Digital Signal Processing & Its Application
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Lecture 18d
Important Characteristics of Ideal Filters

Now we need to ask ourselves, is this a stable system? In fact, we need to ask ourselves 3 questions. Most importantly, is this realizable? That means can I use finite resource to realize it and translate it into a hardware, software or hardware cum software structure. And to answer that, we would need to answer 3 questions. Is it causal or if it is not causal, can I make it causal by shifting?

So, I do not mind waiting for the output for a few sample times. But if I am willing, that means I am willing to introduce a delay. Of course, not forever, I am willing to wait for a finite number of sample times. So, can I introduce a finite delay and make it causal or is it causal inherently can is, one of these, is one of these 2 things true? So in fact, let us now put down the questions that we need to answer.

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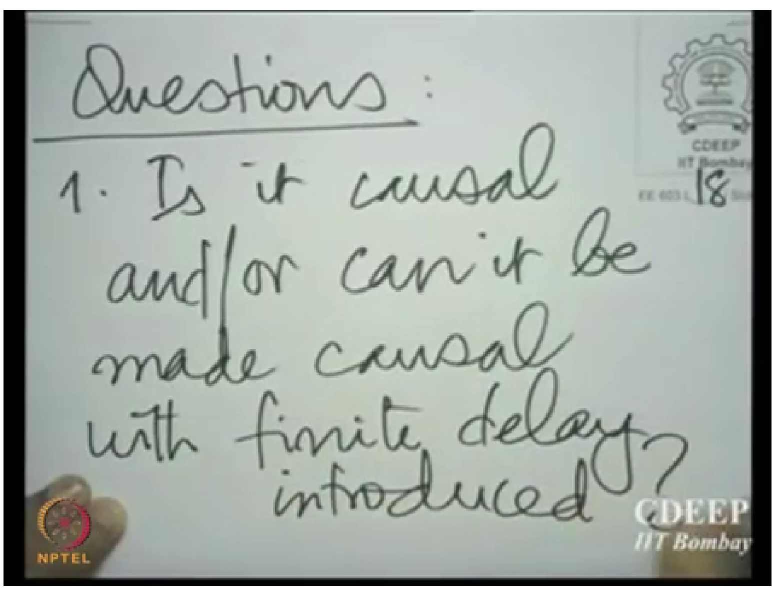

Number one, is it causal and, or can it be made causal with finite delay introduced? Second question, is it stable? And that is one of the questions that was asked in the classroom today. Is this filter stable?

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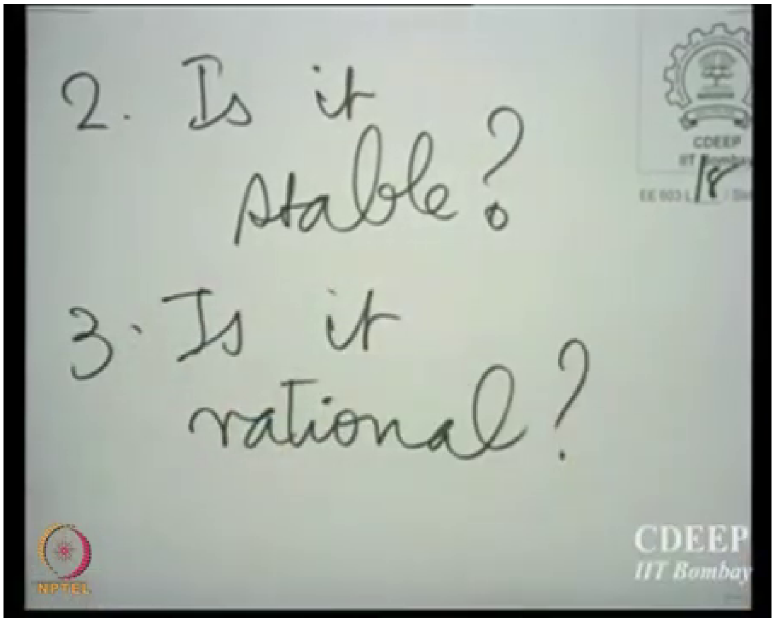

And the third, of course, you want to realize it. So, is it rational? Let us take a specific case of omega c to make a point. We can of course answer in general, later. But let us take the special case of omega c equal to pi by 2 and answer these questions.

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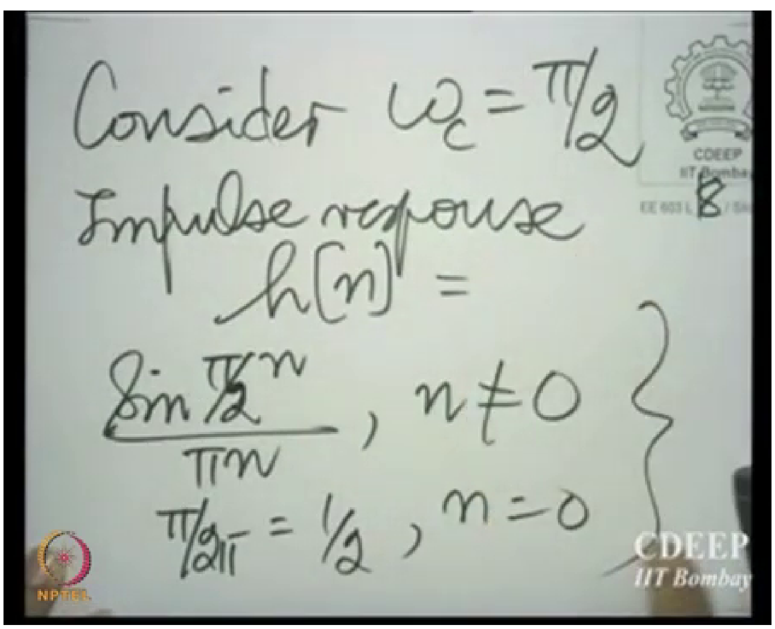

So, the impulse response is as follows, sin pi by 2 n by pi n for n naught equal to 0. And pi by 2 by pi, which is half for n equal to 0. And let us actually write down the simplest response for a few samples, let us write it down all the way from minus 10 to plus 10. So in fact, if you look at this expression, it is very interesting. At every even location, you have a pi by 2 times an even multiple of n. And that would give it essentially a multiple of pi.

So, at any multiple of pi sin pi is 0. And of course, other than n equal to 0, the numerator would therefore be 0 at even locations. At the odd locations, sine pi by 2, n would either take the value plus 1 or minus 1. In fact, could alternate for n equal to 1 it could take the value plus 1 and at n equal to 3 it would take the value minus 1. So, n equal to 1, 1. So every time there would be a 1 minus 1 alternation.

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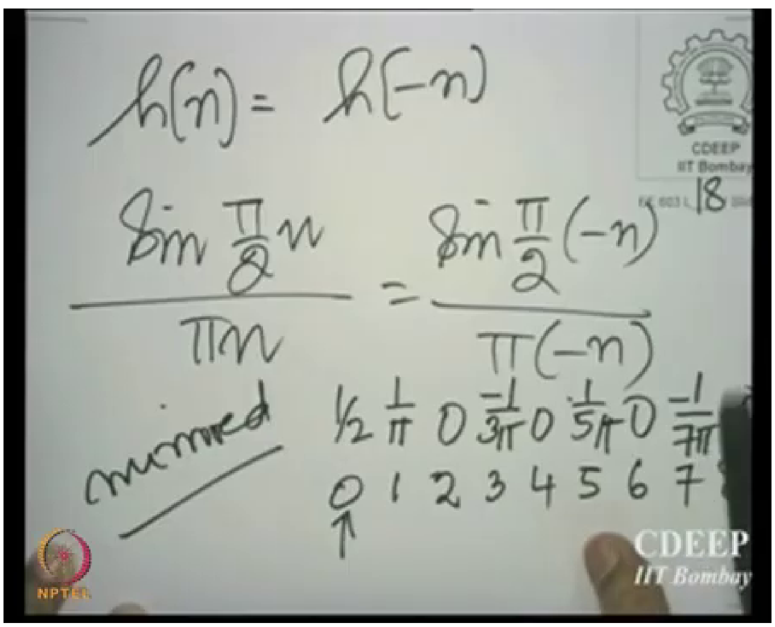

And let us therefore write down the impulse response for a few values. In fact, we can again simplify it, you see h n is equal to h minus n. It is an even impulse response, that is very easy to see. That is because sine pi by 2 n by pi n is equal to sin pi by 2 into minus n by pi minus n, that is very easy to see. And therefore, we can write down the impulse response only for a few values say 0 to 10 as I said 0, 1, 2, 3, 4, 5, 6, 7, 8, you can go on.

At this point it will be 1 by pi, it is going to be 0 here, minus 1 by 3 pi 0 there plus 1 by 5 pi 0 minus 1 by 7pi 0 and so on. And of course, mirrored here, on the negative side is going to be mirrored because it is an even function. So it is very interesting, you have essentially something like half 1 by pi 1 by 3 pi 1 by 5 pi 1 by 1 by 7 pi and so on.

You see, and of course, it is very obvious that this response would go on forever, on the positive side, and therefore also on the negative side, and the negative side could go on forever. So, by bringing in a finite delay, I am never going to be able to make it causal, so it is very clear, because of this, even nature of the response.

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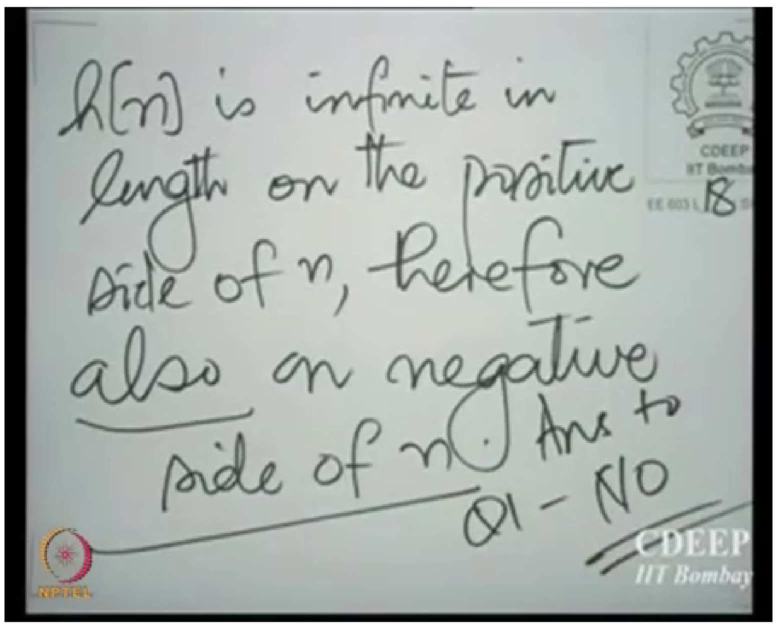

h n is infinite in length on the positive side of n, therefore also on the negative side. So we answer to question 1 is no. We cannot make it causal by a finite delay. Of course it is not causal, and we cannot make it causal by bringing in a finite delay. No matter how much finite delay I bring into the system, it will still remain non causal that is quite clear. So, we've answered the first question. Now let us answer the second question.

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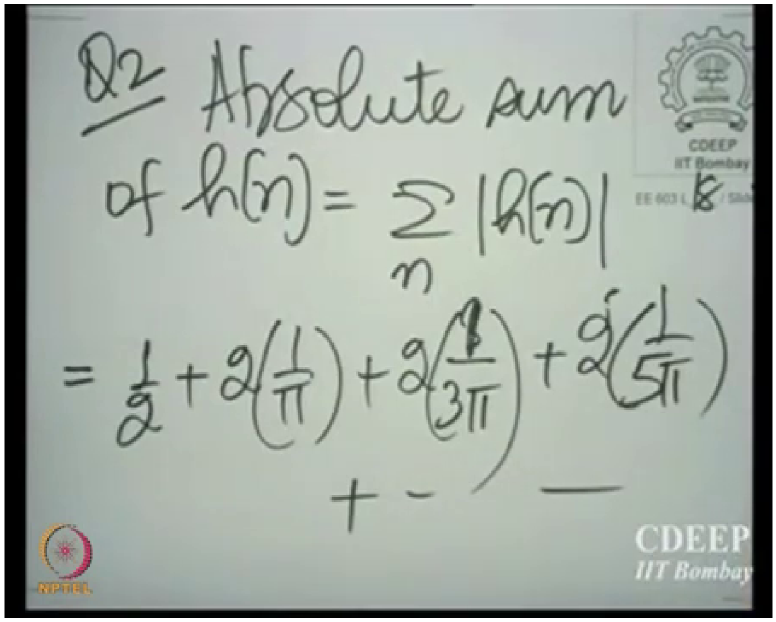

In fact, that is, to answer the second question, we need to take the absolute sum. That is the only way we can answer, we will need to take the absolute sum of h n, which is of course mod h n summed over all n. And that is very easy to see, it is half plus 2 times 1 by pi, plus 2 times 1 by 3 pi plus 2 times 1 by 5 pi and so on. I am saying 2 times, because you need to take the positive and negative odd locations together, take the location 1 and minus 1 that contributes the factor 1 by pi, take the location n equal to 3 that contributes to factor 1 by 3 pi, location 5 and so on. All the even locations it is 0.

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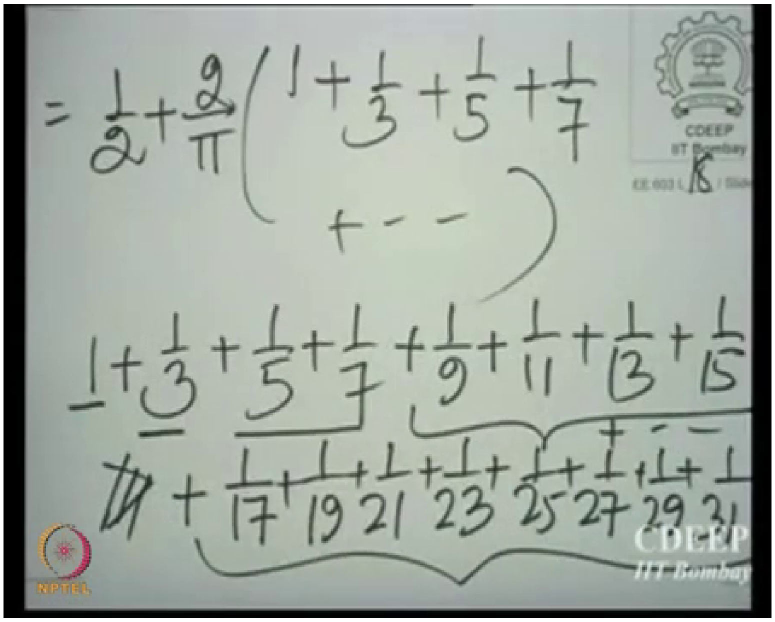

Now, we can pull this together, we can combine these and that is of course, half 2 by pi times 1 plus 1 by 3 plus 1 by 5 plus 1 by 7 plus, and so on forever. So, our concern is with that series, 1 plus 1 by 3 plus 1 by 5 plus 1 by 7, and so on so, that is the series where we need to see if it converges or diverges.

And you see, what we need to do is to note that, 1 plus 1 by 3, plus 1 by 5 plus 1 by 7 and so on, is definitely greater than, well, you know, now you know here, what we need to do is to group two at a time. So, we take this is a group, let us write a few more terms, 1 by 9, 1 by 11, 1 by 13, 1 by 15. And, you know, we can go on with this forever.

We will group this, this, this, these 4, so we will form groups like that. The first one, the second one, 2 subsequently, 4 after that, 8 after that, so the 8 after that would be, you see, you could I mean you could write them down you know 17, 19, 21, 23, 25, 27, 29 and 31. Now here we grouped all these 8 together, so 8 then, and then 16 and so on, you can keep doing that, does not matter. So, the question is, should we group?

Anyway, let us see what, what we can do with this grouping. So, when we group it like this, it is very clear that 1 is greater than half, 1 by 3 is greater than 1 by 4, 1 by 5 and 1 by 7 are each greater than 1 by 8. And therefore, this is greater than 2 times 1 by 8.

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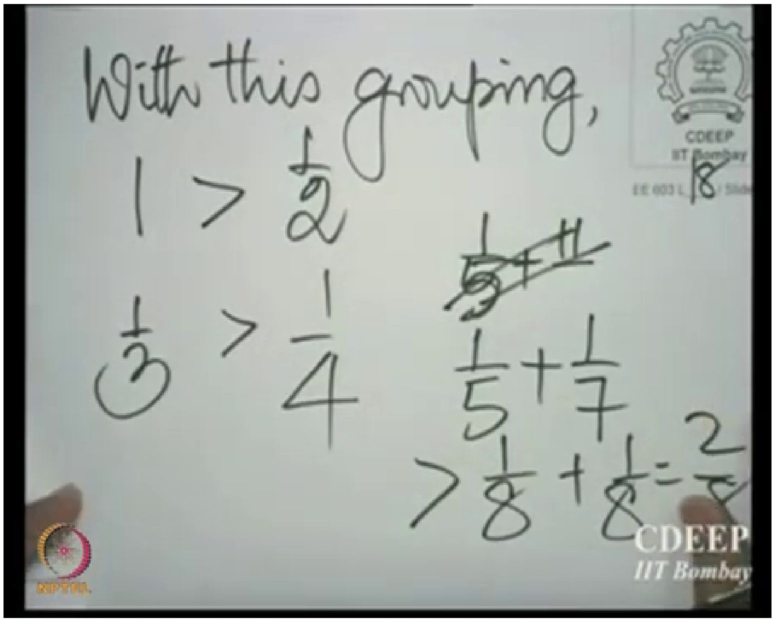

So, what do we have here, this, with this grouping, we have 1 is strictly greater than half, 1 by 3 is strictly greater than one forth. 1 by 9 plus, 1 by 5, plus 1 by 7 is strictly greater than 1 by 8 plus 1 by 8 that is 2 by 8. And 2 by 8 is 1 by 4 again.

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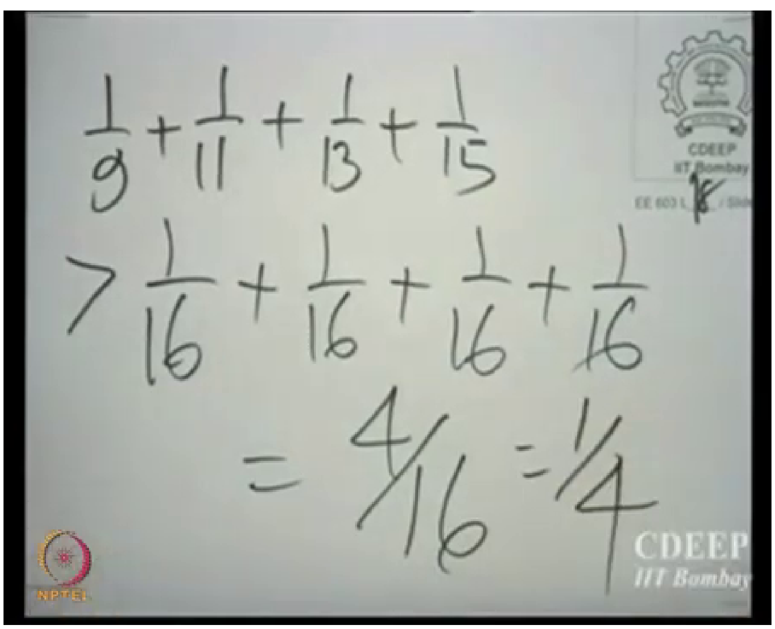

And the next group, 1 by 9 plus 1 by 11 plus 1 by 13 plus 1 by 15 is strictly greater than 1 by 16 plus 1 by 16 plus 1 by 16 plus 1 by 16, that is 4 by 16, which is 1 by 4 again, so as we take these groups, 1 first and a group of 2 and a group of 4, 8, 16, 32, and so on, so forth.

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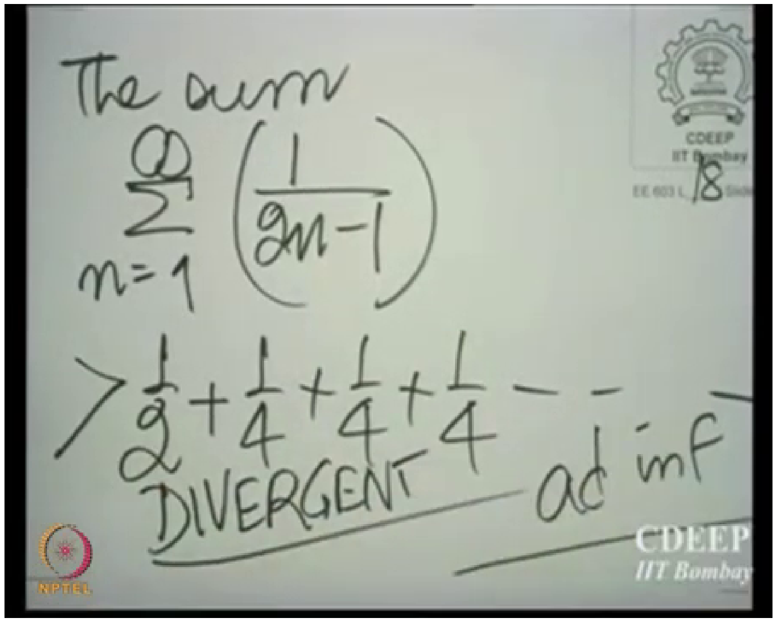

We see that this sum, the sum, summation n going from 1 to infinity, 1 by 2 n minus 1, that is the sum that we are taking is strictly greater than half plus one fourth plus one fourth plus and this is ad infinitum. And of course, this is divergent; any doubt from this, everybody understood this. So very clearly, the sum is divergent.

Now, of course, there are many ways to prove the sum is divergent; this is not the only way. There are many other possible, in fact; I leave it as an exercise to you to find other unique groupings, distinct groupings that also prove its divergence. When a series is divergent, there are many ways to prove, often there are many ways to prove that is divergent.

All that you have to show is that the series is greater than an ever increasing quantity. And that can be done in many ways. So I leave it to you as an exercise to find other ways to prove that this is divergent, but the point is that it is divergent. And because it is divergent, the system is unstable.

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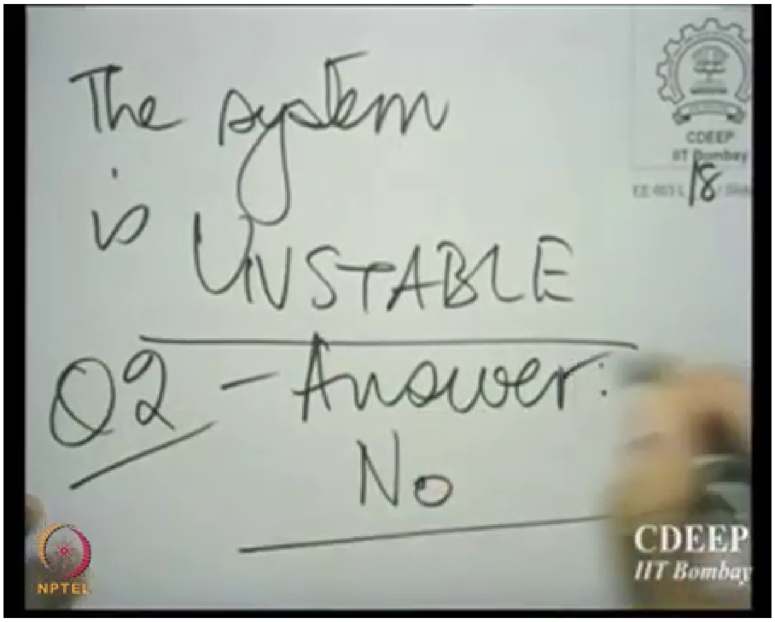

It is very clear; the system is unstable. Question 2, the answer is no. Now we will try and answer question 3.

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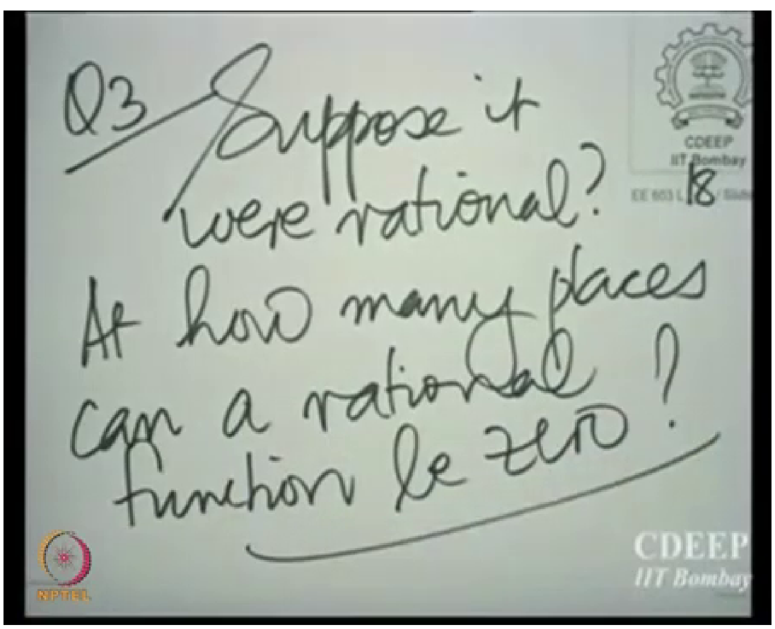

Suppose it were rational, at how many places can a rational function be 0? Can a rational function be 0 on a continuum on the frequency axis. A rational function has only a finite number of points at which the function can become 0. Those are called the 0s, we have call them the 0s. It is very easy to see the rational function cannot, at least a rational function 1 variable cannot be 0 on a continuous region of the Z plane.

A rational function is a ratio of 2 series, finite series in Z and it can become 0 whenever the denominator is not 0, but the numerator is 0. And if the numerator is a finite series in Z and if you equate it to 0, there is only a finite, in fact, you can even say how many places at most it can be 0. If it is a finite series with say 20 terms, it could become 0 only at 19 places at best.

A polynomial in 1 variable cannot have more than as many 0s as the degree of the polynomial. Now here you are asking for a continuum of the frequency axis over which the response is 0. So it is very clear, this cannot be rational.

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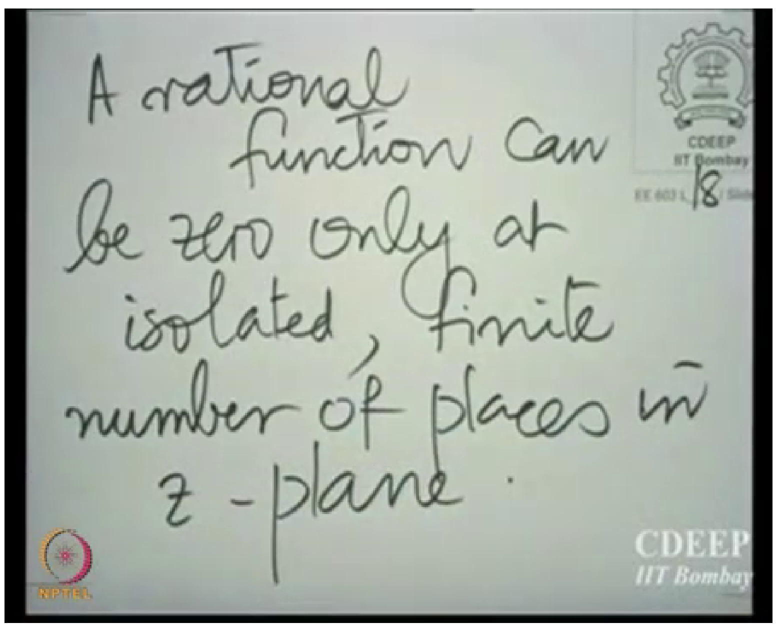

So, the answer is the rational function can be 0, only at isolated finite number of places in the Z plane. And of course, the unit circle is a part of the Z plane. So here you are talking about rational function, which has, you know if this function could be rational, if this filter were rational, you are saying there is a rational function, it has a frequency response, that means you can evaluate the rational, that rational function region of convergence, the rational function includes the unit circle, and on the unit circle, you are asking for a continuum of values on which rational function takes the value 0, which is not possible, function cannot be rational.

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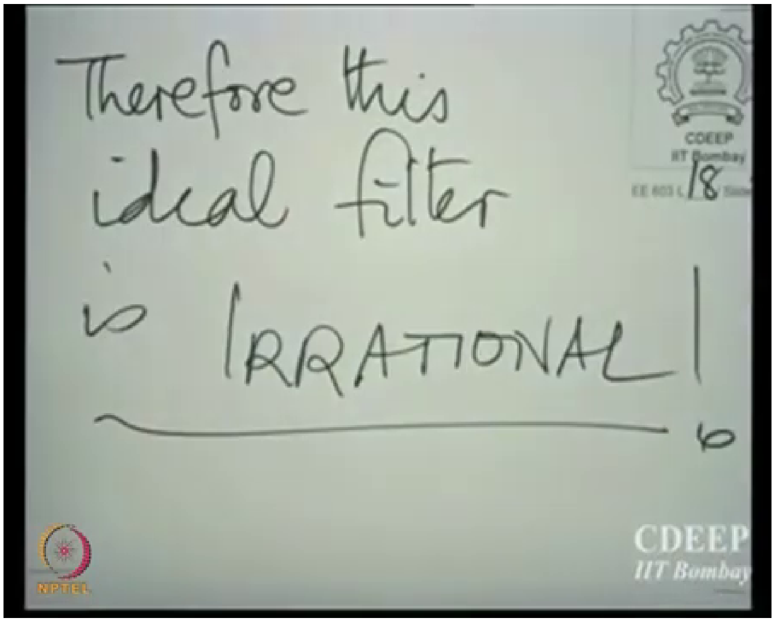

Therefore, the ideal filter is irrational from pre counts we have disqualified the ideal filter. It is not causal and cannot be made causal by introducing finite delay. It is not stable; it is not rational. And because it is not rational cannot be realized. So, we know what we are dealing with now. In fact, this is a very clear explanation on why at all we should spend so many lectures subsequently, beyond this one, to deal with design of filters; we could have concluded the design with this lecture today, had the answer to these questions been yes.

If the ideal filter were non causal, but could be made causal by delay, if the ideal filter was stable, and the ideal filter were rational, we need not have had to have so many lectures at all, we could have concluded here. The ideal specifications could be met with a realizable system. Unfortunately, that is not the case. It is disqualified on 3 counts, bad enough for us to spend much more time than we expected to on the question of design.

But what does that show is that for irrational systems, you could very well have a frequency response, which is piecewise constant as you have here, which is defined almost everywhere on the unit circle, but the system is not stable. And that answers the question that was raised in this class. In fact, it also adds to what we did in the previous lecture. In the previous lecture, we had seen that when you have a rational system, its causality is determined by whether or not the contour or the concept mod Z tending to infinity is included in the region of convergence.

The stability of rational system is seen by whether the unit circle is a part of the region of convergence that cannot be applied for irrational systems. Here we have an example of an irrational system, which seemingly has a frequency response. And that means a unit circle is a part of where that function is defined. Of course, you notice that this frequency response is discontinuous. And that answers half the question.

In fact, I also put it as a challenge, this is a difficult challenge. I put as a challenge before you to see if there is a connection between this continuity of the frequency response and instability of the filter. Is there a connection? Here we notice that the ideal response that we have desired is discontinuous. Does that discontinuity has something to do with the instability that we see, that is a challenge before you.

Notwithstanding this challenge, and notwithstanding this discussion, we now need to as they say, roll up our sleeves and get down to the problem of design. And we shall do that starting from the next lecture onwards by putting down what is, what is very clear is that we cannot have this kind of specifications that we have put down here for the ideal filter, if we put these filter specifications, the filter cannot be realized. So, we will have to first put down a set of realistic specifications, and then proceed to the problem of design. And we shall do that beginning with the next lecture.