Digital Signal Processing & Its Applications
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Lecture No. 03 b
Idealized Sampling, Reconstruction and Aliasing

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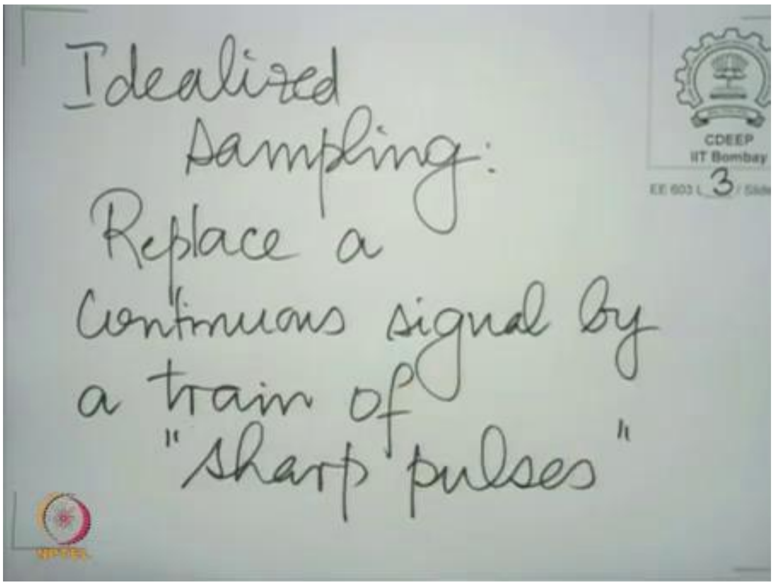

Idealized sampling means to replace a continuous signal by a train of pulses. Sharp pulses, if you would like to call it that. Let us sketch what we are saying. What we are saying is, if you have this continuous signal and if you sample it here.

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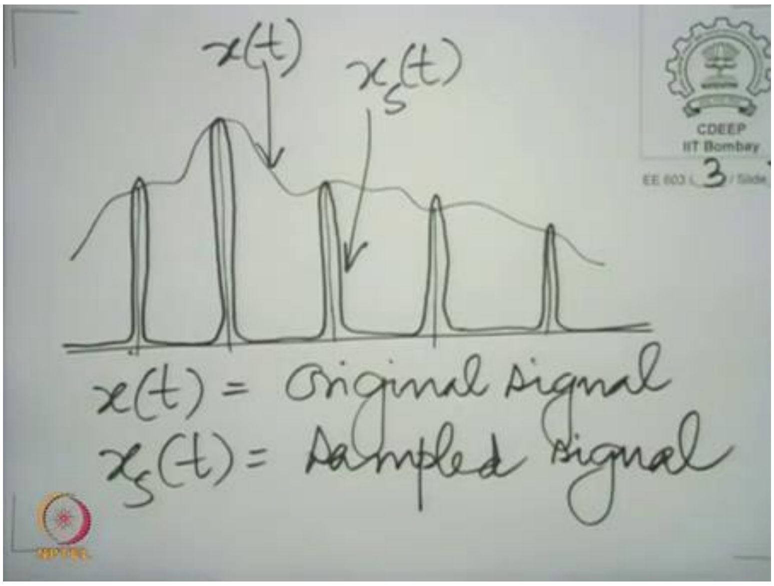

And here and then here, and here, here, what you are doing is just ultimately considering this signal. Let me call this X S of T. and let me call the original signal X T. Now, what is the purpose of reconstruction? The purpose of reconstruction is to get back X T from X S T, simple. And where is the problem? Now we know the problem, the problem is not that you do not have an X T, which you can get back, you have just too many of them. If X T, even if X T were one Sine wave, this process of sampling has created an infinite number of Sine waves, which could have generated those samples.

So, the problem is not that you have no X T. The problem is you have just too many X T, which could have all resulted in the same set of samples. Now the difficulty is to identify which is the correct X T. And that is where we need to see when we will be in a position to do so. You see, a simple approach to identifying the correct X T is to go back to what we did on the frequency axis.

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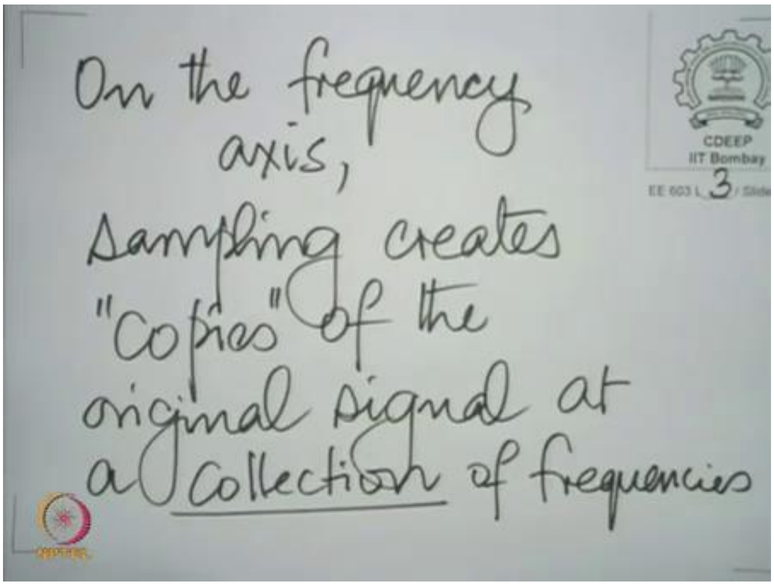

On the frequency axis, what sampling does, sampling creates copies at a collection of frequencies. In fact, let me try and illustrate this to you with an example, or at least with some intuitive explanation.

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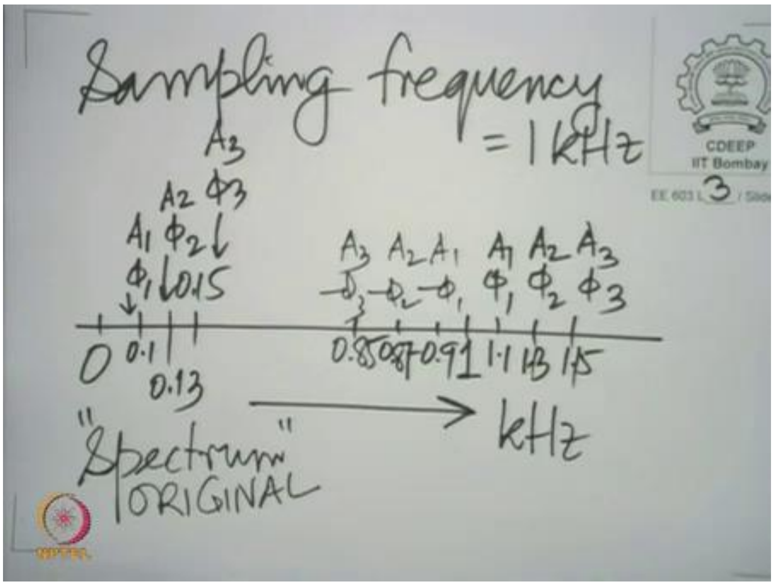

So, we have done the exercise for one Sine wave, but let us do it when you have a few Sine waves. So, suppose we have a few Sine waves, we have a 0.1 kilohertz Sine wave, we have a 0.13 kilohertz Sine wave, and a 0.15 kilohertz Sine wave. So, what is the copy that is created around 1 kilohertz when you sample at 1 kilohertz again, you have. So, what I am saying is you have these three. In fact, let us give their amplitude some values. So, we could say this has amplitude A_1 phase $¢_1$, this Sine wave with amplitude A_2 phase $¢_2$, and this one has amplitude A_3 phase $¢_3$.

Then, if the sampling rate is 1 kilohertz or sampling frequency is 1 kilohertz, if you prefer to call it that. Then you get a copy at around 1 kilohertz, you get 1.1, you get 1.3, and you get 1.5. And of course, the amplitudes are A_1 $¢_1$, A_2 ¢ phases amplitude and phase I mean A_2 $¢_2$, A_3 $¢_3$ here, and you also get copies behind. So, you get 0.9, 0.7 oh well, I am sorry, this should be 1.1, this should be 1.13. So, in fact 1.15, this would be 0.9, and this would be (1 - 0.13) that is 0.87. And this would be (1- 0.15), and that is 0.85.

The amplitudes here would-be A_1, A_2, and A_3 respectively, but the phases would be - $¢_1$, - $¢_2$, - $¢_3$. And you can keep doing this exercise, you can do it around 2 kilohertz, you can do it around 3 kilohertz, and so on, but the central idea is that you have this original, what we call spectrum. Now, we will bring in the word spectrum, spectrum is the decomposition of the original signal in terms of the Sine waves that comprise it, you have this original spectrum. So, let me repeat what is a spectrum? Essentially a spectrum is a Fourier transform.

What is the Physical Meaning of the spectrum? The spectrum is essentially the collection of Sine waves specifying the frequencies, amplitudes and phases. Now, here, we have a discrete set of frequencies, we have 0.1 ,0.13,, 0.15 we could have a continuum of frequencies. So, remember, if the signal is a periodic, then we need to have a continuum of frequencies, the frequency axis could be continuously occupied in a small segment, then you have a band of frequencies occupied.

In fact, you talk about the band of frequencies on which the signal lies. So here, for example, we have taken a case where there are only 3 frequency, 3 Sine waves 0.1 ,0.13, 0.15. So, the band of frequencies is between 0 and 0.15. But of course, you could have a continuous occupancy of the frequencies all the way from 0 to 0.15. So, we will say, in either case, that the band of frequencies occupied is between 0 and 0.15, the difference is here we have a discrete set of frequencies occupied, you could have a continuous access occupied all the way from 0 to 0.15. Whatever it is.

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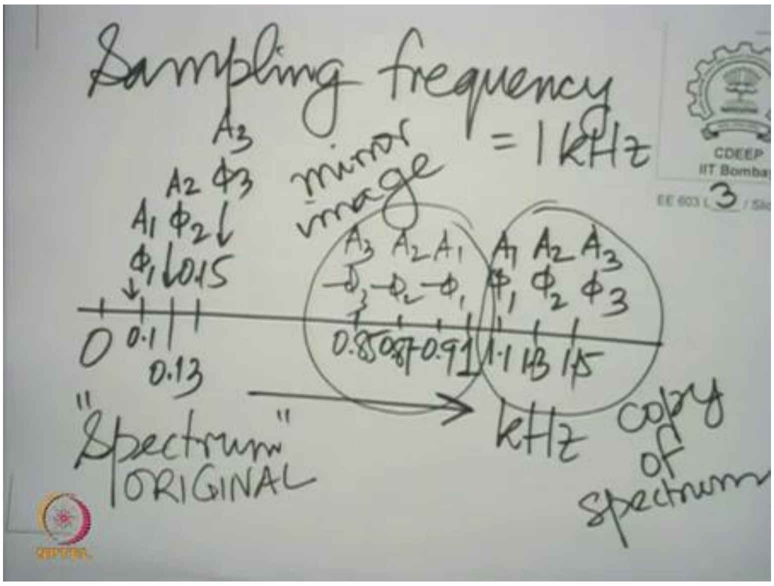

What is the effect of sampling? The effect of sampling is that the original spectrum is copied as it is after 1 kilohertz, after 2 kilohertz, after 3 kilohertz, and so on, and is copied in mirror-image fashion. So, you can visualize this has been mirror-imaged, and in mirror imaging, you have reversed the phase as well. So, you have a copy of the original spectrum here. And a mirror image of the spectrum here. And this is repeated at every multiple of the sampling frequency. So, around 2 kilohertz, you have a copy of the spectrum after 2 kilohertz, and you have a mirror image of the spectrum behind 2 kilohertz.

In mirror imaging, the phase is also reversed. So, it is very easy. How do you reconstruct X T from X S T. If you look at the spectral domain, it is very easy. In the spectral domain, what we need to do is to retain this and throw away this, throw away not only this, all such copies. Let us write that down. So very simple idea.

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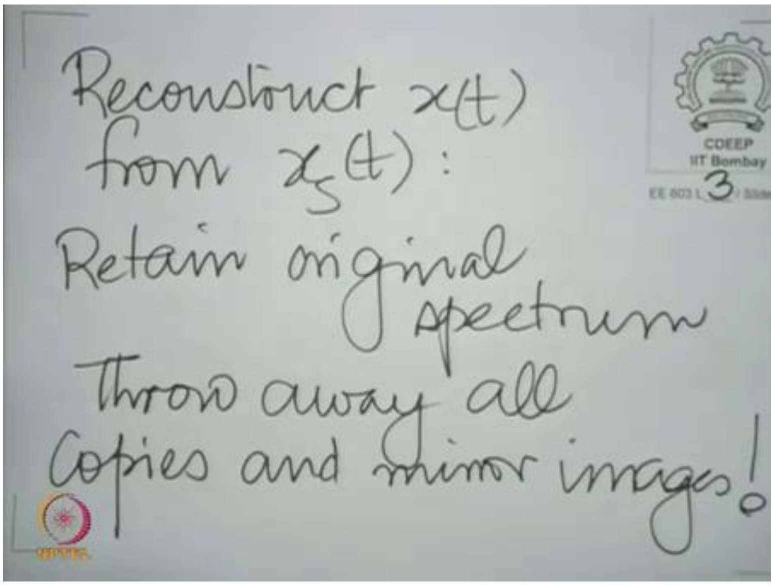

Reconstruct X T from X S T means retain the original spectrum and throw away the copies. All copies and mirror images in fact. Now, the answer is very simple. When can you do this? You can do it. If the copy is in the mirror images do not mix with the original spectrum. Let us write that down to.

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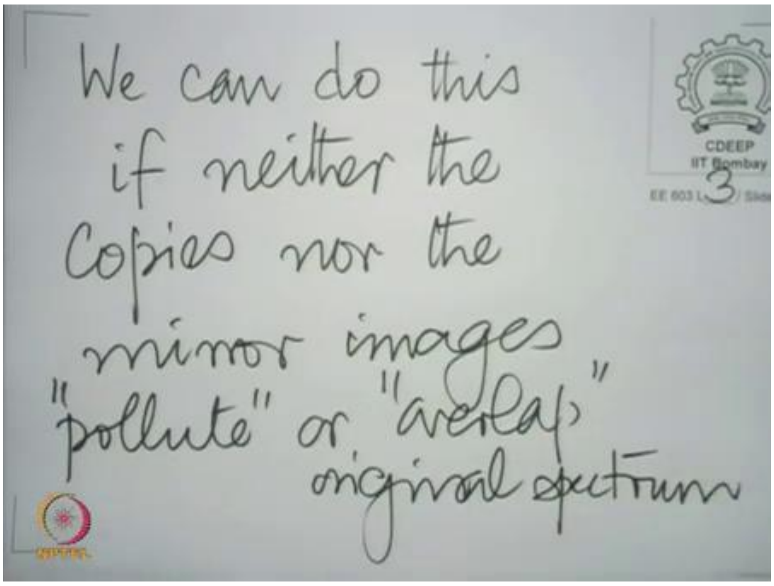

We can do this if neither the copies nor the mirror images pollute or overlap the original spectrum. And again, now we have a simple answer, which is the troublemaking copy or mirror image. In fact, if you think about it, it is only the first one which is the troublemaker.

And that too not the first mirror, not the first copy. It is the first mirror image which is a troublemaker. That is the one which comes first as you move away from the original spectrum. So, let us write that down.

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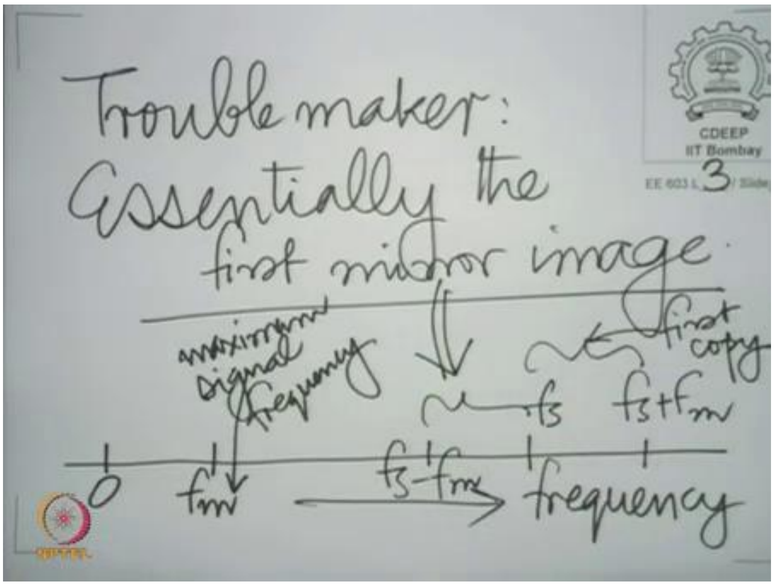

The troublemaker is essentially the first mirror image. So, you see again, let us write this down on the frequency axis. And this time, let us use more general symbols. Let us not confine ourselves to this 1 kilohertz 0.1, 0.11 and so on. Let us use 0 to F M, this is the maximum frequency or the band occupancy. We will say that this whatever Sine waves there are in the signal are all between 0 and F M and the sampling frequency is F S.

So, the first mirror image occupies the range between F S and F S plus F M. I am sorry for the first copy. The first copy occupies the range between F S and F S plus F M. And the first mirror image occupies this range and F S minus F M. So, now, we are very clear, as long as this point is after this point, we are okay.

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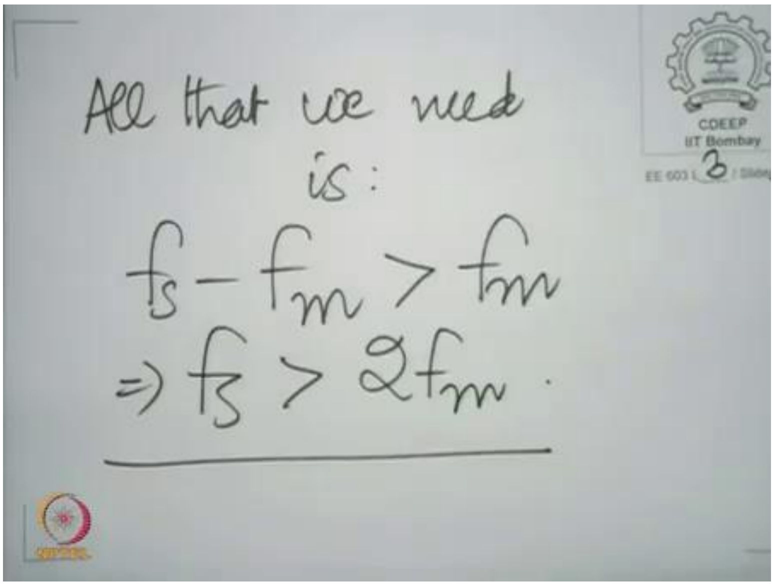

So, all that we need F S minus F M should be greater than F M, or F S should be greater than twice F M. Now, this is a very simple principle, this is called the Nyquist principle of sampling. In fact, I dare say many of you probably know this principle. You know, this is called the Shannon, Nyquist ,Whittaker there are so many names associated with this principle.

In fact, many people have arrived at this conclusion by using different approaches. We have also used a simple approach to come to this conclusion, perhaps little intuitive, maybe not that rigorous, though, we can complete the same discussion with rigor. And I have left the part of that as a challenge for you. Whatever the case, I am saying that the principle is very simple.

One simple way to get back X T from X S T is to take the lowest part of the frequency axis has containing the original spectrum, and to ensure that all the copies and the mirror images are well spaced away from the original spectrum, they have not polluted the original spectrum.

So, now, this brings us to the situation when will there be a problem when this does not happen? In other words, if that first mirror image happens to overlap on the original spectrum there is trouble. And what is that trouble called? That trouble has a name in the literature.

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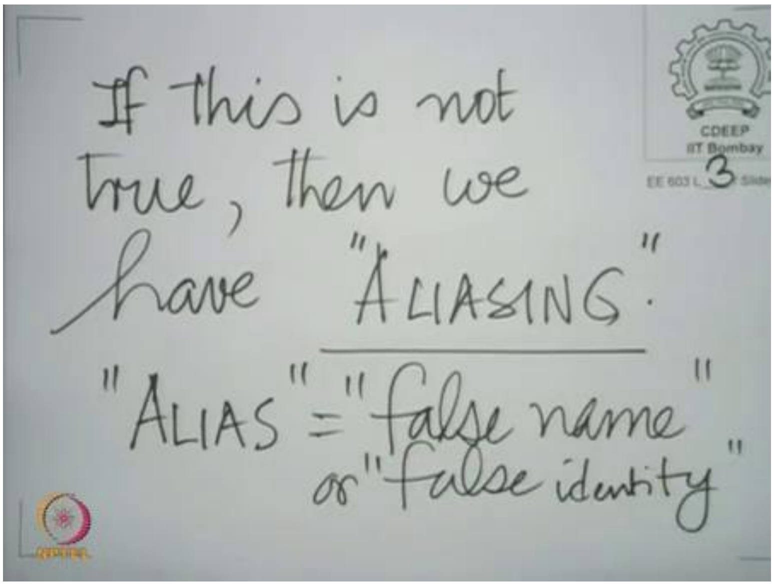

If this is not true, then we have “ALIASING”. Now, I want to spend a couple of minutes on the term ALIAS. You know how we use the word ALIAS in common discussions in common parlance, quite frequently. There are two situations in which the word ALIAS is used. One in the context of troublemakers and the other in the context of highly gifted people, writers, poets. So, you know that some writers or poets take on ALIAS? I do not need to give you examples. There are several writers either in the English language or in the Indian languages, who like to take on different pen names, as they call them. They call ALIASES.

And then you also have many people who create trouble in society who take up ALIASES, or even if they do not take up ALIASES, the authorities give them ALIASES or the authorities know them by their ALIASES. Now, the reason why I mentioned this is that the spirit behind this is also true in signal processing. ALIASING is not necessarily always bad, it can be done in very desirable situations or it can be done in very undesirable situations. In this first course, we shall treat ALIASING as undesirable in a way. Because we want a very simple methodology for reconstruction.

We do not want to complicate our reconstruction process too much. We want the original spectrum to be unpolluted. Now, I will explain why the word ALIAS is appropriate. You see, take the frequency let us say take the frequency 0.7 kilohertz. Right? Now, suppose you sampled at a rate of 1.4 kilohertz exactly. If you had 0.7 kilohertz as the signal frequency or if you had a signal whose maximum frequency component went all the way up to 0.7 kilohertz, you should have sampled at more than 1.4 kilohertz.

Suppose you happen to sample exactly at 1.4 kilohertz. Where would the first mirror image come at (1.4 -0.7) that is at 0.7 So, the original signal and it is first mirror image overlap at 0.7. So, what happens is that the first mirror image assumes a false identity, actually it is not a 0.7 kilohertz signal, but it pretends to be the 0.7 hertz signal which the original signal is.

So, in other words, you have a false or another man posing to be a 0.7 kilohertz frequency and interfering with the original 0.7 kilohertz signal frequency. That is why we call it ALIASING. There is an imposter there is an intruder there. So, you can take this argument further. Now, suppose the original signal had components all the way up to 0.9 and you have sampled it only at 1.4. Now what is going to happen? (1.4- 0.9). So, you have 0.5.

Now, you have a component at 0.5 kilohertz coming from the original signal and you have a component at 0.5 kilohertz contributed by the first mirror image. Actually, that component at 0.5 kilohertz is a mirror image of the component at 0.9 kilohertz in the original (1.4-0.9), that is 0.5. So, you have an original 0.5 kilohertz component, and you have a false component at 0.5 kilohertz coming actually as a mirror image of the 0.9 kilohertz component as a consequence of sampling at 1.4 kilohertz.

So, you have an imposter and the original one sitting together at 0.5 kilohertz. Assuming a false identity and interfering with the original signal, we can understand it like that. Now, this is the bad part of it. We cannot reconstruct if we choose to use the simple approach of taking a region around 0 frequency to reconstruct. But I want to put a second challenge before you here. Let me explain the challenge to you, but I would not answer it too far here, because it is a challenge after all. The second challenge is.

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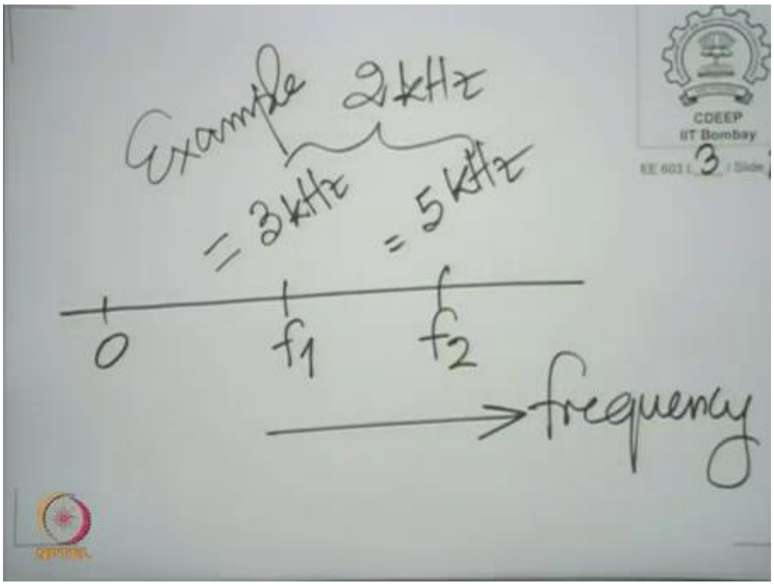

Suppose, I know that I have a band occupancy of the signal. That means if I look at the Sine waves in the signal, the frequencies where there are nonzero amplitudes are between some F_1 and F_2. Let us for the sake of example, take F_1 equal to 3 kilohertz, F_2 equal to 5 kilohertz.

So, what I mean is, I have a band of only 2 kilohertz occupied actually. But if I go by the highest frequency occupied, it is 5 kilohertz. So, of course, I am guaranteed that if I take a sampling rate of more than 10 kilohertz here, and of course able to reconstruct the original signal by retaining all the frequencies up to 5 kilohertz and throwing out the rest. That is perfectly possible.