Digital Signal Processing & Its Application
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Lecture 17
Examples - Causality and Stability on ROC

A warm welcome to the seventeenth lecture on the subject of Digital Signal Processing and its Applications. We use this lecture to build a few more ideas in the context of system theoretic concepts in the Z domain. So, in the previous lecture, and in the lecture before that we had looked at the rational Z transform in detail. In the previous lecture, we also saw an example of an irrational Z transform.

And we also explained, you know what we meant by rational Z transform in detail, what the consequences were in terms of invertibility and so on. Now, what we need to do is to look at systems which have rational impulse responses, in other words rational systems. And we wish to characterize those systems from the point of view of other properties that an LSI system may or may not have. So, in fact, let us quickly recapitulate.

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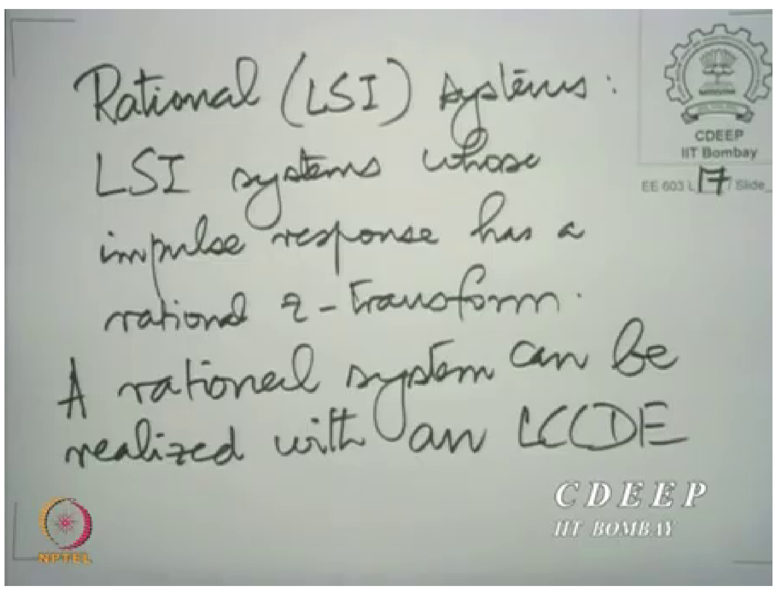

We have so far talked about rational systems. Rational systems are of course LSI, they are LSI systems whose impulse response has a rational Z transform. Incidentally, we also talked about the importance of rationality, from the point of view of being able to convert a system into a realization. We saw last time that a rational system can be realized.

In fact, a rational system can be realized with an LCCDE, Linear Constant Coefficient Difference Equation. And a linear constant coefficient difference equation can be translated into a hardware or a software structure, we saw that too. In fact, we saw a specific form of realization last time, there are other ways in which the same rational system can be realized and in fact, we shall be spending quite some time on the subject of realization of rational systems, later.

However, at this time, we have assured ourselves, we have convinced ourselves by example, that it is possible to realize any rational system by converting it into a corresponding linear constant coefficient difference equation, and then realizing that linear constant coefficient difference equation. Now, what we did not carefully note last time was that the system that we had realized was also causal.

Stable is not an issue, you know, you can realize both stable and unstable systems, but realizability also does depend on whether the system is causal or not. I mean, there are two ways of looking at it. One is that, if you are dealing with the system in real time, and unless the system is causal, it cannot be realized, because you cannot have future samples available to you, when you are processing the current sample, that is one way of looking at it.

The other way of looking at it is even if you did, the structure of the system must be such that there is a unique way in which you can proceed with the computations of the samples. If that is not true, there is a problem. Now, instead of trying to explain this in general, we would look at the requirements on a rational system for it to be causal and stable. And thus, convince ourselves of the context in which we want to remain in the future now.

You know, instead of trying to look at all possible things that we may not be able to realize it is much more meaningful to look at what we want. Namely, we want a causal system; we want normally a stable system if you want it to have a frequency response. And in that case, we need to characterize the rational system by looking at its poles and 0 and anything else that characterizes it, to see if the system is causal and stable. Now, let me put down the problem that we are trying to address clearly.

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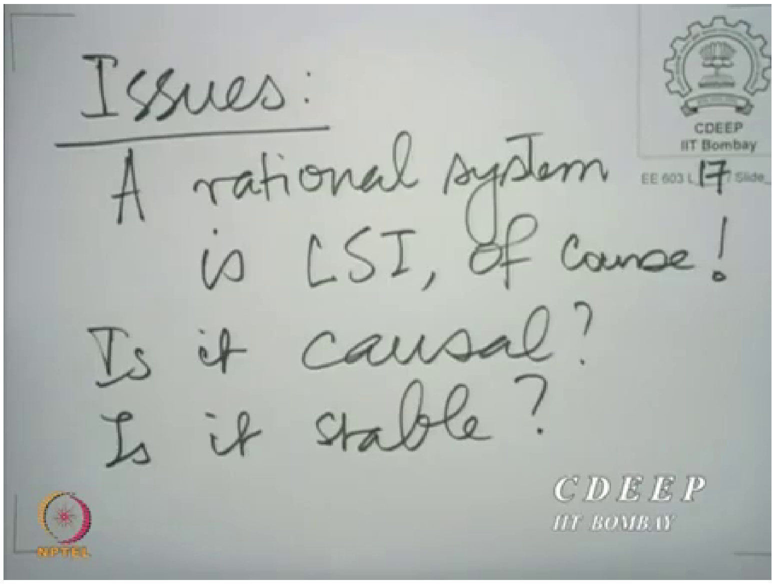

Issues, a rational system is of course, linear shift invariant. But is it causal, is it stable, may or may not be. In fact, it is not at all difficult to see that there are examples of all possibilities. Causality and stability are independent properties, you may have 4 kinds of rational systems, those that are not causal, but stable, those that are causal but not stable, those that are neither causal nor stable, and those that are both causal and stable. In fact, I go back to the examples of the rational transform that we have dealt with to date.

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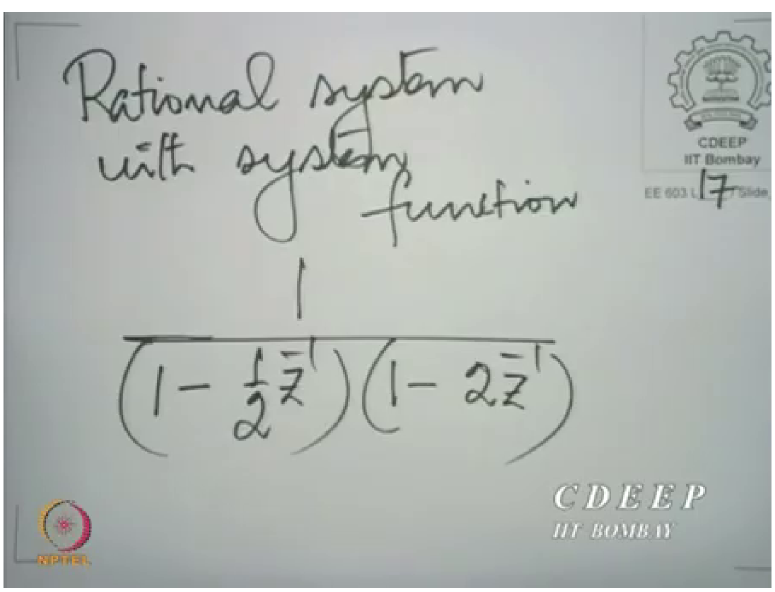

So, let us look at the rational system with system function 1 by 1 minus half Z inverse 1 minus 2 Z inverse. Now, please note, the moment I write this, you should immediately ask me, what is the region of convergence. I told you, we often forget that a rational system must be characterized by both an expression and a region of convergence. So, here I have 3 possible regions of convergence, let us look at each of them, in turn.

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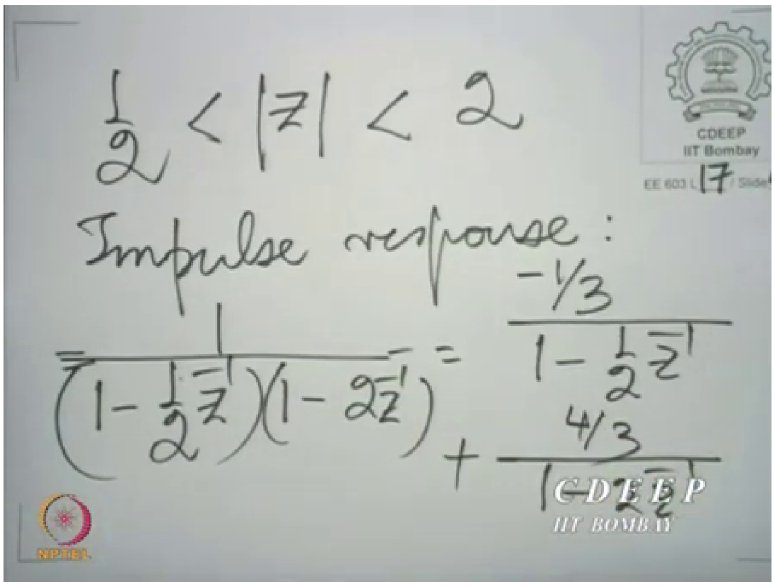

Let us look at the region of convergence mod Z between half and 2. In which case, the impulse response, I leave it to you to complete the working but I will straightaway write down the answer, the impulse response would become some constant. I mean, you know, in fact, let us do one thing, let us complete the working by decomposing this into partial fractions first.

So, you have a term in half Z, 1 minus half Z inverse, and you have a term in 1 minus 2 Z inverse. So, when I want the term in 1 minus half Z inverse, I multiply by 1 minus half Z inverse and put Z equal to half. So, I have 1 minus 2 by half or 1 minus 4 that is 3 minus 3. And then I multiply by 1 minus 2 Z inverse and put Z equal to 2.

So, I have 1 minus one fourth in the denominator, that is 3 4 so 4 by 3. In fact, just for convincing ourselves, let us verify that this is indeed the decomposition. You do not need to do this every time but I am just doing it a couple of times to convince you that you are doing the right thing.

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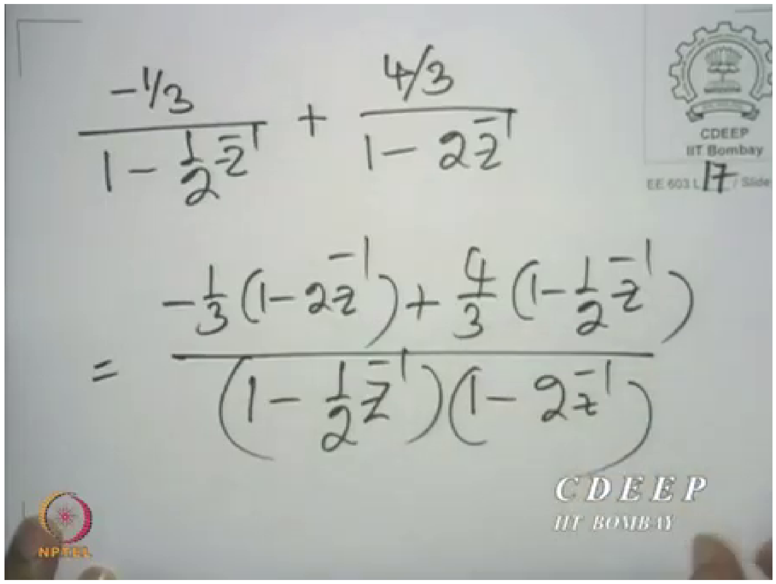

So, minus 1 by 3 divided by 1 minus half Z inverse plus 4 by 3 1 minus 2 Z inverse can be put together, 1 minus half Z inverse 1 minus 2 Z inverse minus 1 by 3 1 minus 2 Z inverse plus 4 by 3 1 minus half Z inverse. And we can see that the constant term is 4 by 3 minus 1 by 3 that is 1. The coefficient of Z inverse is minus 2 by 3, plus 2 by 3 that is 0.

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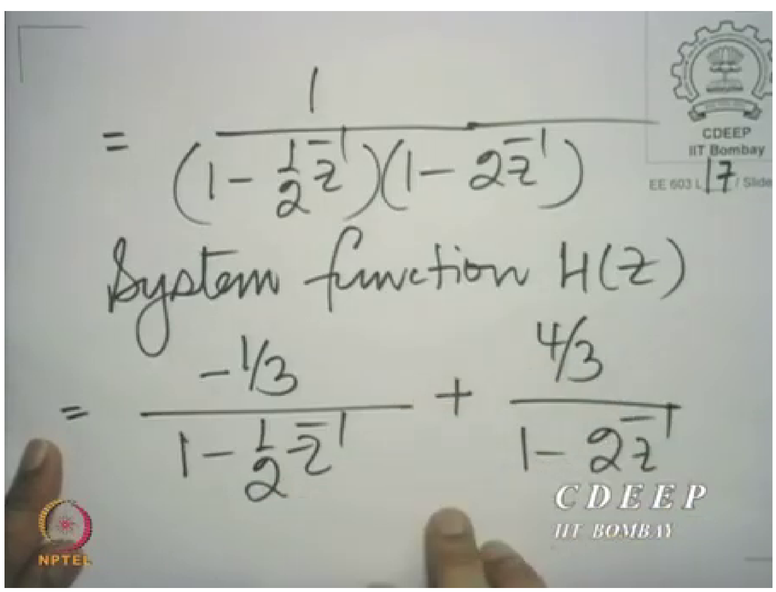

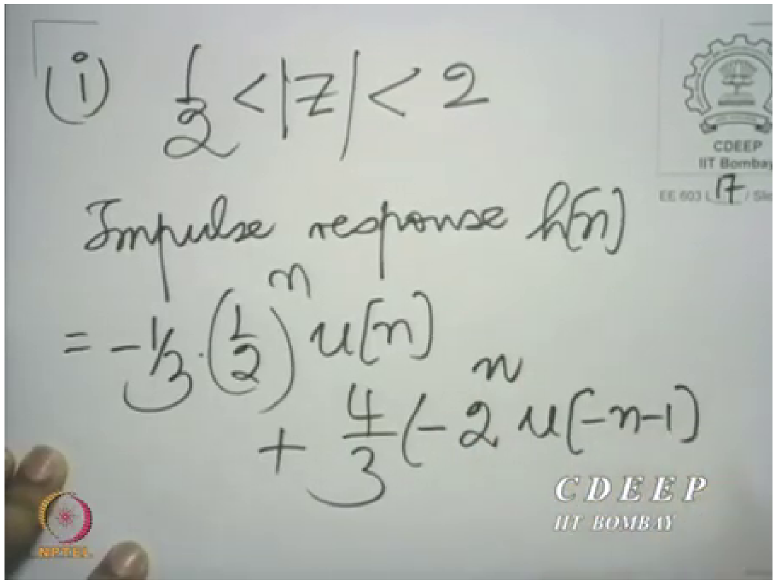

And therefore, we have this equal to 1 by 1 minus half Z inverse 1 minus 2 Z inverse. Anyway, now we see that the system function can be rewritten as minus 1 by 3 1 minus half Z inverse plus 4 by 3 1 minus 2 Z inverse and we need to invert this for each case of the region of convergence. So, as I said let us take the first case as we have considered, namely, mod Z between half and 2.

Now, when mod Z is between half and 2, for this mod Z is greater than half, so you have a right sided sequence coming out of this. And for this mod Z is less than 2, you have a left sided sequence coming out of this. So, all in all, you get impulse response h n to be minus 1 by 3 half raised to the n u n plus 4 by 3 into minus 2 raise to the power of n u minus n minus 1. In fact, let us sketch the simplest response sequence, to get an understanding.

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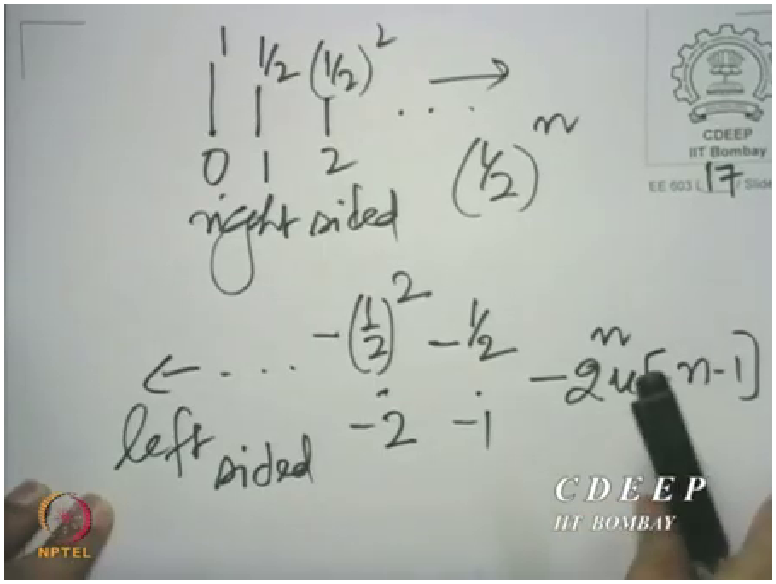

You see half raise to power of n starts from 0, so you have 1 half, 1 half squared and so on. This is half raised to the n. Similarly, 2 raise to the power n u minus n minus 1 begins at minus 1 and at minus 1 it takes the value minus 2 raise to minus 1, so minus half, at minus 2 it takes the value minus half squared, and so on so forth. So, this is 2 raised to the power of n minus 2 raised to power of n u minus n minus 1. This is right sided; this is left side and you can put them together.

You take minus one third times this sequence, plus 4 by 3 times this sequence. Of course, luckily, they do not overlap. So anyway, it means you just multiply the corresponding samples either by minus 1 by 3, if they are between 0 and infinity or by 4 by 3 if they are between minus 1 and minus infinity, easy. So, we know what the impulse responses are. And in fact, here we can find the absolute sum of the impulse response with great ease.

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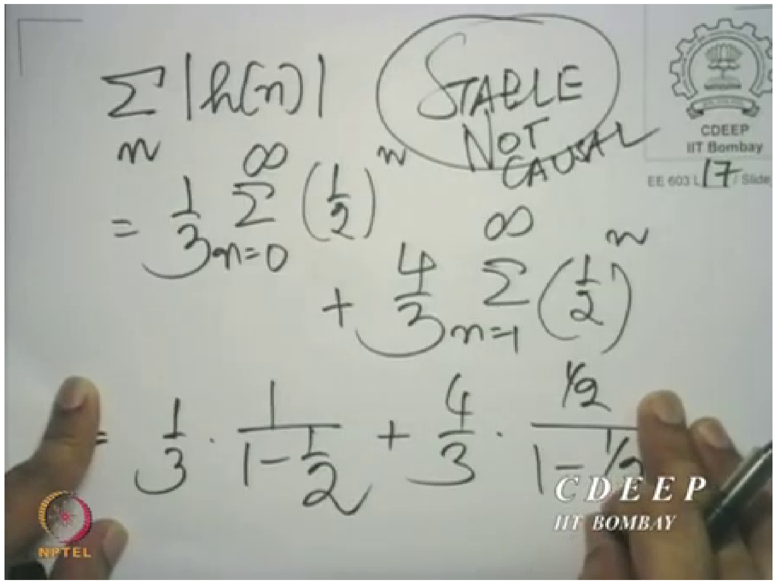

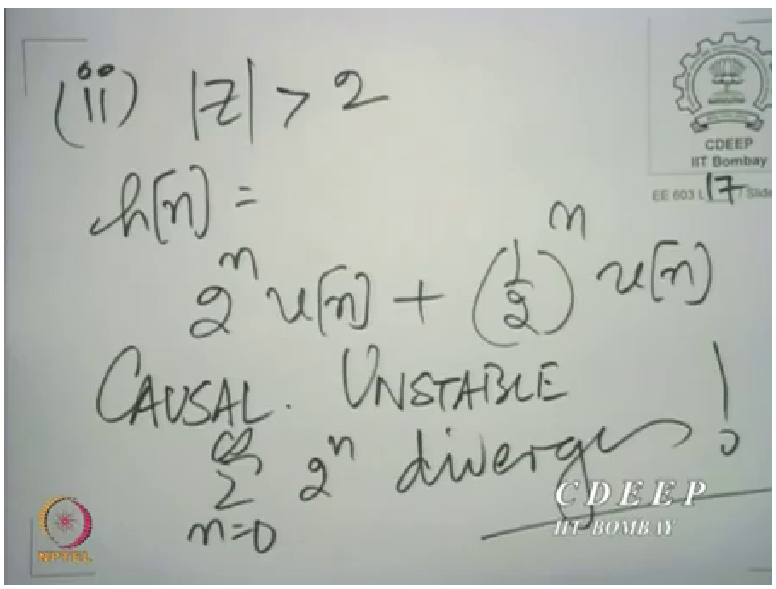

We can, we can easily see that the absolute sum is very easy to calculate, it is 1 by 3 summation n going from 0 to infinity half raised to the n plus 4 by 3 summation n going from 1 to infinity, again half raised to the n. And that is 1 by 3 1 by 1 minus half plus 4 by 3 into half into 1 minus. And of course, you can simplify this. But anyway, it is absolutely summable. So, clearly the system is stable. Now, we take the second region of convergence mod Z greater than 2.

And there of course, you have the impulse response to be very simple 2 to raise the power of n u n plus half raised to the power of n u n. Incidentally, the system that we have just seen is stable but not causal, that is easy to see because the impulse response is not 0 for all n less than 0. So, system is not causal. But here we are, we have a causal system here, but clearly unstable. That is because summation n going from 0 to infinity 2 raise to the power of n diverges, that arises from here that term diverges. So, the system is unstable.

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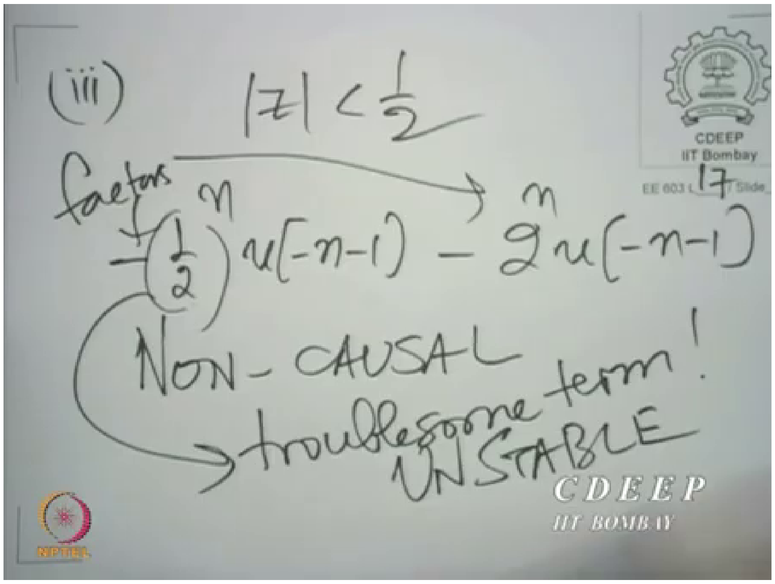

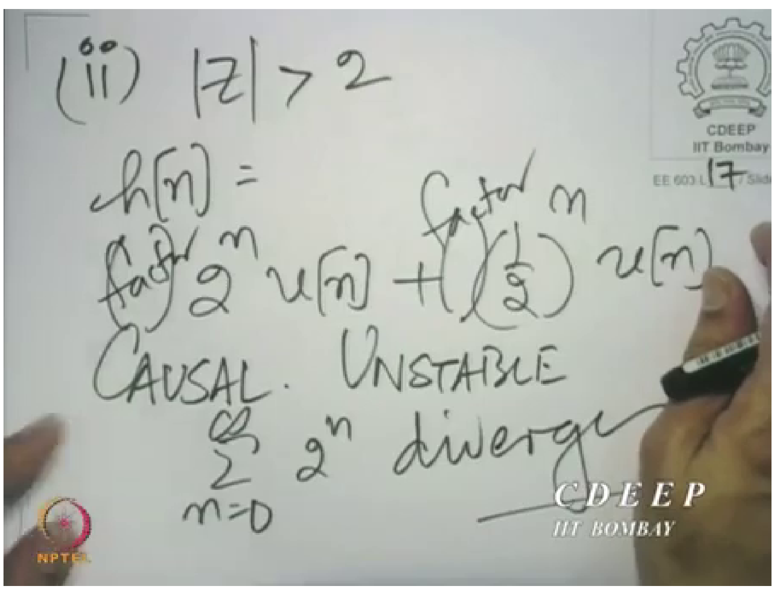

Now we take the third region of convergence. Mod Z less than half, whereupon we have minus half raised to the power of n u minus n minus 1 minus 2 raised to the power of n u minus n minus 1. Of course, by the way, a little correction here, I should put the factors along with it. 4 by 3 and, so you know, so there is it is a, it is a linear combination. So, say factor here and factor here.

I am just in general writing, so you know 4 by 3 and minus 1 by the factor should come so that correction should be there.

Here, again, the factor should come 4 by 3 and minus 1 by 3, that is okay, that does not seriously affect the result. But clearly, this is totally a left sided sequence, this is non causal. And here it is not this, but this, this term that creates the problem. Half raised to the power of n u minus n minus 1 is the troublesome term. It in fact becomes a growing exponential. So, this is the troublesome term.

The system is unstable, because of this term. It is not summable, it is not absolutely summable. So, we have an example of all 3 kinds, non causal and unstable, causal but not stable, and not causal but stable. So, both negatives, 1 negative and 1 positive, you know, now both positives, we want to see an example of both positive that is, of course, very easy.

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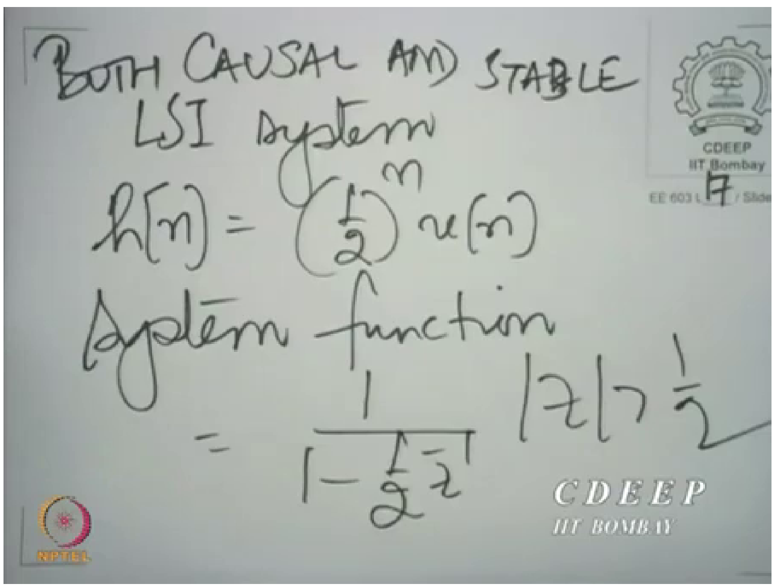

So, if you have the system, LSI system with impulse response h half raise to the power of n u n, the system function is 1 by 1 minus half Z inverse mod Z greater than half. And this is both causal and stable. So, now we have an example of all 4. In fact, we will take this example or this pair of examples to draw some conclusions.

You see, if you notice, among the 3 regions of convergence, for the first example, it was only one region of convergence that gave you a stable system, the region of convergence, which included the unit circle where the frequency response was defined. In the other 2 cases, there would be no frequency response defined. That is because the unit circle is not included in the region of convergence.

So, it looks like the unit circle has something to do with stability. In fact, we will soon see that that is really what matters in a rational system, whether the unit circle is a part of the region of convergence or not. Secondly, we noticed that causality really related to being right sided, that is to be expected. If you want the system to be causal, the impulse response must be 1 sided, and if at all 1 sided right sided.

And otherwise, of course, it must be of finite length and all the samples must be at 0 or afterwards, so right sidedness is required. Now when you have right sightedness, then the region of convergence is outwards, beyond a certain circle. If a sequence is left sided, the region of convergence is inwards, inside the interior of a certain circle.

So, being interior and being exterior, this matters in causality. And how far exterior can you be all the way up to infinity. So, whether or not Z tending to infinity is included has something to do with causality. Now we will formalize this.