Digital Signal Processing & Its Application
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Lecture 17b
Theorems - Causality and Stability on ROC

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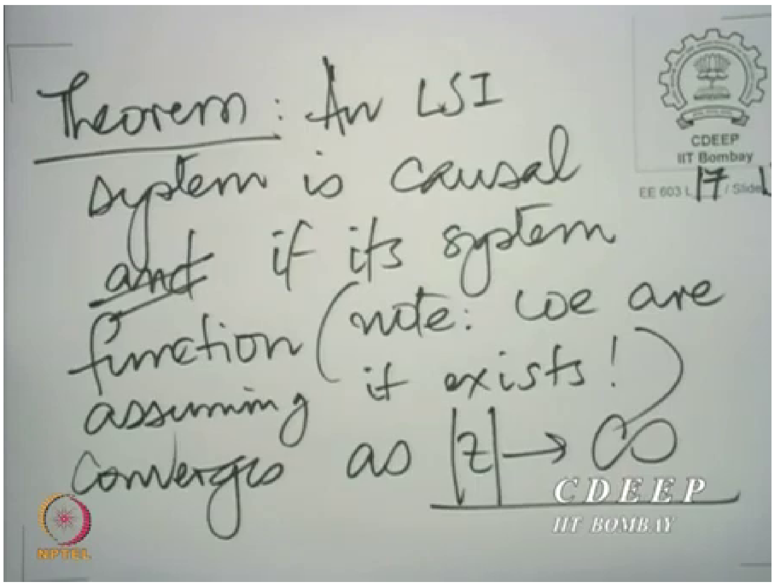

You see first let us prove the simpler of the two results. Let us prove that, an LSI system is causal and rational, well actually it does not have to be rational. I would not write and, I will say it is causal. If it is system function, note we are assuming it exists. That means the impulse response has a Z transforms. If its system function converges as Z tends to infinity or you can say as mod Z tends to infinity. That is very easy to see.

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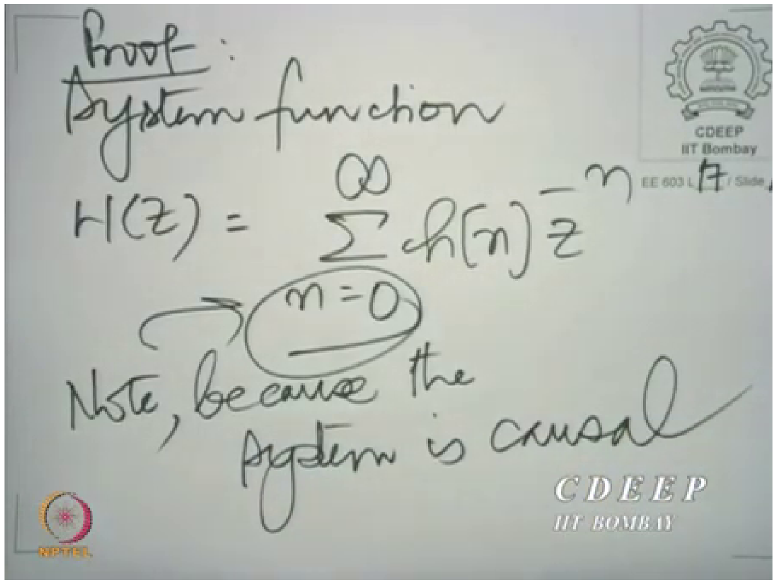

In fact, if a system if an LSI system is causal, and if it has a system function, then the system function must be proof, the system function H Z must be summation n going from 0 to infinity h n Z raise the power minus n. Note because the system is causal. Because the system is causal, you are sure that there are no samples before 0.

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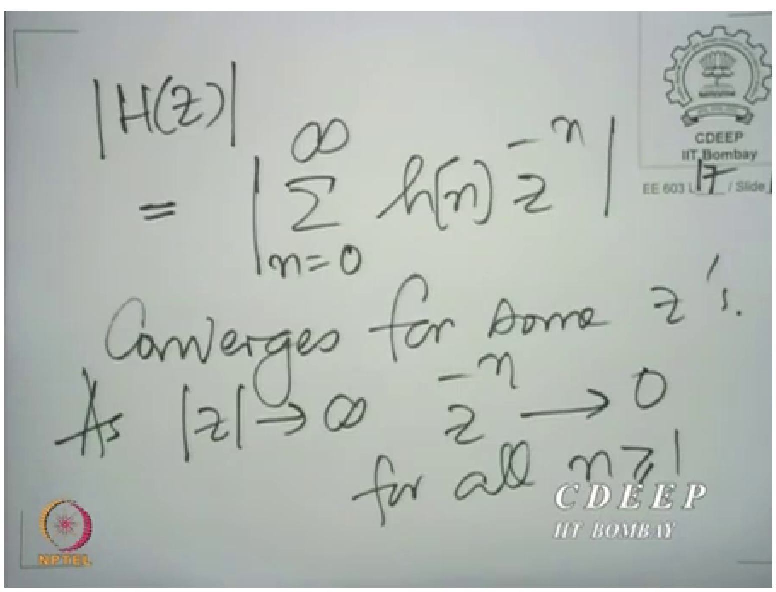

Now, you see as Z, as mod Z tends to infinity, we can consider more H Z, mod H Z is mod summation n going from 0 to infinity mod h n Z raise to the power minus n. Now this converges for some Z's, so you know that system function exists. Now, as mod Z tends to infinity, Z raise to the power of n tends to 0 for all n greater than equal to 1. That is easy to see. And therefore, all the terms vanish, except for n equal to 0.

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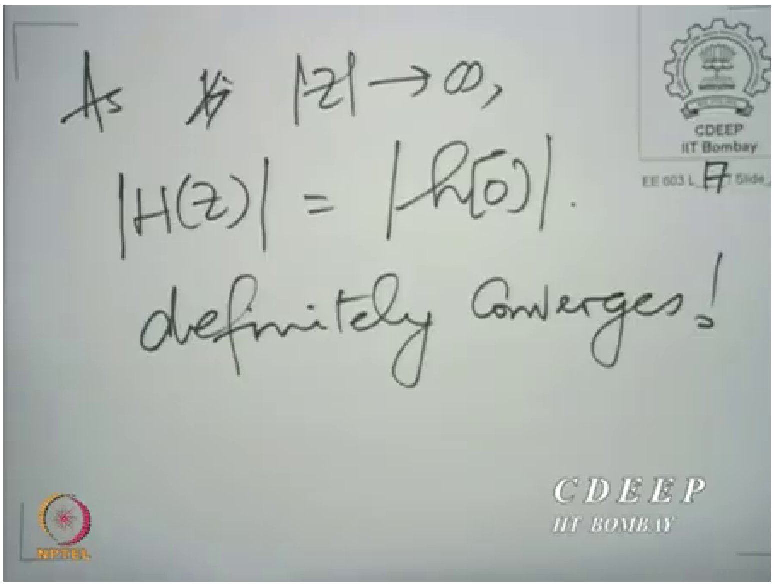

As Z tends as mod Z tends to infinity therefore, mod H Z simply becomes mod h 0. So, it definitely converges. Now this is the if part, what about the only if part. So can a system, you see so what we have said, what we have proved here is that if mod tend, you know, if we, if a system is causal, then its system function must converge as mod Z tends to infinity. So, you know here we are.

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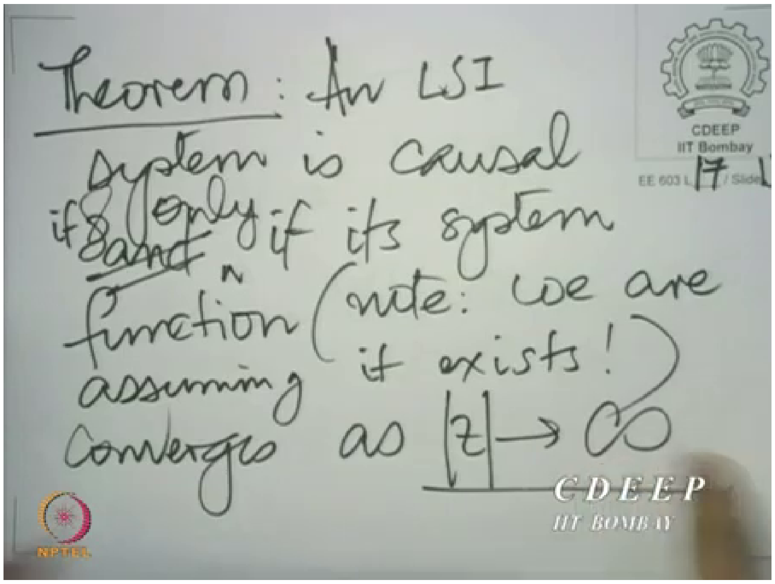

In fact, what we have proved here is only if in the theorem, but what we do wish to prove is if and only if. What we have just now shown is that if a system is causal, which assures us that h n is 0 for n less than 0, then the, then the Z transform, we assume the Z transform exists must converge for Z tending to infinity we have shown that. Now we need to prove the converse. The converse is, if the Z transform converges as Z tends to infinity, then the system has to be causal. Is that clear, is it clear what we are trying to do it must improve both ways.

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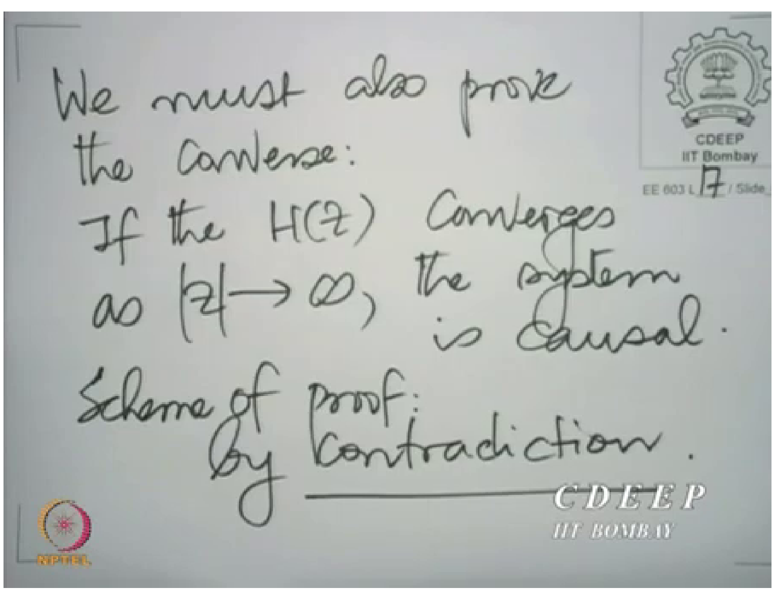

Now, I give you the scheme of the proof, I leave a little bit of the proof to you. I give you the scheme of the proof and I leave you to complete the details. We must also prove the converse, if the Z transform, if the system function converges as mod Z tends to infinity the system is causal. Now, the scheme of the proof is, essentially that we can prove it by contradiction. So, we assume that H Z converges as mod Z tends to infinity.

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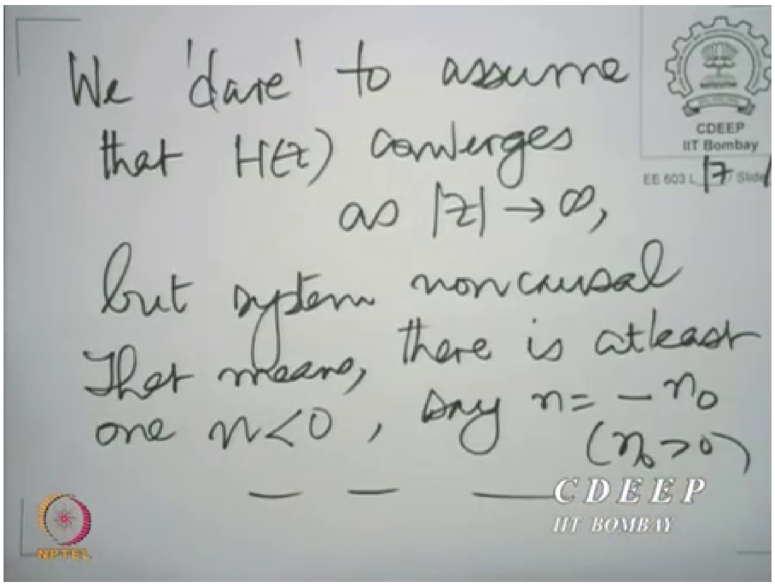

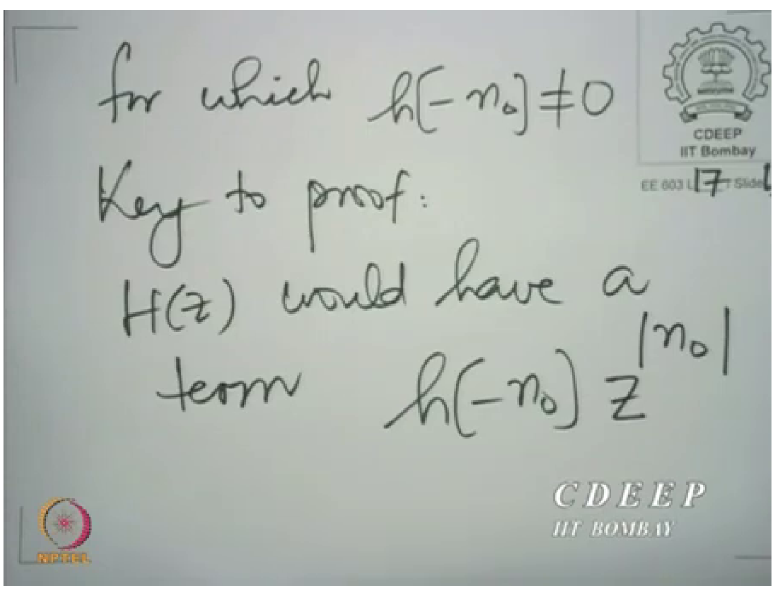

But we dare to assume that the system is not causal, which means that it has some non-zero samples on the negative side. That means, there is at least one sample 1 n less than 0, say n equal to minus n0, of course n0 is greater than 0 here for which the impulse response is not 0. The key to the proof is that H Z would have a term h minus n0 Z raise to the power mod n Z, arising from that impulse response sample.

And of course, remember not just one there can be several sums, there can be I mean if a system is non causal, it can be several there can be infinite such samples. The key to the proof is there is a system is causal, then can there be negative non-zero samples given that such a term comes up, such a term has a Z to the power mod n0, so a positive power of Z. And what happens as Z tends to infinity, what happens is mod Z tends to infinity here.

For any 1 such term, it is obvious that that term will blow up without bound. Now, the key or the crux of the proof is, and this I am leaving to you, leaving you to reflect upon is, if you have multiple such samples, can it possibly happen that this sum of positive power or linear combination of positive powers of Z can go to 0 all over the contour mod, mod Z going to infinity. You see mod Z going to infinity is a contour, not a point.

So, it is a concept, it is a contour, it is a, it is a growing boundary, right. Now, on this boundary, is it possible that all these positive powers of Z can be linearly combined to be 0 all over that infinite contour, can it happen? Well intuitively it is clear that it cannot and I leave it to you to reflect more and see if it can at all. In any case, what I do reflect, what I will do leave it to you to reflect upon and conclude is that if the system is rational and causal, then this can certainly not happen. So, I leave it to you to complete the details of the proof based on this concept.

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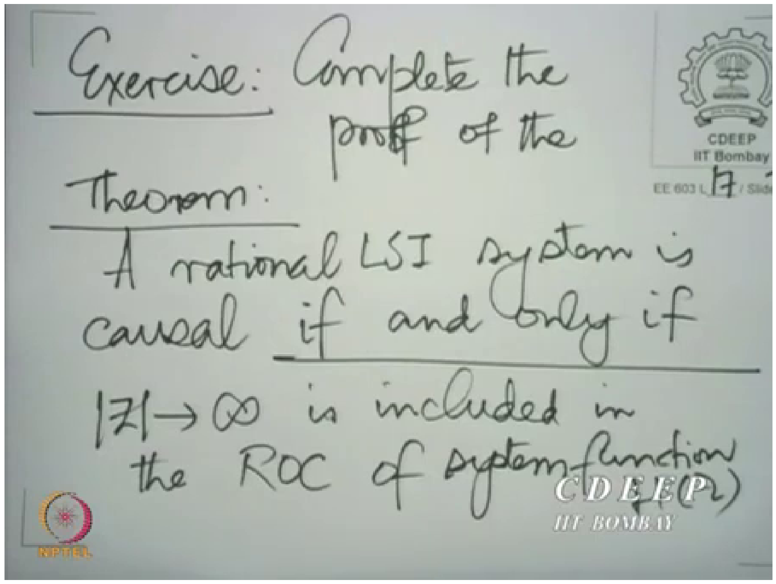

So, exercise, complete the proof of the theorem. A rational LSI system is causal if and only if mod Z tending to infinity is included in the region of convergence. This is the system function of that LSI system, of course. Now we ask the question in the context of stability.

Now for stability, if you have rational systems, we have seen immediately what kind of impulse response to expect. In fact, in some sense, rational systems have now become very predictable to us. There are only 2 types of terms so in fact, I would say only 3 types of terms that can come out of rational systems.

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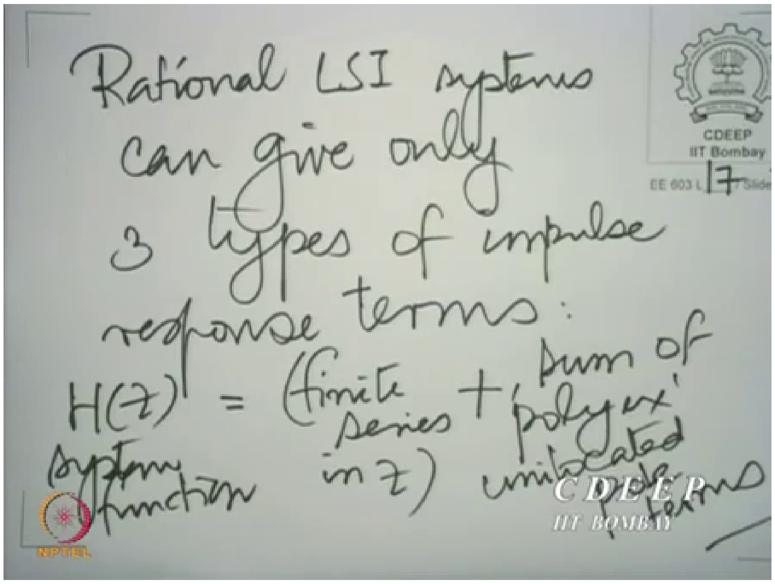

Impulse response terms, you see, that is because, if you have the rational system function, H Z it can always be decomposed as a finite series in Z or Z inverse plus a sum of all polyex terms, polyex unilocated pole terms. Polyex means, essentially terms which correspond to a polynomial in n multiplied by an exponential term. And how do they look in the Z domain, they essentially look like the denominator you have 1 pole repeated more than once or more than once.

And the numerator you have a degree 1 less than the degree of the denominator, that is a, that is how the Z transform of polyex term looks. So, in the partial fraction expansion, you have a finite series plus a sum of such terms, uni-located poles and the poles may occur with multiplicity of more than one and in the numerator you have a degree up to 1 less than the degree of the denominator and the time domain this corresponds to a polyex term.

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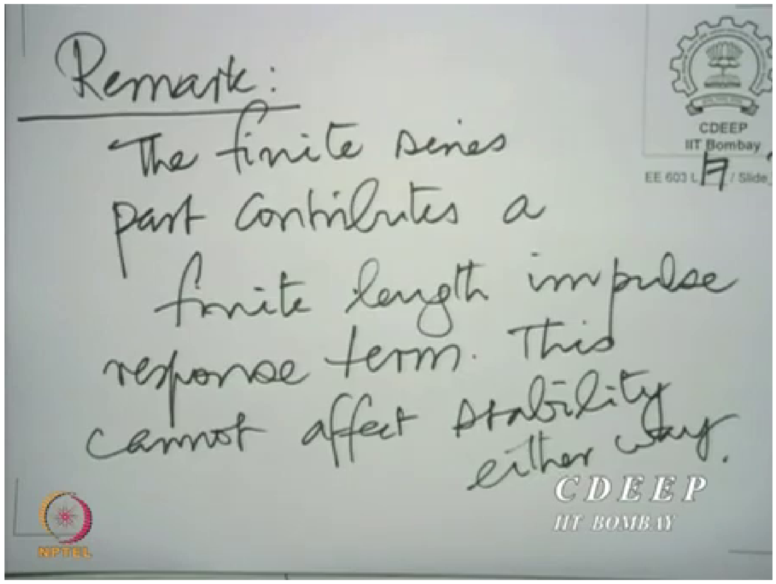

Now, you see we, I make some remarks. The first remark I make is the finite series part. The finite series part contributes a finite length impulse response term, this cannot affect stability, this cannot affect stability either way. So, if the system is, if the other terms make the system stable, this cannot make it unstable.

That is because at most this will contribute a sum which is equal, you know, if you take the absolute sum of all these samples, that is finite, and they cannot perturb the absolute sum of the impulse response to more than what their own absolute sums, you know. So, it is very clear that this cannot affect the impulse, the stability of the system, either.

The system is unstable, it will continue to be unstable even if you have these terms, this finite series here, so either way it has no effect. So, we can even neglect that finite series, now we look at the polyex terms. Now what we show is that if any one of the polyex terms diverges, then the sum has to diverge.

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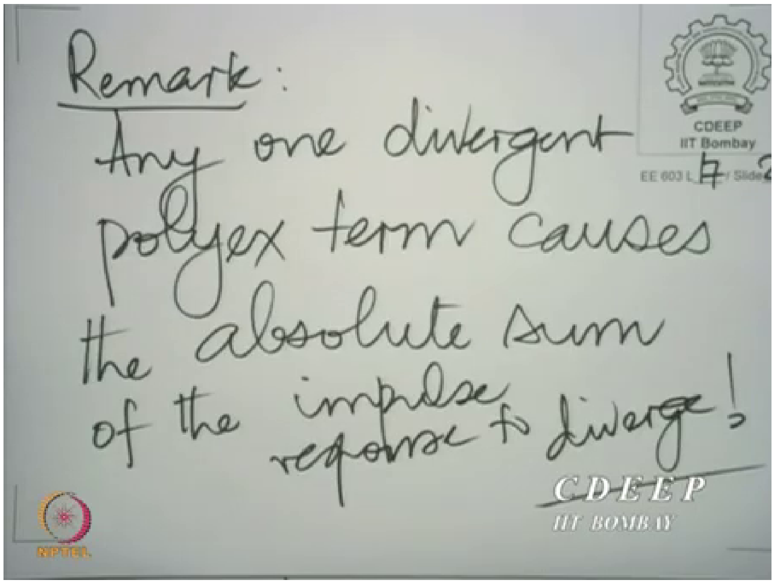

So, the next remark that I make is any one divergent polyex term causes the absolute sum to diverge. Now, to understand this is a slightly deeper issue, why does this happen?

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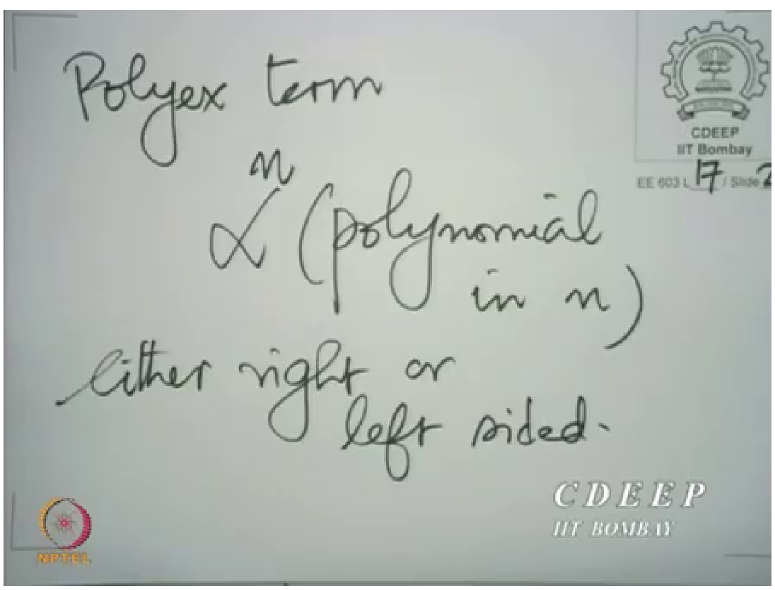

What is a polyex term, a polyex term is a term of the form alpha raise to the power of n times a polynomial in n, either right sided or left side. Now, you see a polyex term diverges, diverges means it is sum tends to infinity. That happens if and only if mod, you see if this exponential grows or the exponential does not decay, that is a better way of saying it.

So, if the exponential remains steady there is a problem that is mod alpha equal to 1 also is a problem. But if the exponential grows, so if it is left sided the exponential will grow if mod alpha is less than 1 and if it is right sided the exponential will grow if more alpha is greater than 1.

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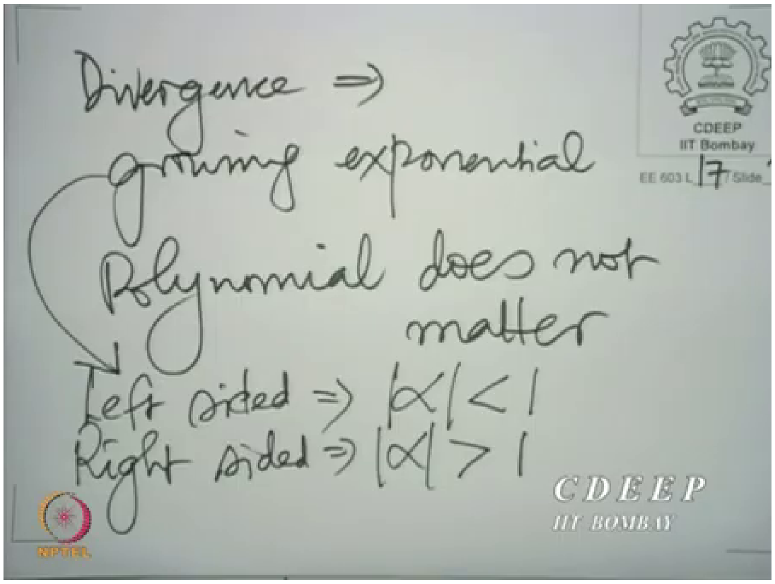

So, divergence means, growing exponential, the polynomial does not matter. You see the polynomial can never dominate the exponential either way, if the exponential decays, even if the polynomial is growing, which it would anyway, either way, it does not affect the decay. An exponential always overpowers a polynomial, either way. So polynomial again does not affect divergence or convergence, it is the exponential which does.

So, you see, left sided means, if it is a growing exponential and left side that means mod alpha is less than 1. If it is a growing exponential and right sided it means mod alpha is greater than 1. For example, if alpha is half, then the left side exponential here will be divergent, if alpha is 2, the right side exponential will be divergent. Now, why is it that we are saying that any one divergent term, 1 bad apple spoils the rest, why is it that 1 divergent term kills all, that is because polyex terms are linearly independent.

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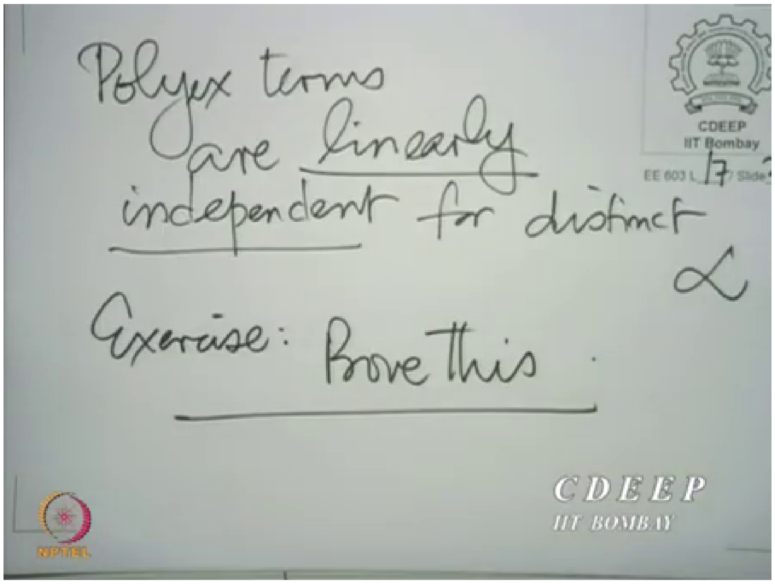

Polyex term are linearly independent for distinct alpha. That means, you can never have two polyex terms which combine to 0 all over the n axis, if they have distinct alpha. I leave it to you to prove this, exercise prove this. What I mean by that is, you cannot have two polyex terms with different alphas, which add up to 0 all over the integer n.

If the alphas are distinct, there sum cannot be 0 all over n. So, one cannot cancel the other, that is what we are saying. If one of them is a rotten apple, it is bound to cause the others to rot as well. It cannot be canceled by anyone else that is the problem. So now where are we, you see if there is one rotten apple, that means if there is one polyex term which diverges, our stability is gone.

When will that polyex term diverge, when either mod alpha is equal to 1, if mod alpha is equal to 1 that means, the pole lies on the unit circle, so pole lying on the unit circle gone, system unstable. Now, if the pole if, mod alpha is greater than 1, then you want a left sided term. What do you mean by left sided term, the region of convergence must be in inside mod Z equal to alpha, mod Z equal to mod alpha.

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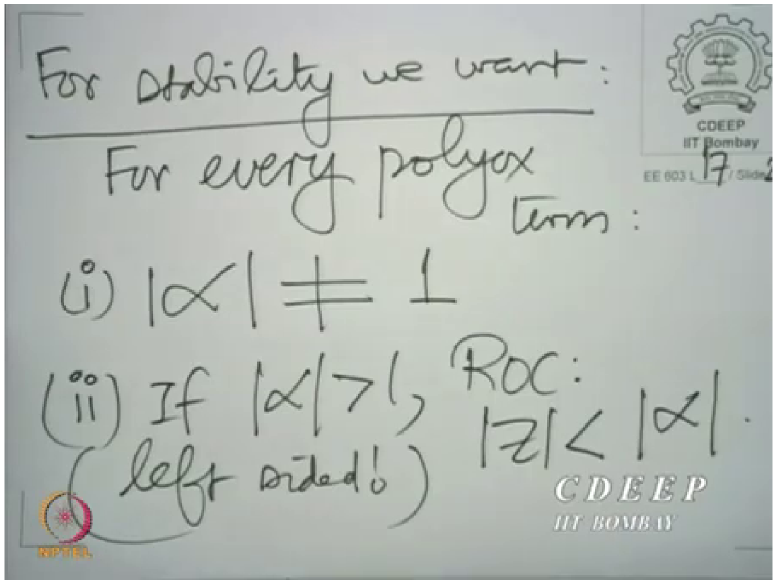

So, let us write down. For stability we want, for each, for every polyex term mod alpha not equal to 1 mod alpha equal to 1 immediately disqualified. If 1 alpha greater than 1 ROC mod Z is less than mod alpha because we wanted to be left sided.

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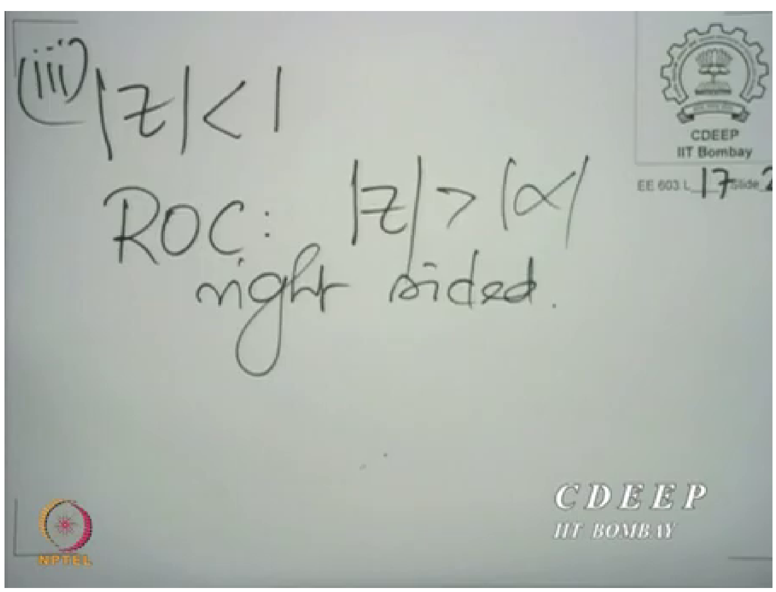

If mod Z, if mod alpha is less than 1, then we want the ROC to be mod Z greater than mod alpha, because right side So, what are we saying in effect, there cannot be a pole on the unit circle. If there are any poles with magnitude greater than 1, the region of convergence must be to the interior of such poles. If there are any poles with magnitude less than 1 the region of convergence must be to the exterior of such poles.

So, where can the region of convergence be, it must be between the biggest pole or the pole of largest magnitude with magnitude less than 1 and inside the pole of smallest magnitude with magnitude greater than 1. And what does that mean that means the unit circle must be in the region of convergence. Unit circle in the region of convergence is sufficient and necessary if the system is rational and stable. So, we write the theorem and I leave it to you to put down a formal proof based on what we have just discussed and with that, then we conclude this class.

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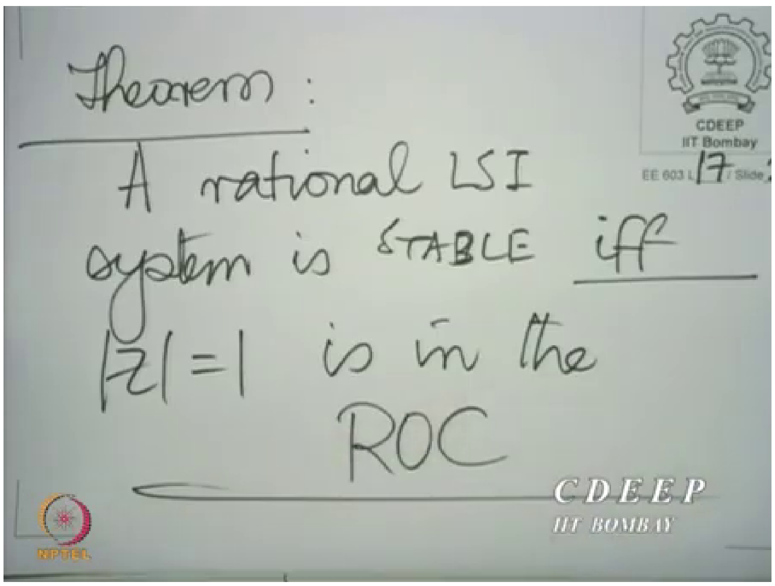

The theorem is, a rational LSI system is stable if and only if, if i double f, mod Z equal to 1 is in the region of convergence. And of course, now we know what happens when you want the system to be both causal and stable. If you want the system to be both causal and stable, the unit circle must be in the region of convergence, mod Z tending to infinity must be in the region of convergence and the region of convergence is simply connected regions you can have pieces.

That means all the region from the unit circle up to mod Z tending to infinity must be in the region of, there cannot be any poles outside the unit circle all the poles must be inside the unit circle. So, we conclude with that remark that if a system is both causal and stable and rational of course, then all its poles must be inside the unit circle. We shall proceed to see more from this point onwards in the next lecture. Thank you.