Digital Signal Processing & Its Application
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Lecture 15b
Double Pole Example and their Inverse Z Transform

What we have been strategically doing all this while, is taking simple examples of rational Z transform, where we are immediately able to decompose into partial fractions. But the trouble comes when you, for example, have a double pole at a point, repeated pole. Let us take an example.

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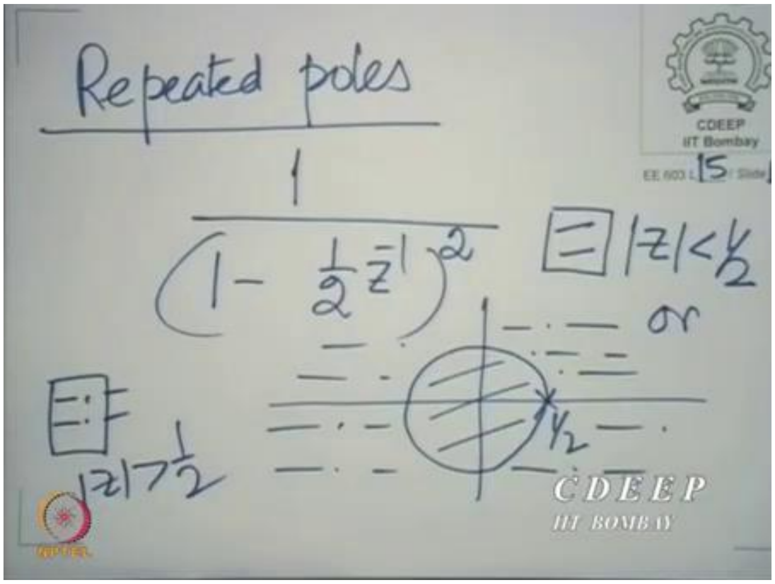

Let us begin with the simple example of just 1 repeated pole and that is it. So, you have 1 by 1 minus half Z inverse the whole squared. And, of course, you have two possible regions of convergence, you know there are, there is only one singularity here that is half. So, you could either have this as the region of convergence, or you could have this as the region of convergence, mod Z less than half or mod Z greater than half.

And we need to be able to find the corresponding sequence for each of these regions of convergence. Now we invoke here the property, we can do it in two ways, we can either treat 1 by 1 minus half Z inverse the whole squared as a product of 2 Z transforms, each of which is 1 by 1 minus half Z inverse. But you know, this is specific to this case.

Suppose you had 1 by 1 minus half Z inverse, the whole to the power 6, what exactly would you do? You know, so we need to evolve a general strategy. And if you take a product of Z transforms, you are going to start convolving sequences, convolution is in general, a more complex operation then what we are going to do. We will invoke the property of taking the derivative of a Z transform.

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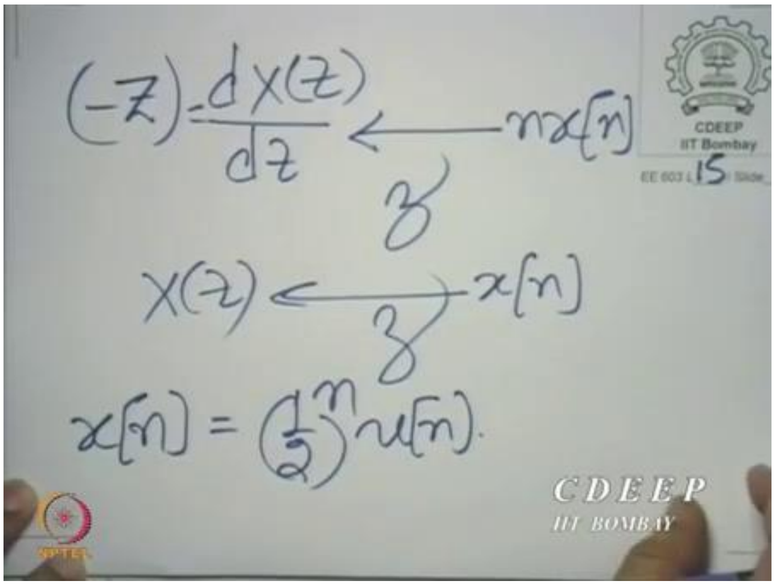

So, we invoke the fact that minus Z times dx Z dZ corresponds to n times x n, where the Z transform of x n is X Z, we invoke this property now. And we take; we take the specific x n equal to half raise the power of n un. Whereupon remember the Z transform, we have dealt with this Z transform before.

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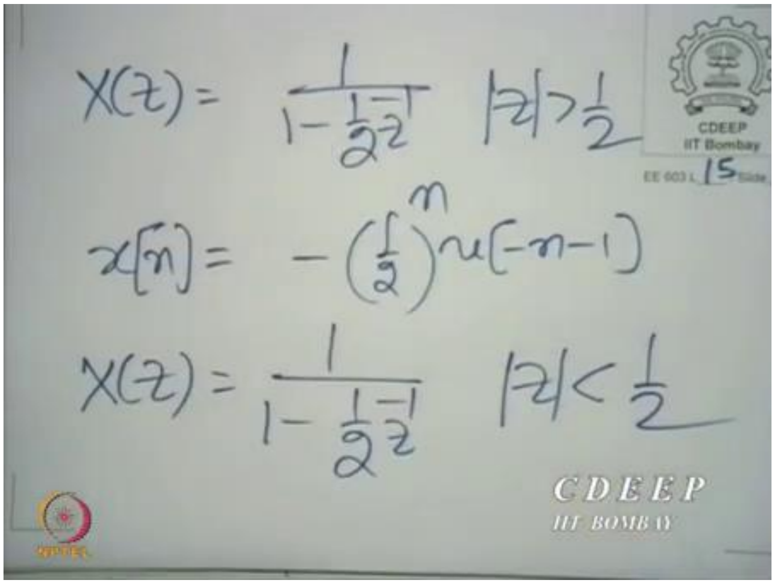

So, X Z would then be 1 by 1 minus half Z inverse with mod Z greater than half. Of course, we can also take x n to be minus half raise the power of n u minus n minus 1, whereupon X Z would be the same expression, but with more Z less than half as the region of convergence, so we will keep both of these in mind.

It is not really terribly a cause of worry whether we have this region of convergence or this region of convergence after we take the derivative. So, let us take the derivative.

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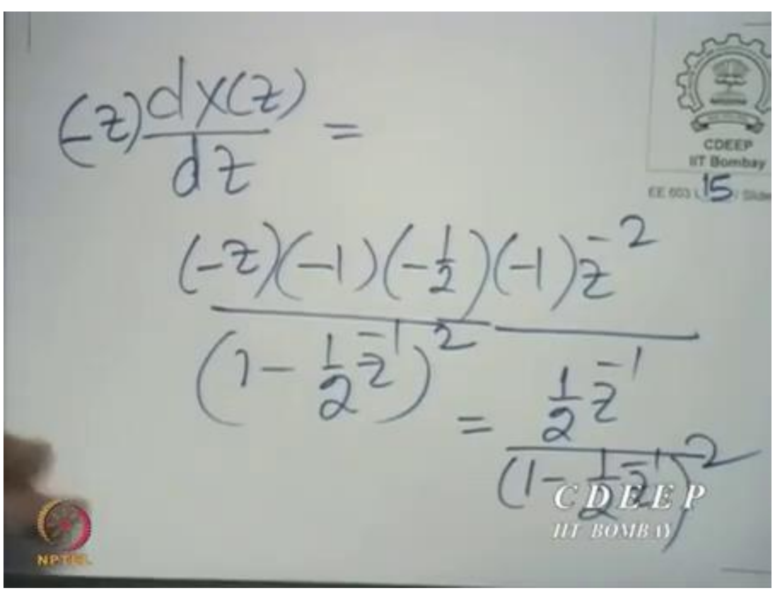

So, the derivative of X Z, as you see here is minus 1 1 minus half Z inverse raise to the power minus 2 times minus half minus 1 Z raise to the power minus 2. Minus 1 1 minus half Z inverse squared or this raise the power minus 2, and then the derivative of what is inside that is minus half times minus 1 Z raise to the power minus 2.

And if you multiply this by minus Z, you get minus Z in the numerator as well. So, let us simplify this. You see very clearly, here we have 4 minus signs that makes it a plus. You have half Z inverse, 1 by 1 divided by 1 minus half Z inverse squared. Now how do you get, you see what we have here is what we wanted except for the factor of 1, except for the factor of half Z inverse in the numerator.

Now, having a half is not a problem, you can remove the half, you know, except the Z transform is a linear operator, so that is not a problem, you can just remove the half by multiplying by 2. What do you know about the Z inverse? When you want to remove a Z inverse, you need to multiply by Z, multiplication by Z is equivalent to advancing the sequence by one step. That means replacing n by n plus 1.

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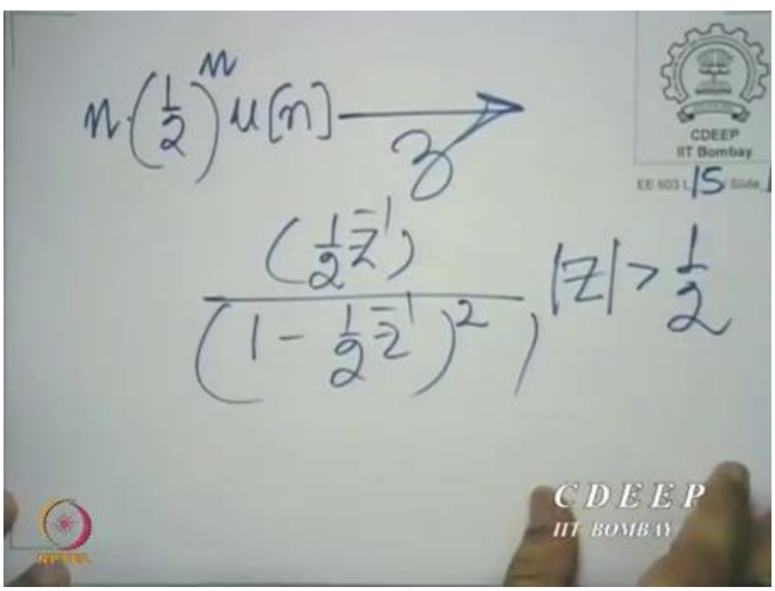

So, it is very clear to us that n times half to the power of n u n would have to Z transform half Z inverse divided by 1 minus half Z inverse the whole square with the region of convergence mod Z greater than half. This follows in a straightforward way from what we have done so far.

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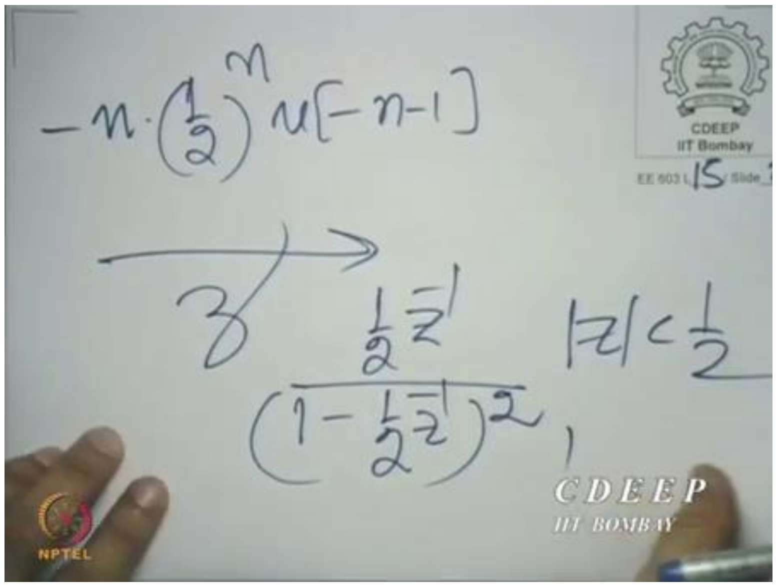

And, you know, in continuation n times half raised to the power of n u minus n minus 1 negative of this as the Z transform, same thing but mod Z less than half, simple. Now, let us call these sequences, let us give these sequences a name. So, let depending on the region of convergence, either you have the first or the second sequence for half Z inverse by the same denominator.

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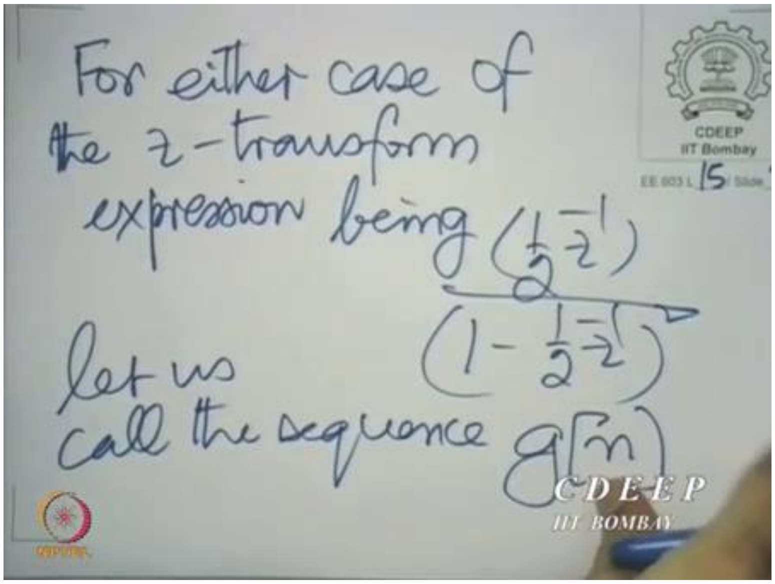

Let us say either case of the Z transform expression being half Z inverse divided by 1 minus half Z inverse squared, let us call the sequence g n. So, what we are saying is, you see the strategy is the same, whether you have the first region of convergence or the second the strategy is multiply by 2 and then advance.

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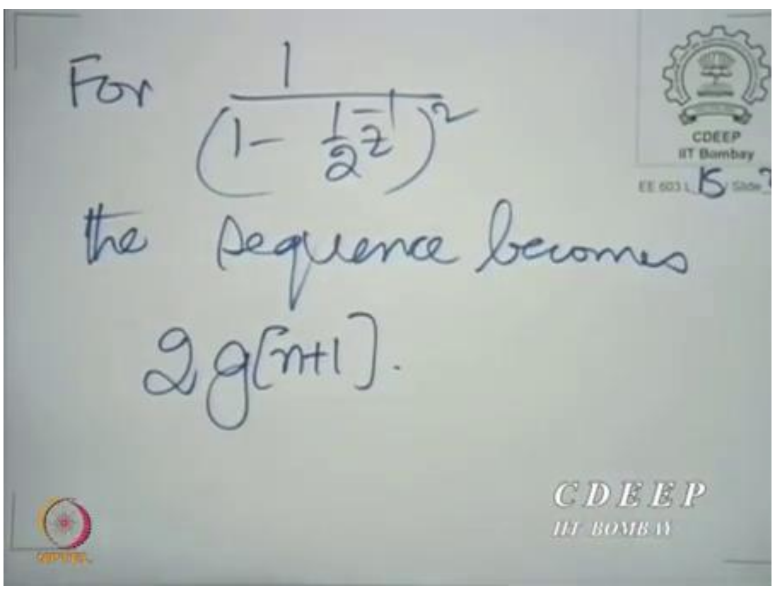

So, for 1 by 1 minus half Z inverse the whole squared, the sequence becomes 2 times g n plus 1, simple. Now, this is a general strategy that we can use. In fact, I leave it to you as an exercise to prove the following or rather to work out the following. So, I leave it to you as an exercise.

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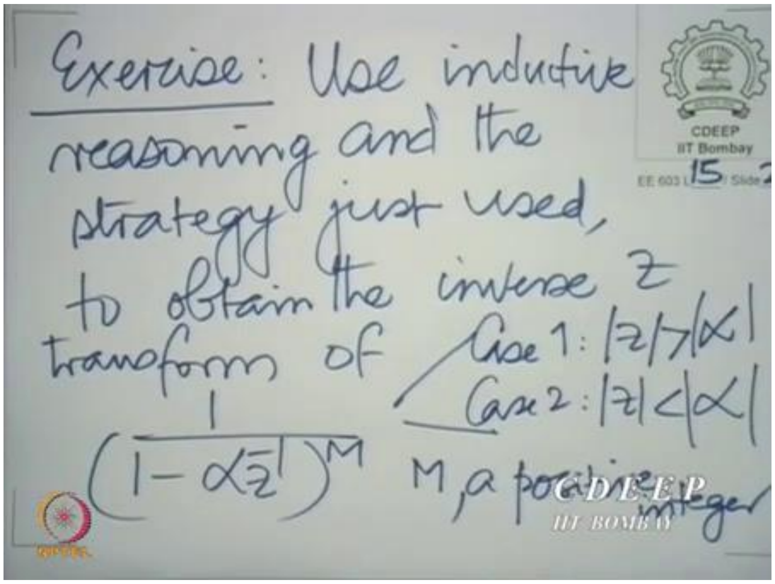

Use inductive logic, inductive reasoning and the for the strategy just gone by to obtain the inverse Z transform of 1 by 1 minus alpha Z inverse the whole to the power M, M is a positive integer. And for 2 cases, case 1 mod Z greater than alpha, case 2 mod Z less than alpha, of course, mod alpha. Now you see once we have done this, we are more or less in good shape for dealing with any sequence any, any Z transform that has multiple poles in the denominator.

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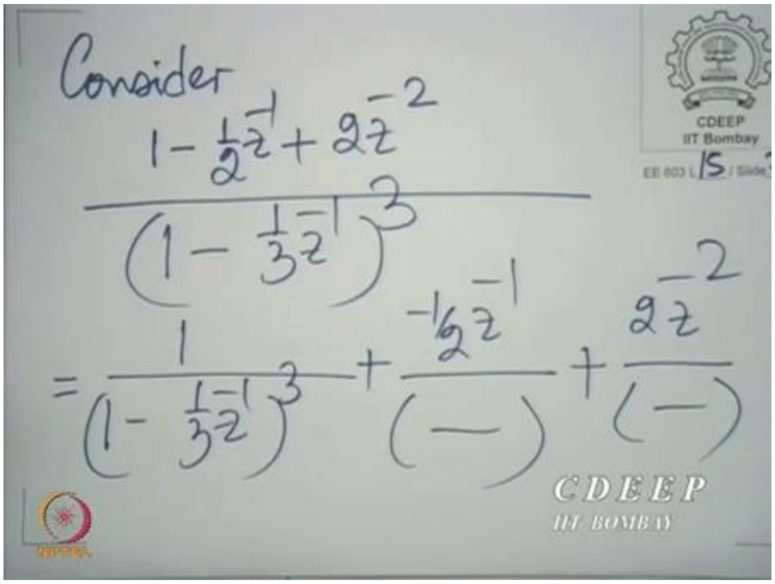

For example, suppose we happen to have multiple poles, suppose we had a triple pole at alpha equal to half these are, you had a term of the form, let me write it down. You had a term, so consider, for example, say 1 minus half Z inverse plus 2 Z to the power minus 2 divided by 1 minus one third Z inverse the whole to the power 3.

Now you can always rewrite this, you can rewrite it as 1 by 1 minus one third Z inverse to the power 3 plus minus half Z inverse by the same thing plus 2 Z to the power minus 2 by the same thing. And if you know the inverse that transform for this with the given region of convergence, you know it for this as well, all that you do is multiply that inverse Z transformed by minus half and then delay it by 1 step, you know it for this to multiply that inverse Z transform by 2 and delay it by 2 steps.

So, you see, when you have a partial fraction expansion now, you have to be careful. If you have multiple polls, a partial fraction expansion needs to allow a degree less than the number of repetitions of that poll, or number of occurrences of that poll in the numerator, just less than. So, a term, typical term in the partial fraction expansion, when you have a triple poll, as you have seen at Z equal to one third would have a second degree term in the numerator.