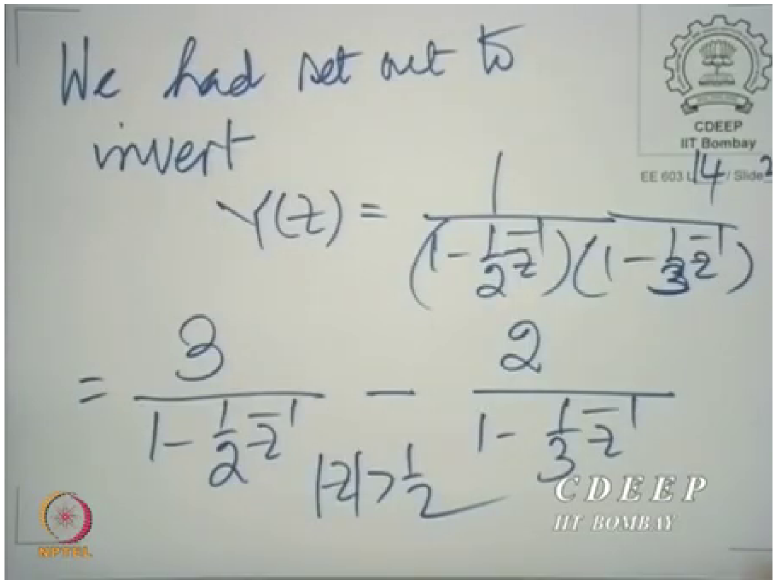

So, you see, the system that we saw a minute ago is indeed an example of a rational system. And now, of course, let us complete the job of inverting the Z-transform for which we had set out. See, we had set out to invert $Y(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$. And we had seen that this is essentially $\frac{3}{1-\frac{1}{2}z^{-1}} - \frac{2}{1-\frac{1}{3}z^{-1}}$. And of course, the region of convergence is $|z| > \frac{1}{2}$.

And now, you see, this is a sum of two Z-transforms, take this term, when $|z| > \frac{1}{2}$, $\frac{3}{1-\frac{1}{2}z^{-1}}$ from the linearity of the Z-transform must correspond to the sequence $3\left(\frac{1}{2}\right)^n u[n]$. And since $|z| > \frac{1}{2}$, $|z|$ is of course greater than $\frac{1}{3}$. So, for this sequence two with this, this expression with $|z| > \frac{1}{2}$ can also be thought of as the same expression with $|z| > \frac{1}{3}$, $|z| > \frac{1}{3}$ is of course true.

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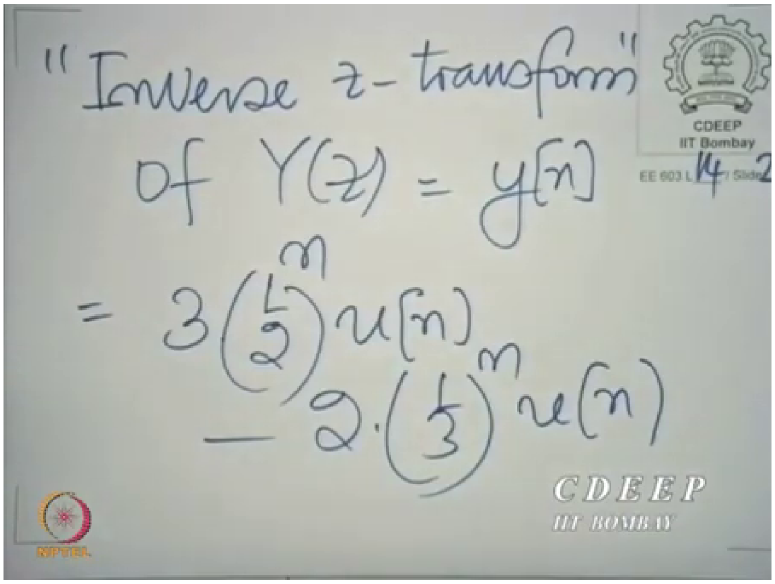

Therefore, the inverse Z or the inverse, the sequence corresponding to this or we might call it the "inverse Z-transform" of $Y(z)$ which is $y[n] = 3\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[n]$. From the linearity of the Z-transform. Now, this example brings a very important point to light.

You notice that in this example, the output Z-transform is the product, it is the convolution of the sequences $x[n]$and $h[n]$, it is the convolution of the input with the impulse response. But it also happens to be a linear combination of the input and the impulse response. This is something very peculiar to rational Z-transforms. In fact, you also see why this happens.

You notice that we inverted the Z-transform by decomposing into partial fractions. When you decompose into partial fractions, what you are saying is, each of the terms in the denominator give rise to one term in the partial fraction expansion. And therefore, a product of various terms denominator leads to a sum of the same or a linear combination of the same term.

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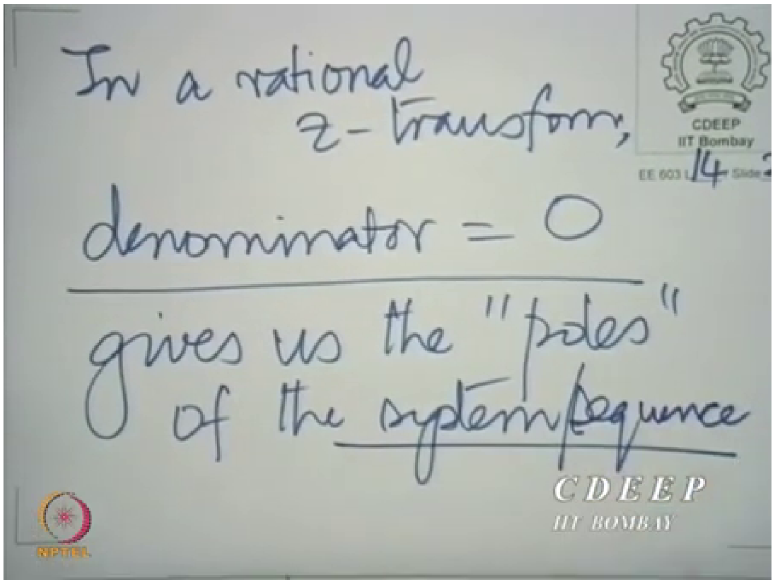

Now we introduce one more idea here. In a rational Z-transform denominator equal to *0* gives us what are called the "poles" of the system. We will explain in a minute why they are called the poles. Well, I should not call it the system necessarily, system of sequence, as the case may be. We say system if that rational Z-transform is the Z-transform of the impulse response. And we say sequence if we are talking about the Z-transform of a sequence.

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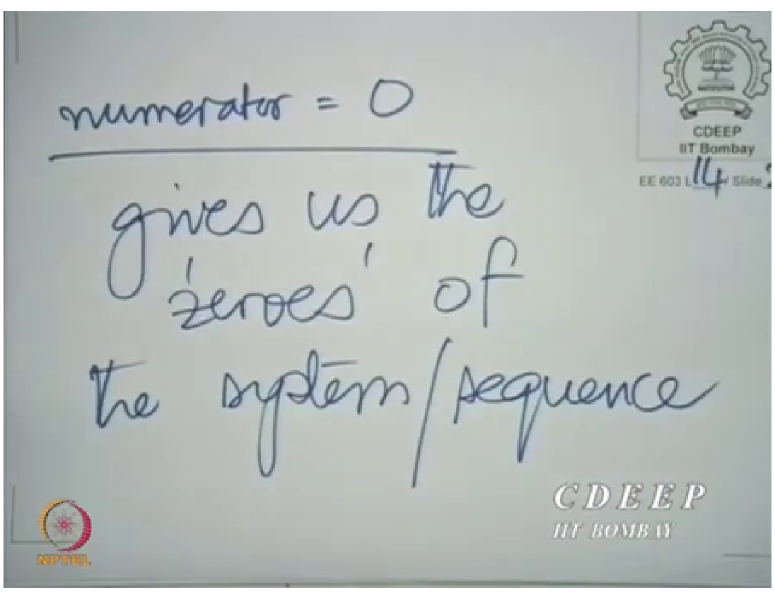

Now, the numerator equal to *0* gives us the "zeros" of the system or the sequence. And we will explain in a minute why this nomenclature. In fact, let us visualize the magnitude of the Z-transform, you see the magnitude of the Z-transform is always non-negative.

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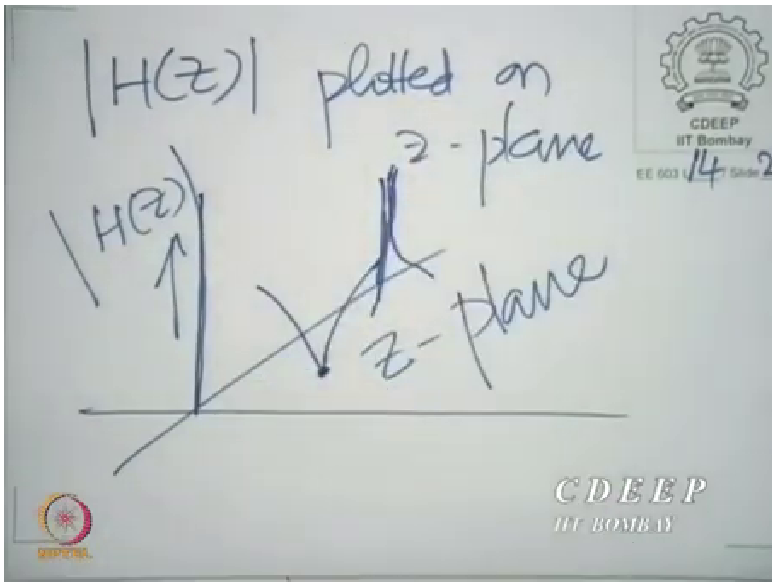

So, you can visualize the magnitude of the Z-transform plotted on the Z-plane, so I draw a *3*-dimensional or a seemingly *3*, or pseudo *3*-dimensional picture here. I have the Z-plane and the vertical; remember the vertical is only in one direction, it is $|H(z)|$. Now, wherever there is a *0*. You know can visualize $|H(z)|$ now as a tent on this Z-plane.

That is easy to visualize because you see a tent means it must be only, you cannot have a tent under the ground. So, tent must be only above the surface. And that is true because $|H(z)|$is non-negative. So, this tent, what happens to the tent, wherever there is a *0*, the tent joins the ground at that point. What happens to the tent, wherever there is a pole, the tent rises upwards, you see goes towards ∞. That is why it is called a pole.

The point where the denominator becomes *0* leads to the tent reaching the sky by virtue of a pole and the point where the numerator equal to *0*, the tent goes and touches the ground. That is why this terminology, poles and zeros. Anyway the poles are very important. If you have a linear shift invariant system, now, now we have defined what a rational system is.

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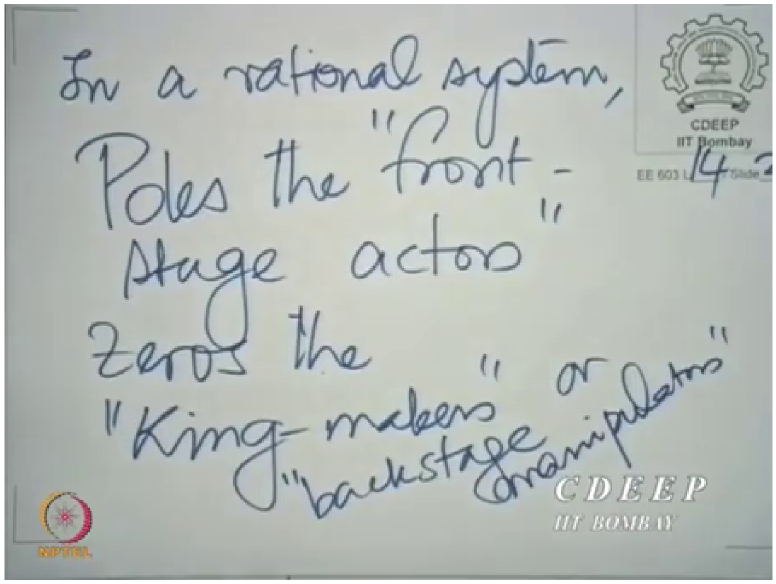

So, in a rational system, we call the pole the front stage actors and the zeros the kingmakers or backstage manipulators. You see, when you make a partial fraction expansion of a rational Z-transform, it is the poles which tell you what terms you should have. What do the zeros do, the zeros influence the coefficients of these terms. So, zeros do not explicitly figure in the partial fraction expansion.

They implicitly figure they, they are kingmakers, they say, you know who should go forward, who should, so backstage. The actual actors which you see on front stage are the poles. The poles are what determine what kind of response that system has, impulse response. And in fact, if you look at the poles of the input, and the poles of the system, together, they tell you the poles of the output.

What role do the zeros play, the zeros determine how important, how unimportant each pole is, because they influence the terms that the numerator is the partial fraction expansion. So, zeros do not explicitly figure in the nature of the response. But the zeros are important. Just as in a drama, the front stage is important, the actors on the stage are important, as are all the people manipulate, see, if they do not, if the people at backstage suddenly turn the lights off, the actors are of no good.

That is what the zeros do. Sometimes the zeros kill one of the poles, they do that. Turn the lights off on that pole. So, the zeros are backstage manipulators. Anyway, the zeros and poles together determine the nature. And in fact, here we can see that you see, when you have an input and an impulse response, both of which have rational Z-transform, the poles of the input and the poles of the z, of the impulse response together determine the poles of the output, that is easy to see because in fact, the poles of the output are a union of the poles of the input and the poles of the impulse response.

That is very easy to see. Because the Z-transform the output is the product of a Z-transform the input and the Z-transform the impulse response. And therefore, it has to be, the poles have to be the union of the two. Of course, but the zeros can play a role. They can, they might come in as I said, turn the lights off, they might cancel some of the poles that can happen.

If they do not cancel all the poles will be there and then also tells us now, you know, when you do a partial fraction expansion, it also tells us that the output is going to have terms which come from the input and which comes from the impulse response. In a way, in a rational situation, in a situation where both the input and the impulse response are rational, the output can also be thought of, in some sense broadly not, do not take this literally, but broadly as a linear combination of the input and the impulse response.

So, convolution also leads to a linear combination. Broadly, do not take this literally. We shall see more of this in the next lecture. Thank you.