Digital Signal Processing & Its Application
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Lecture 20 B
Relationship between S - plane and Z - plane

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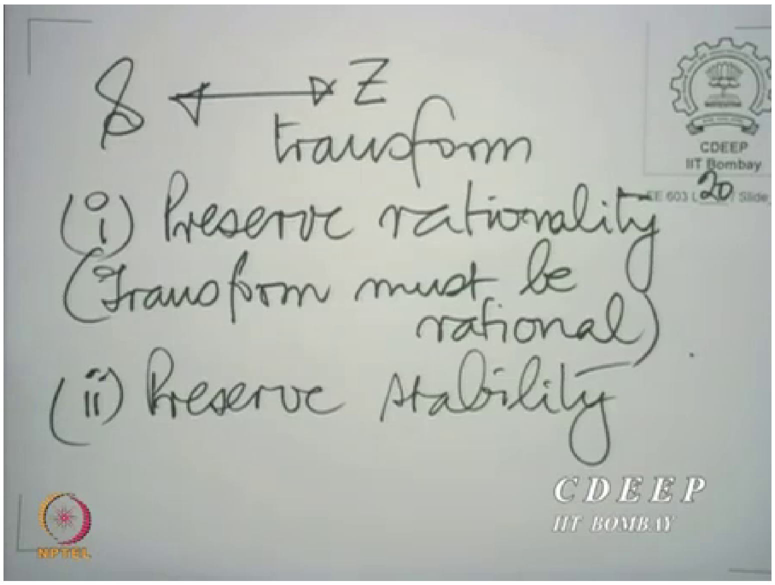

In fact, the s to z transform should do the following, preserve rationality that means the transform should be rational on its own. You see, because an analog filter system function is expected to be rational. Rationality corresponds to realizability even in the analog domain. So the analog system function itself is expected to be rational. Now, you want to retain that rationality when you go into the discrete domain by making an s to z transformation.

So the only way you can do it is if you replace s by a rational function of z. So that transformation must be rational. Secondly, the transformation must preserve stability. And now we want to interpret what preservation of stability means. Preservation of stability means a proper mapping from different parts of the s plane to different parts of the z plane. In fact, let us now put down specifics there.

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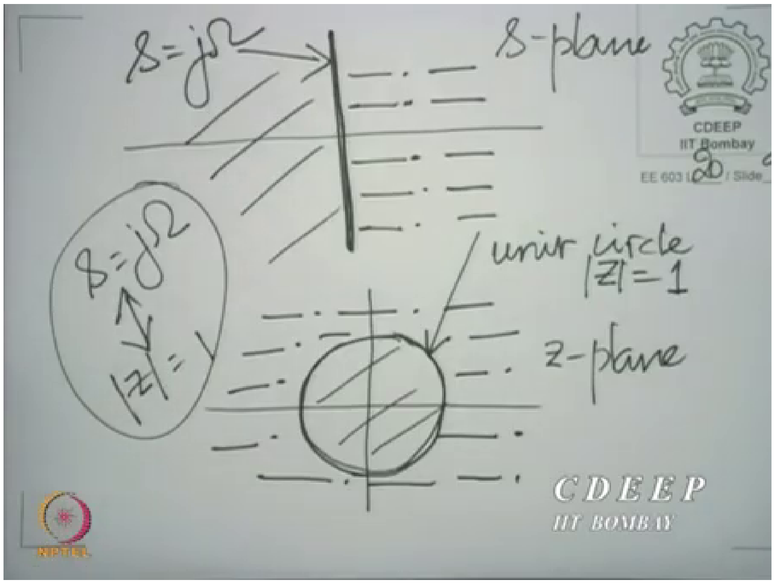

So, you see, what does it mean? It means in the s plane, let us mark the regions of importance in the s plane and the z plane. So you have the z plane here, and you have the s plane there. The contours of importance are the unit circle in the z plane or $|z| = 1$. And the imaginary axis in the s plane, $s = j\Omega$. We have seen that $s = j\Omega$ corresponds to sinusoids. So, the imaginary axis and the unit circle must be mapping one on the other.

That is a requirement, of course, that is actually not a requirement, by itself it has nothing to do with stability. It is a requirement when you have a frequency response already constructed in the analog domain, it moves to a corresponding frequency response in the discrete domain. But what is important for stability is that the left half of the s plane go into the interior of the unit circle, and the right half of the s plane go to the exterior of unit circle. This is what is important for stability.

Now if this happens, if the right half of the s plane maps into the exterior of the unit circle, and if the left of the s plane maps to the interior of the unit circle, and of course, imaginary axis maps to the unit circle then we will very easily be able to show that one cannot result, it cannot happen that we design a stable analog filter and it becomes an unstable discrete time filter. We can show that without too much of difficulty.

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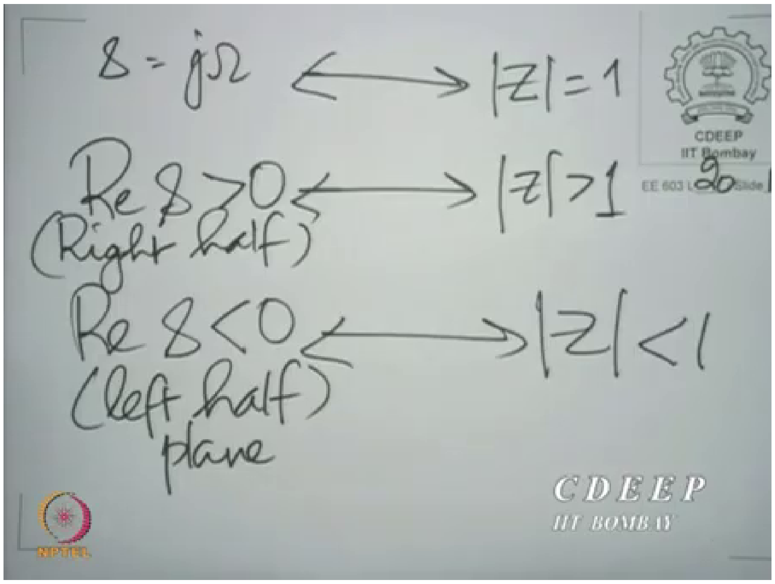

In fact, let us prove that. So what we are saying is $s = j\Omega$ must map into $|z| = 1$, $Re\, s > 0$ must map to $|z| > 1$ and the $Re\, s < 0$ that is the left half plane, must map to $|z| < 1$. This is the right half. Now remember what we are saying is, we are making a transformation on s. s is the independent variable in the system function in the analog domain.

Suppose it could happen that you have, now you see, remember, we are retaining now again, we have assumed the filter to be causal.

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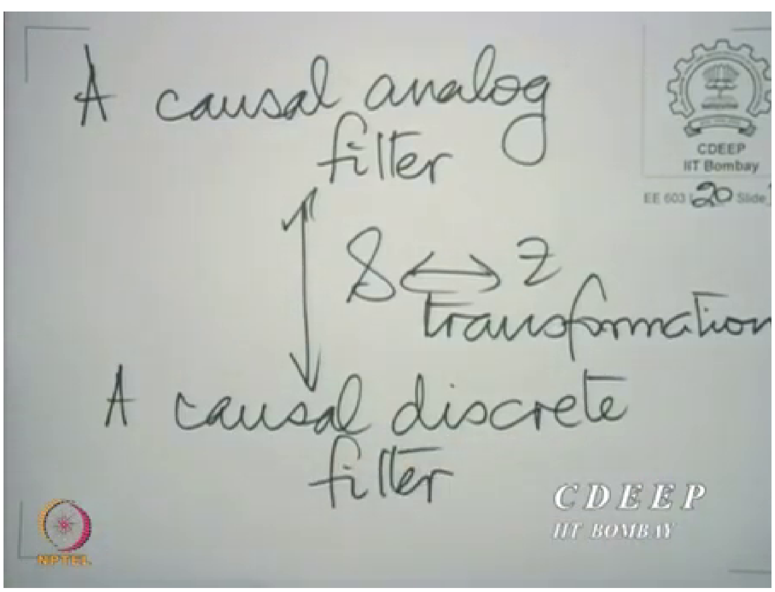

So let us, we should make a note of that, we should emphasize that. What we are saying is we take a causal analog filter and use this s to z, whatever this transformation be, we will have to now come to that transformation but we must know what we want out of that transformation. So a causal analog filter must move to a causal discrete time filter or discrete filter. Now, if the discrete filter is causal we know what to expect from its poles.

We are assuming the discrete filters causal; we want it to be rational because it should be realizable. And we have a causal rational discrete time system we know what to expect of its poles. If it is to be stable, the poles must lie inside the unit circle, they have no choice. Suppose a pole happen to go outside the unit circle.

Now, what is a pole? A pole is a point where that system function diverges, goes to infinity, and that must have come from some point in s, under the transformation some point in s must have brought you to that point in z Now a point on the, if the point in z happens to be outside the unit circle, it could have only come from a point on the right half plane in s.

It could not have come from a point in left half plane of s, if we have ensure that there is a close relationship of left half plane to interior of the unit circle, imaginary axis to unit circle and right half plane to the exterior of the unit circle. If you have ensured that there is this joint relationship, then you cannot have a pole on the exterior of unit circle because that should have come then from a point in the right half plane.

It could not have come from a point in the left half plane. But you cannot have a pole in the original system in the right half plane, because it is stable. Of course, that filter has to be stable. So if we ensure that this is satisfied, that left half goes to interior, imaginary axis goes to unit circle, and right half goes to exterior, we are guaranteed that a stable analog filter will go into a stable discrete time. Is that clear to everybody? Yes?

So that is of course, but you see we have of course, now as a consequence of that because the left half goes to interior, right half goes to exterior, naturally the unit circle and imaginary axis will have a one to one relationship. However, you do not want any arbitrary relationship there. We had made a remark on that too. We said that relationship must also be monotonically increasing.

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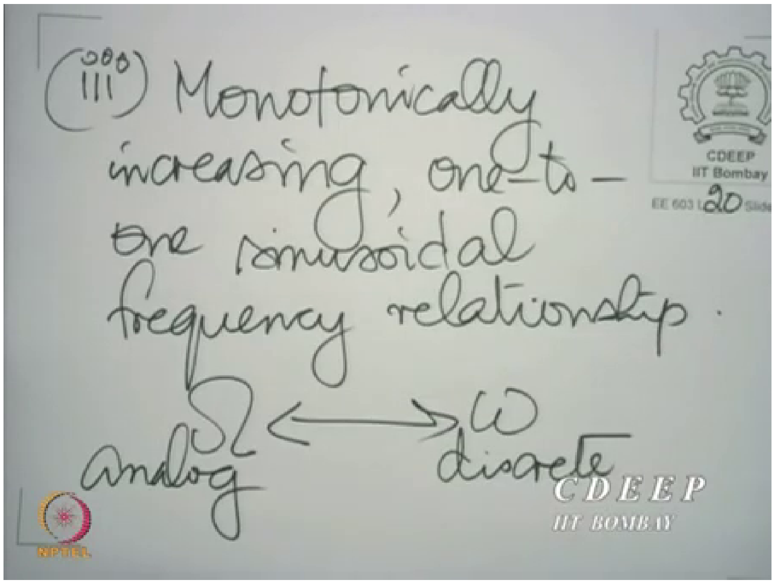

Relationship, so this is the condition number, we have already given two conditions so far. The third condition is that there is a monotonically increasing sinusoidal frequency relationship, increasing one to one, sinusoidal frequency relationship between $\boldsymbol{\Omega}$ and ω. This is the analog sinusoidal frequency and this is the discrete sinusoidal frequency, normalized of course. ω is the normalized discrete frequency.

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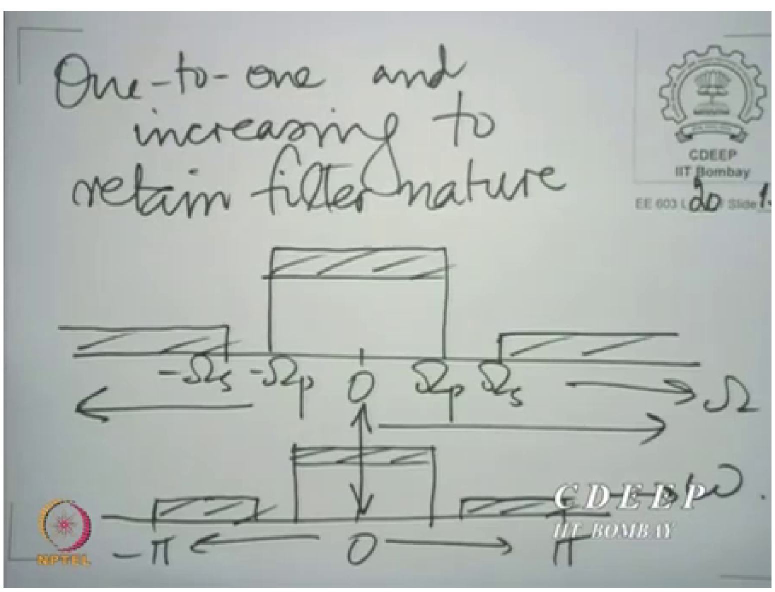

Now why does it need to be one to one, we have also put down the explanation. It needs to be one to one because we want to keep the nature of the filter intact, one to one and increasing to retain the nature of the filter. You see, for example, in the analog domain, if you happen to have a low pass filter with its pass band and stop band edges, as we talked about the last time, and tolerances and transition bands.

Then under this transformation 0 must come to 0, as we move from 0 towards ∞ we must go from 0 to π. Therefore ∞ must be in 1 to 1 with π. As we go from 0 to $-\infty$, we should be moving from 0 to $-\pi$, and therefore $-\infty$ should be one to one with $-\pi$. And because of the increasing nature we are going to have the same pattern translated, or be it with a non-linear distortion.

It has to be non-linear, there is no option. But the nature of the filter is retained. So we noted that this transformation has no choice, but to be non-linear, the transformation between $\boldsymbol{\Omega}$ and ω, or the analog sinusoidal frequency and the normalized discrete angular frequency. It has no choice but to be non-linear, so all these are the conditions that this transformation must satisfy.

And now we need to use our intuition and our insights to construct such a transformation. We also have a hint from where to begin. The transformation must essentially be an approximation of the derivative operator in terms of shifts. And what simpler example of a derivative approximation can we have than to subtract the current or the past sample from the current sample.

You see, what is a derivative? A derivative is essentially a rate of change, and the smallest unit of time on which we can measure change in this discrete system is one sample time. So, if we look at the change in one sample time we have an approximation to the derivative. So, we are tempted to consider the following candidate for this transformation.