Digital Signal Processing & Its Application
Professor Vikram M. Gadre
Department of Electrical Engineering
Indian Institute of Technology, Bombay
Lecture 15a
Examples of Rational Z Transform and their Inverses

A warm welcome to the 15th lecture, on the subject of Discrete Time Signal Processing and Its Applications and we proceed in this lecture to say more about the rational Z-transform, how to invert it, how to deal with systems whose impulse response has a rational Z-transform, what we can say about systems with the rational Z-transform for the impulse response, and so on.

So, we are going to spend quite a bit of time on the rational Z-transform because that occurs very frequently in our use of LSI systems. Let us recall a couple of points that we talked about in the previous lecture.

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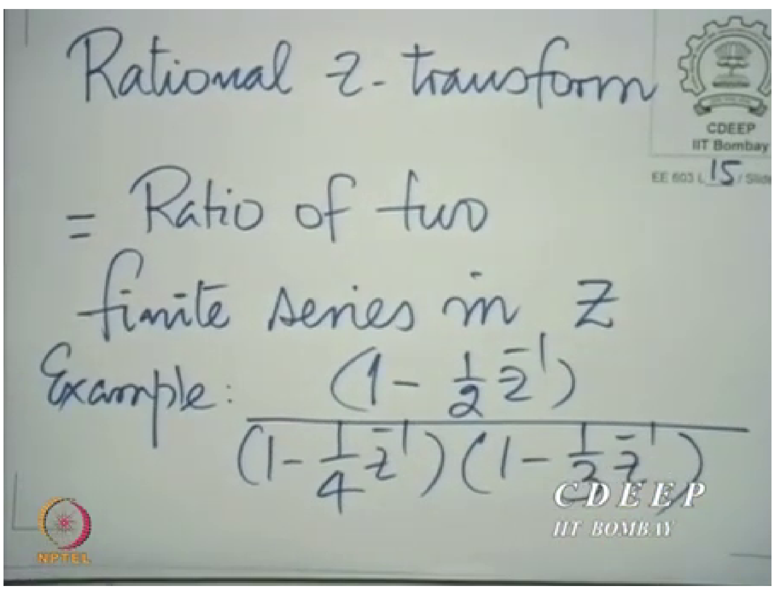

So, in the previous lecture, we had talked about the rational Z-transform, let me just recapitulate what it means. Rational Z-transform is a Z-transform, which is a ratio of two finite series in z. An example can be, $\frac{\left(1-\frac{1}{2}z^{-1}\right)}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$. This is an example of a rational Z-transform.

Now, I must always emphasize that a Z-transform has associated with it a region of convergence, a Z-transform is always an expression with a region of convergence, it is incomplete without either. So, for example, if we look at this particular Z-transform here, $\frac{\left(1-\frac{1}{2}z^{-1}\right)}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$.

We have different possible regions of convergence based on the singularities. In fact, if you look at the denominator here, we have two singularities in the denominator, one singularity at $z = \frac{1}{4}$ and one singularity at $z = \frac{1}{3}$. Now, the points of singularity cannot be included in the region of convergence. So, what we need to do is to identify the possible regions of convergence by excluding these points of singularity. Let us see how to do that.

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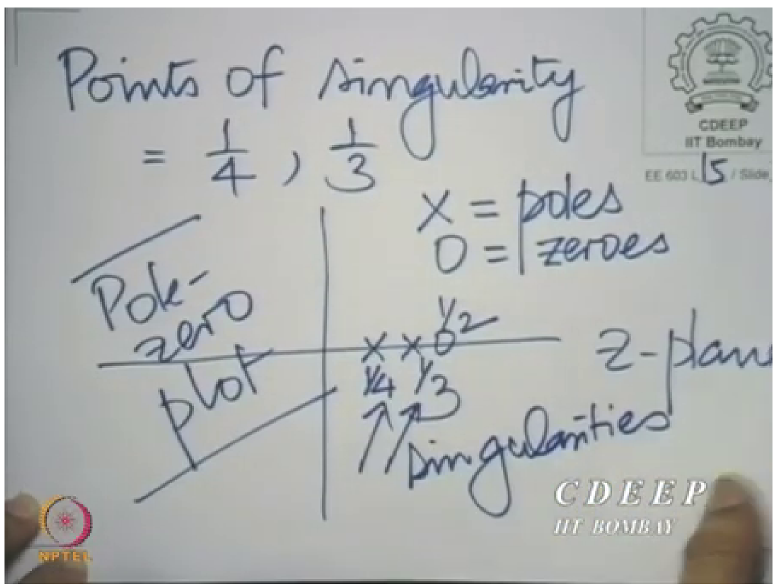

So, we have the points of singularity to be $\frac{1}{4}$ and $\frac{1}{3}$, let us mark them in the Z-plane. We use the following convention. Now, we use "×" to denote the poles, recall that the poles are the points where the denominator becomes *0*. And we use "o" to denote the zeros. So, we have two poles here, and there is a zero at the point $z = \frac{1}{2}$.

This is the, this is called the “pole-zero plot”, for the Z-transform that we saw a minute ago. Now these are singularities, the poles give you the singularities. And therefore, the regions of convergence must exclude these singularity.

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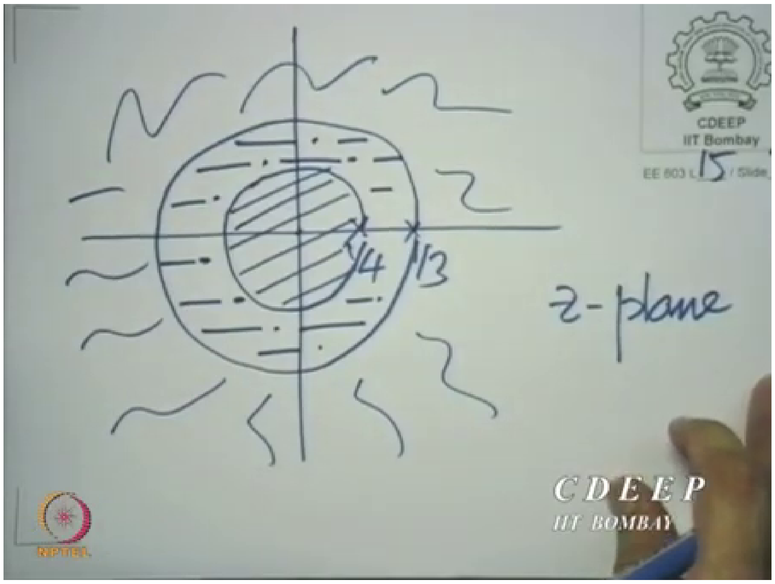

What possible regions of convergence can we have, let us mark them. You have to identify concentric circles. So, this is a Z-plane again, complex plane. You have $\frac{1}{4}$ here, $\frac{1}{3}$there. Let us draw concentric circles with the center of the origin passing through each of these singularities.

Now clearly, if you want simply connected regions; that is if you have regions where any two points can be joined by a line completely within that region, then there are only three possible simply connected regions that these two concentric circles can give. Let us mark them with different shadings. So, we have one like this, this is one possible region of convergence, the other like this and the third one like this. I am just marking it externally, you know. Let me write these same regions of convergence down analytically.

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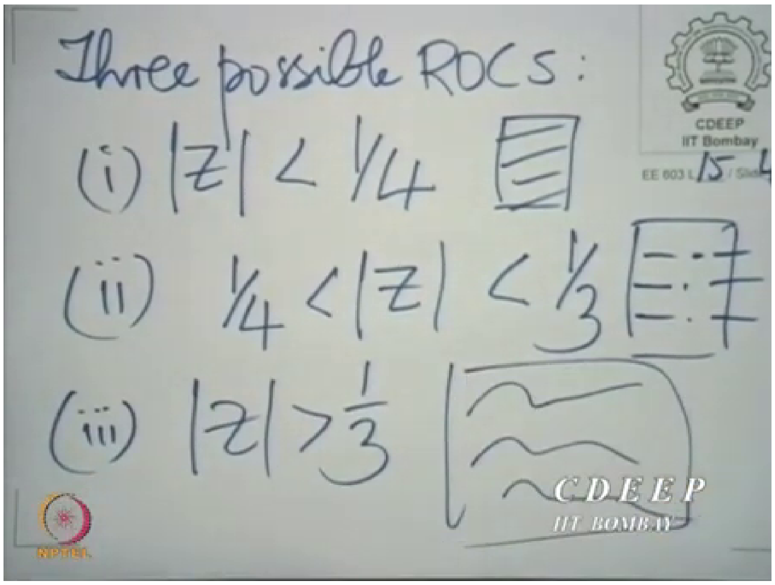

So, we have the three possible regions of convergence. $|z| < \frac{1}{4}$ and that is essentially the region where we have shaded like this. $|z|$between $\frac{1}{4}$and $\frac{1}{3}$and that is the region that we have shaded like this. Finally, $|z| > \frac{1}{3}$and that is the region that we have shaded like this. We expect that each of these regions of convergence would give us different sequences corresponding to the same expression.

In fact, we will see in a minute how we can use the partial fraction decompositions, strategically to arrive at these three different possible sequences. There we go.

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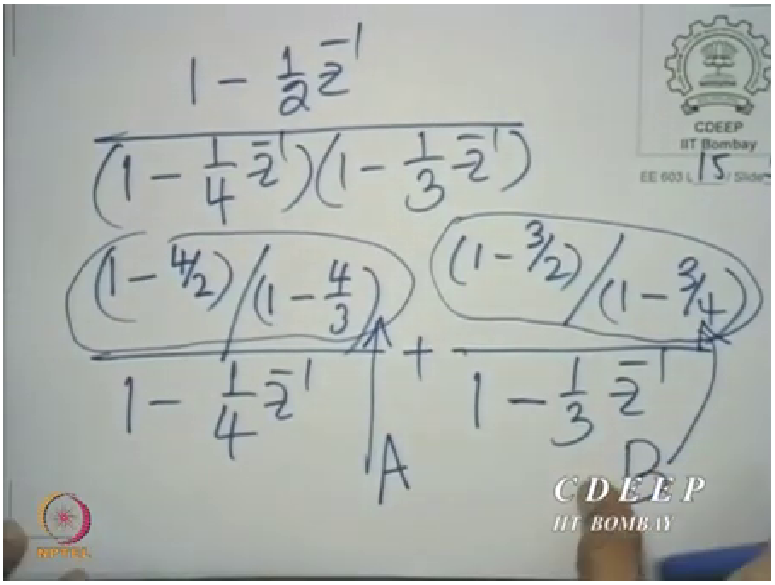

So, we have $\frac{\left(1-\frac{1}{2}z^{-1}\right)}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$ can be decomposed by partial fraction expansion into two terms. To obtain this term, I multiply this z transformed by $\frac{1}{1-\frac{1}{4}z^{-1}}$, and put $z = \frac{1}{4}$. So, I get $\frac{1-\frac{4}{2}}{1-\frac{4}{3}}$.

And here I get by multiplying by $\frac{1}{1-\frac{1}{3}z^{-1}}$ and putting $z = \frac{1}{3}$, $\frac{1-\frac{3}{2}}{1-\frac{3}{4}}$. Of course, you can evaluate these, let me call this whole thing, I do not really, I am not terribly worried what it is, let us call it A. And let us call this whole thing, B. So you see we have two terms now.

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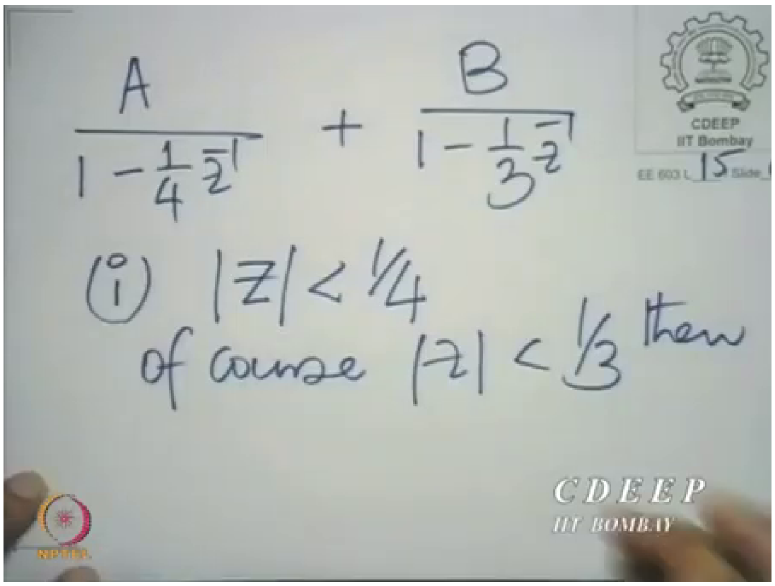

We have $\frac{A}{1-\frac{1}{4}z^{-1}} + \frac{B}{1-\frac{1}{3}z^{-1}}$. And let us take each region of convergence in turn. So, let us take $|z| < \frac{1}{4}$. If, is less than $\frac{1}{4}$, of course, $|z| < \frac{1}{3}$. So, both of these terms would need to be expanded by using powers of z and not z^{-1}. So, you have to rewrite these.

You see the strategy is when you have a region of convergence, look at how it behaves with respect to each term. Here you have two terms, $|z| < \frac{1}{4}$behaves the same way for both the terms. If $|z| < \frac{1}{4}$ it is also less than $\frac{1}{3}$. So, you know you do the same thing to both the terms in this particular case.

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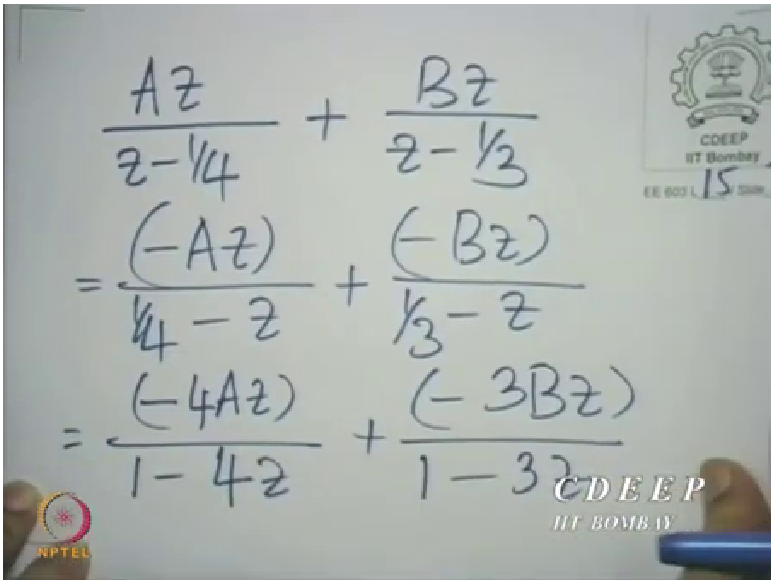

So, you have $\frac{Az}{z-\frac{1}{4}} + \frac{Bz}{z-\frac{1}{3}}$ which you can also rewrite as $\frac{(-Az)}{\frac{1}{4}-z} + \frac{(-Bz)}{\frac{1}{3}-z}$. Or if you like, you could even multiply numerator and denominator by *4*. So, you know, let us let us rewrite. So you have $\frac{(-4Az)}{1-4z} + \frac{(-3Bz)}{1-3z}$. And the key is, because $|4z| < 1$ and $|3z|$is also less than *1*, you can expand these two terms in the following way.

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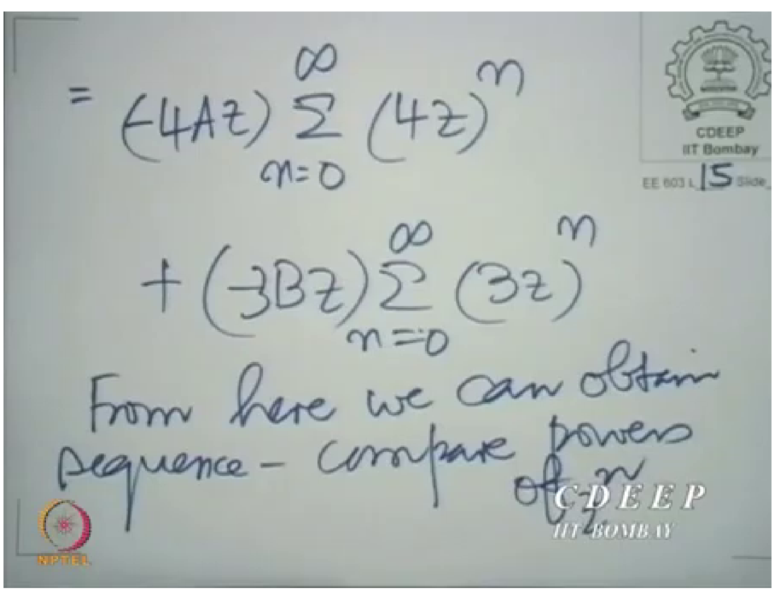

This becomes $(-4Az)\sum_{n=0}^{\infty}(4z)^n + (-3Bz)\sum_{n=0}^{\infty}(3z)^n$. And from here, we can obtain the inverse Z-transform, we can obtain the sequence, compare terms, compare powers of n. So what you need to do is to look at the general expression for the Z-transform and compare powers of z raise to the power of, powers of z, z^n, both of them.

So, I will leave it to you to complete this. In fact, I shall write down the answer. And, you know, leave it to you to prove it.

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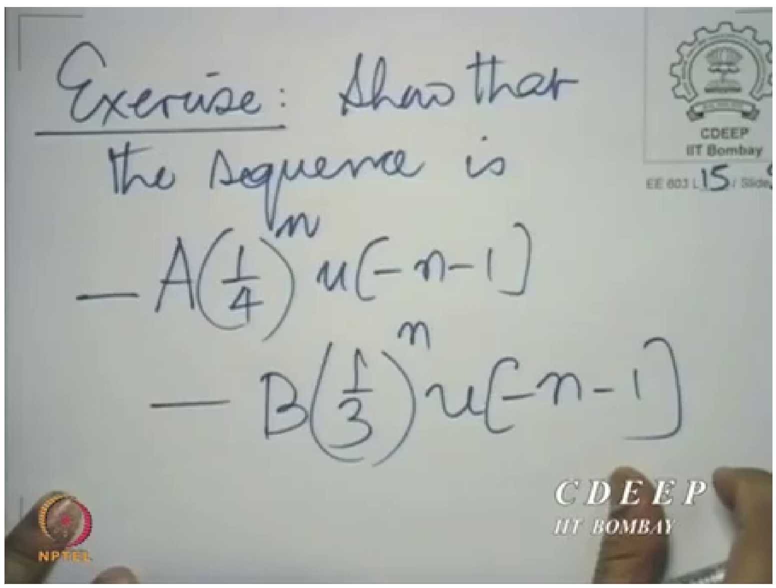

So, I have exercise, show that the sequence is $-A(\frac{1}{4})^n u[-n-1] - B(\frac{1}{3})^n u[-n-1]$. Now we will take the second possible region of convergence and there I am going to have a bit of trouble.

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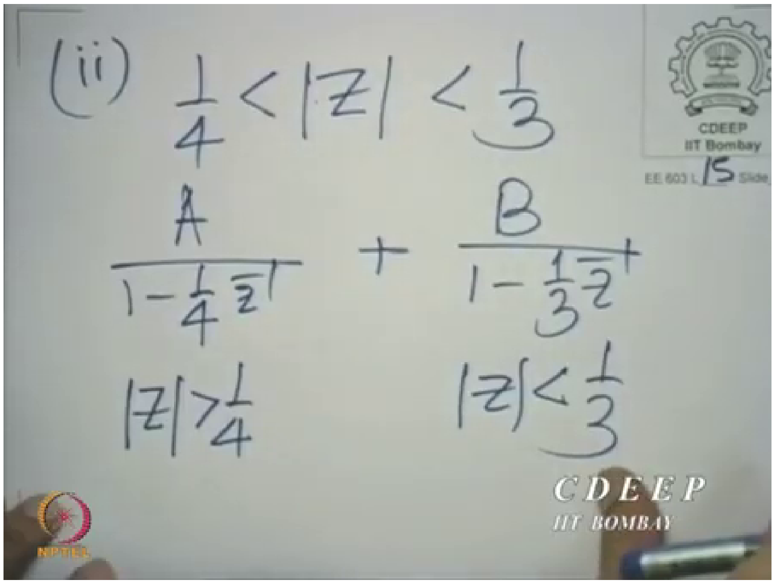

You see, the second possible region of convergence is $\frac{1}{4} < |z| < \frac{1}{3}$. Now this is a little tricky. When this is the case, of course, the partial fraction expansion does not change that holds. So, you have $\frac{A}{1-\frac{1}{4}z^{-1}} + \frac{B}{1-\frac{1}{3}z^{-1}}$. But then $|z| > \frac{1}{4}$ but $|z| < \frac{1}{3}$. So, there is a different way by which we treat this term and this term. So here, you would have to do a different kind of expansion.

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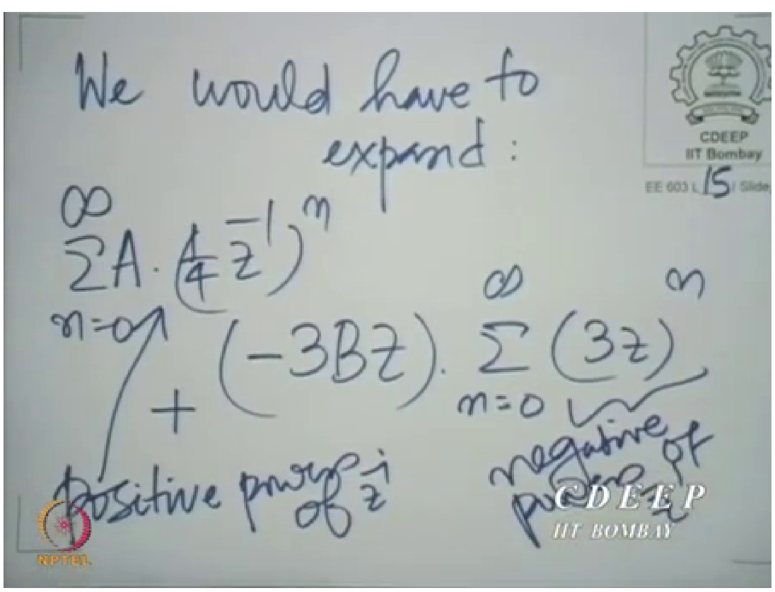

You will have to expand it by keeping the powers of z^{-1} in the first term, so we will have to expand $\sum_{n=0}^{\infty} A(\frac{1}{4}z^{-1})^n + (-3Bz)\sum_{n=0}^{\infty} (3z)^n$. And from here, we would be able to recognize by comparing term by term. Interestingly this term would give us positive powers of z^{-1}.

And this would give us negative powers of z^{-1}. Positive powers of z or negative powers of z^{-1}, positive powers of z^{-1} or negative powers of z, however you like to look at it. But in the expression for the Z-transform, the positive parts of the inverse, give us the samples at positive *n* and the negative powers of the inverse give us the, give us the samples at negative *n*. So, there we go.

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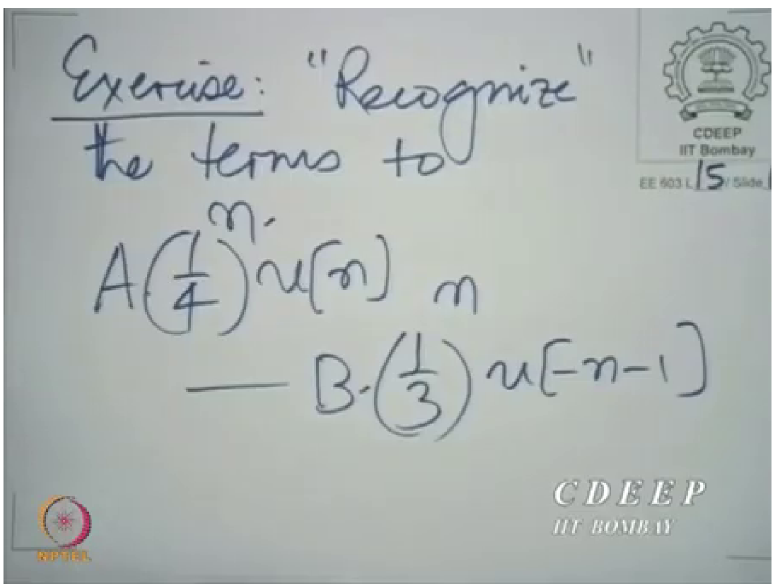

Again, we will leave it to you as an exercise to recognize these terms. Ensure that they come together to form $A(\frac{1}{4})^n u[n] - B(\frac{1}{3})^n u[-n-1]$. Finally, we take the third possibility; the third possibility is where $|z| > 1$. So, of course, if $|z| > \frac{1}{3}$, also $|z| > \frac{1}{4}$.

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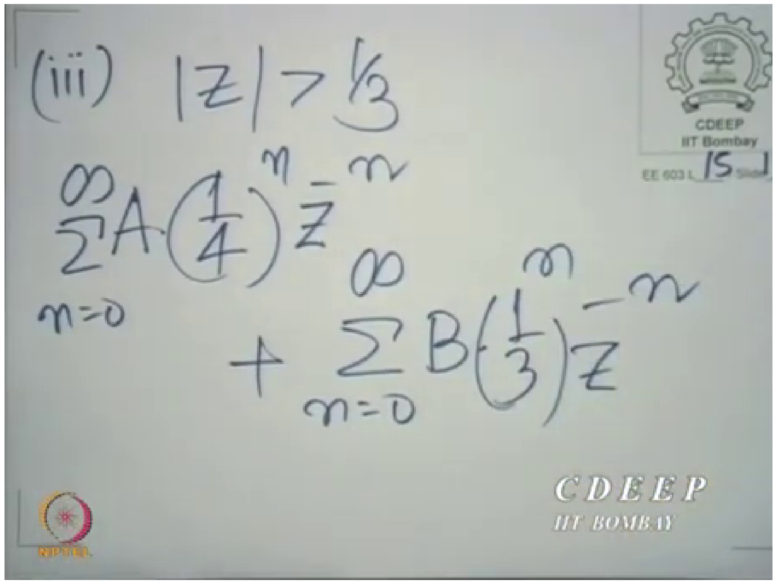

$|z| > \frac{1}{3}$. So, of course, the expansion proceeds as $A(\frac{1}{4})^{n}$, you know, essentially $A(\frac{1}{4})^{n}z^{-n}$, if you please, $\sum_{n=0}^{\infty}$, you know recall, you must always keep the powers of z there. So $\sum_{n=0}^{\infty} B(\frac{1}{3})^{n}z^{-n}$. So this is easy, you know this case, the inverse is very easy to calculate.

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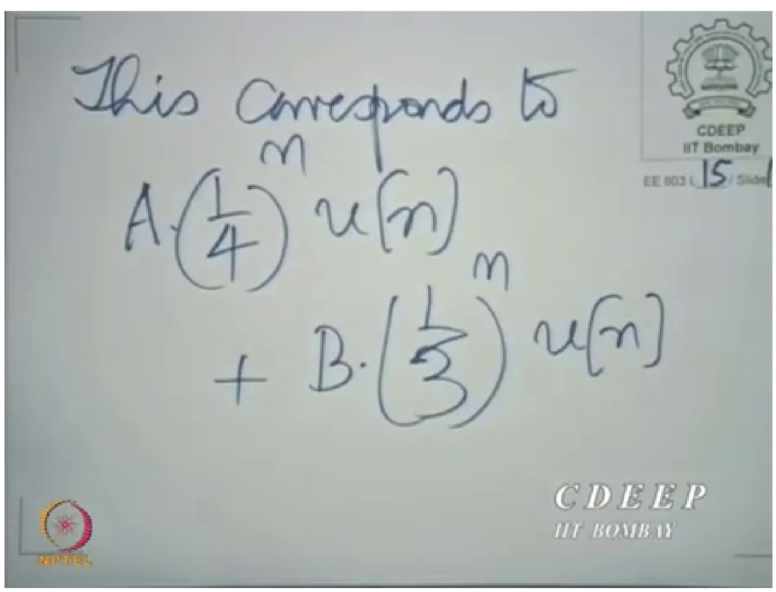

This essentially corresponds to $A\left(\frac{1}{4}\right)^n u[n] + B\left(\frac{1}{3}\right)^n u[n]$. Now, that in the first region of convergence, we got what is called a left-sided sequence, a sequence which is non-zero only for, for only towards the list of a chosen point. In this case, you could take that chosen point to be $n = 0$. So the first region of convergence we have left-sided sequence, what is a left-side sequence, a left-sided sequence is a sequence which is non-zero to the left of a chosen point. Typically, that chosen point is *0*.

Now, the sequence that we got in the third region of convergence is what is called a right-sided sequence it is non-zero to the right of a chosen point. And that of course, chosen point happens to be $n = 0$. In the second, slightly tricky case, we have what is called a double-sided sequence. So, it is neither a left-sided nor right-sided. It is double-sided. In fact, it is doubly ∞. It is ∞ in extent to the left and it is ∞ in extent to the right.

Of course, you can have sequences that are finite in extent to the left and finite extent to the right and of course, they have to be finite length sequences. So finite length sequence is, you can view it as neither left-sided nor right-sided, if you please. But it is definitely not doubly ∞. Anyway, so much so, now, this gives us an idea how we deal with the region of convergence and it also convinces us of the importance of the region of convergence.