Digital Signal Processing and Its Applications
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Lecture 61
Bilinear Transform

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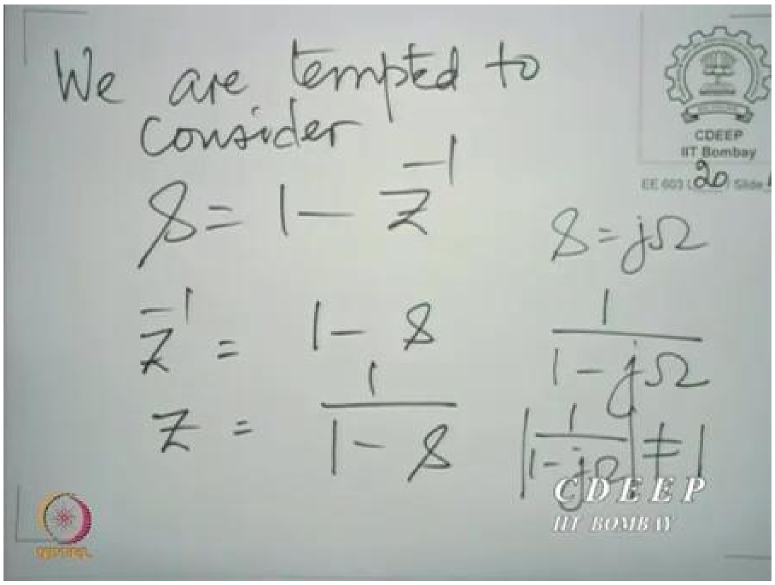

We are tempted to consider s equals to 1 minus z inverse. What it means is we are approximating the derivative by taking the difference between the current sample and past sample. Now we do not need to go too far to see that this is not adequate, as far as our condition scored. In fact, all that we need to do is to write z in terms of s.

So, $z^{-1} = 1 - s$, and therefore $z = 1/(1 - s)$, if we take this transformation to be true. And then we only need to substitute $s = j\Omega$. And then we have $1/(1 - j\Omega)$, does not even have a magnitude of 1. So,$|1/(1 - j\Omega)| \neq 1$.

So, we have failed on a very important count anyway, in fact this transformation you know we should have, this is if this is z, then the imaginary axis should have gone to the unit circle. So, we are failing on that count anyway. And anyway, this is not going to be, so you know simple approximation like this is not going to help us. In fact, let us use some more cues to lead us to a transformation of choice. Let us do it the other way.

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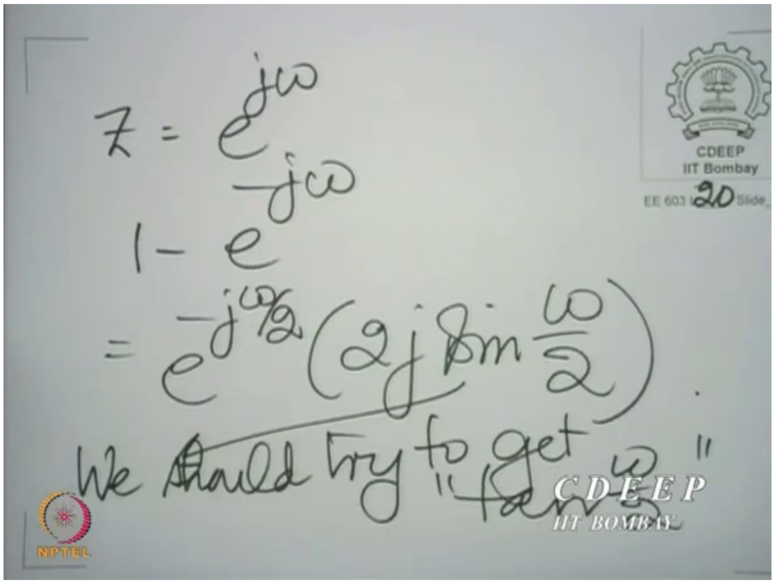

Let us put $z = e^{j\omega}$, in the same transformation and see what we get? We get $1 - e^{-j\omega}$. And that can be rewritten as $e^{-j\omega/2}(2jsin(\omega/2))$. So, you know there are 2 counts on which you know of course we were pessimistic a minute ago and we noted that the transformation was not all right.

But then you know, to move to a solution you cannot remain pessimistic, you have to find some way out of the tricky situation in which you are. And we could now take an optimistic view and say but you know if you whatever you liked here, is that this be an essentially imaginary. So, this is the good part of the, this is the good part of the substitution and this is the bad part.

You know when you put, when you take took a point on the unit circle you should have gone to a point on the imaginary axis, but this would have put you on a point in the imaginary axis it is this which is creating a trouble, here. So, the one thing is we want to do away with this term. Now how do you do away with this term, you cannot do away with this term, but by multiplying by its complex conjugate or dividing by the same term. So, it is clear that we cannot be happy with just a polynomial in z or z inverse.

We need a rational, proper rational function in z or z inverse to replace s. And that rational function in z should be such that it cancels out this factor, that is the first observation. The second observation is that, when ω goes from $-\pi$ to $+\pi$, even if this term were to be absent. You would be taken only from -2 to +2 here, not from $-\infty$ to $+\infty$, from that count also it is inadequate.

In fact, this is the problem with any sinusoid or co-sinusoid that you might want to call it. Just a sine or a cosine term can never take you all over real the axis. Among the trigonometric functions the sine and cosine functions do not take you all over the, all over the real axis. Which function takes you all over the real axis? The tangent function or the co tangent function. So somehow, we need a co tangent or a tangent function here.

Now how do we get a tangent function? You cannot get a tangent function by subtracting z's. So, you can get a tangent function if you divide that is another, you see if you divide a sine function by a cosine function or a cosine function by a sine function you get a tangent function. That is another reason why you would want some kind of proper rational function of z to replace s.

And in fact, if we take tan tangent, we have the answer, $tan(\omega/2)$, when ω goes from -π to +π , $tan(\omega/2)$ would indeed run all the way from $-\infty\ to\ +\infty$. So, if we can go from here you see we should make, we should try to get tan here, $tan(\omega/2)$ here. And how can we get a $tan(\omega/2)$? By dividing $sin(\omega/2)$ by$cos(\omega/2)$. So, let us do that. And in fact, once we allow for that division, we are also likely to do away with this trouble. So, we will be killing 2 birds with 1 stone in the proverbial sense not literally.

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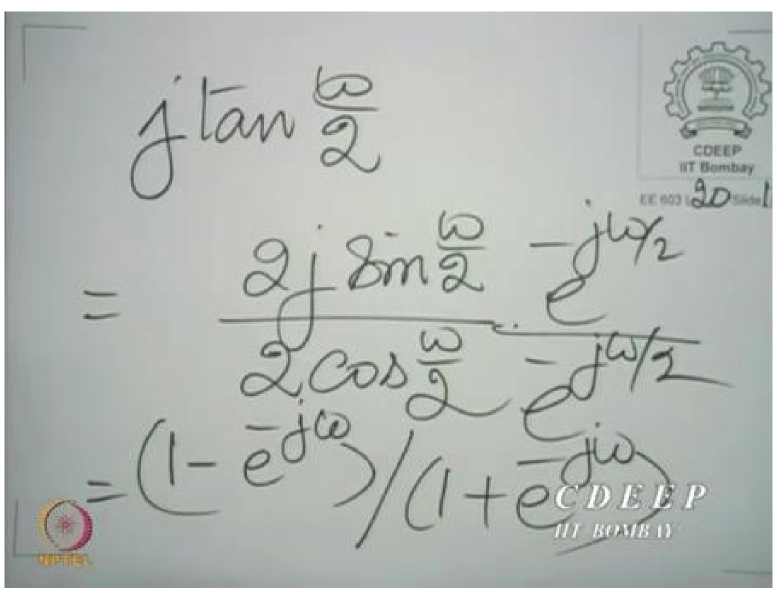

Indeed to get$tan(\omega/2)$, you know what you are saying is you want $j\ tan(\omega/2)$ and $j\ tan(\omega/2)$ can be written as $2j\ sin(\omega/2)\ /\ 2jcos(\omega/2)$. And nothing stops you from multiplying by $e^{-j\omega/2}$

in the numerator and denominator. And in fact, there we have the answer. This is nothing but $(1 - e^{-j\omega})/(1 + e^{-j\omega})$, please check that, so easy. And therefore, we have the answer to the transformation that we want.

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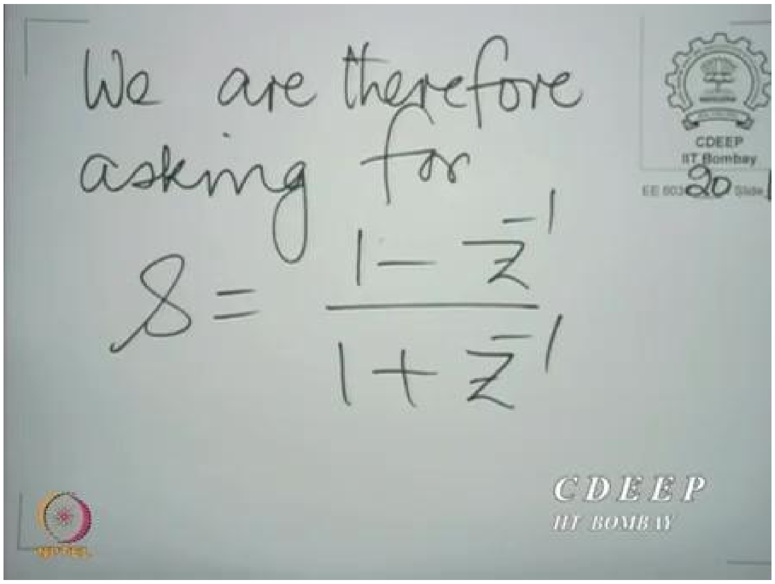

We are therefore asking for $s = (1 - z^{-1})/(1 + z^{-1})$. And now let us check that this transformation in deed does all the jobs that we wanted to do, of course one job we have already verified. Let us start with the thing we already done.

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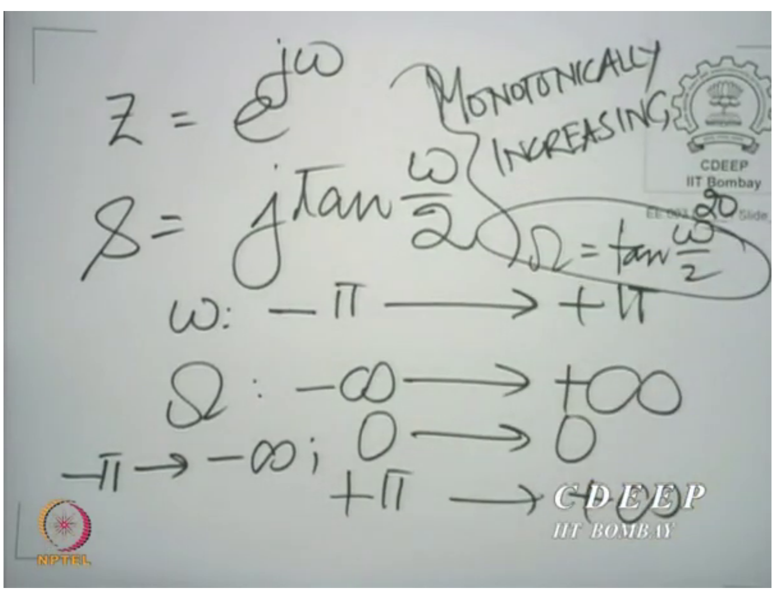

So, let us start with $z = e^{j\omega}$. We will just, it is a formality, but we will just complete it. And we have seen that s becomes $j\, tan(\omega/2)$ here. So, the good thing is ω going from -π towards +π, takes you from Ω going all over from $-\infty$ to $+\infty$, -π goes to $-\infty$, 0 goes to 0 and +π goes to $+\infty$. And this is monotonically increasing all that we wanted out of the mapping between the unit circle and the imaginary axis has been satisfied, is that correct?

As we move from minusπ to plusπ, there is also this mapping is one to one. There is for every this is, so I mean here you have $j\, tan(\omega/2)$, so essentially what you are saying is Ω is $tan(\omega/2)$. That is the mapping that you are establishing here. And this is one to one, that is the beauty of it.

You know for every ω, there is a unique Ω, for every unique for every Ωthere is a unique small ω. In spite of a fact that ω is on a finite interval and Ω is on an ∞ interval, we still have a one-to-one mapping. This is the beauty of this mapping. It is one to one, it is on to both ways, it is invertible and of course it is monotonically increasing, as we wanted it to be.

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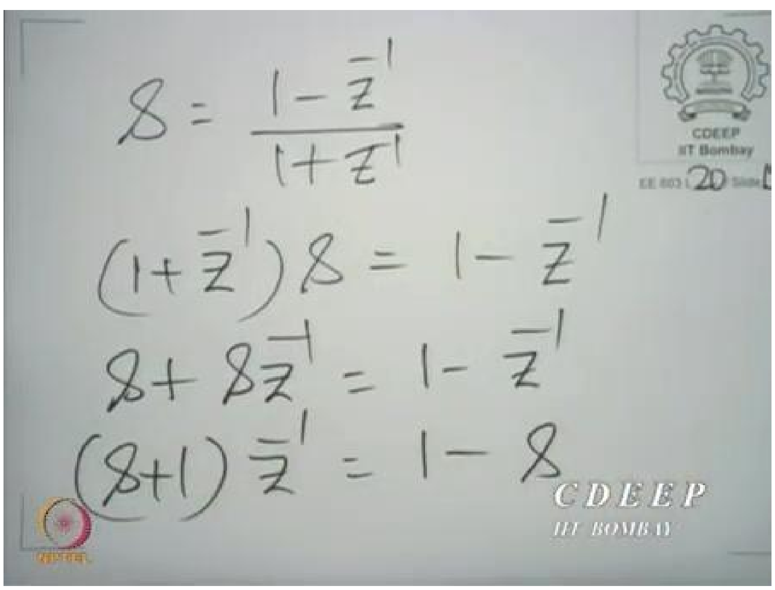

Well, now about the left and right half planes. So let us, let us write the inverse mapping, that is easier to deal with. So, we have $s = (1 - z^{-1})/(1 + z^{-1})$. So of course, I can write $(1 + z^{-1})s = (1 - z^{-1})$. So, I have $s + sz^{-1} = 1 - z^{-1}$ and that tells me that $(s + 1)z^{-1} = 1 - s$.

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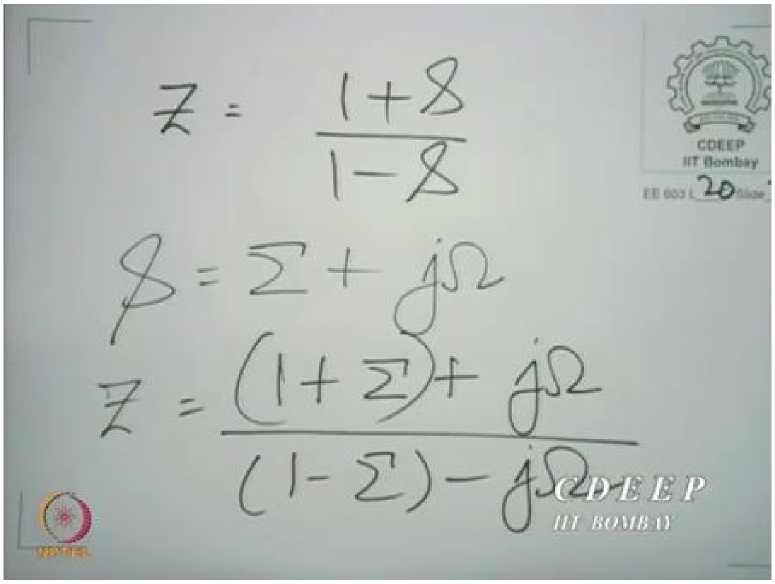

And therefore, $z = (1 + s)/(1 - s)$. Now we can say a lot about the real and imaginary part. Let $s = \Sigma + j\Omega$. So, $z = (1 + \Sigma + j\Omega) / (1 - \Sigma - j\Omega)$. And therefore, we are interested in the magnitude of z, it is the magnitude of z that concerns us. So, let us see what happens to the magnitude of z?

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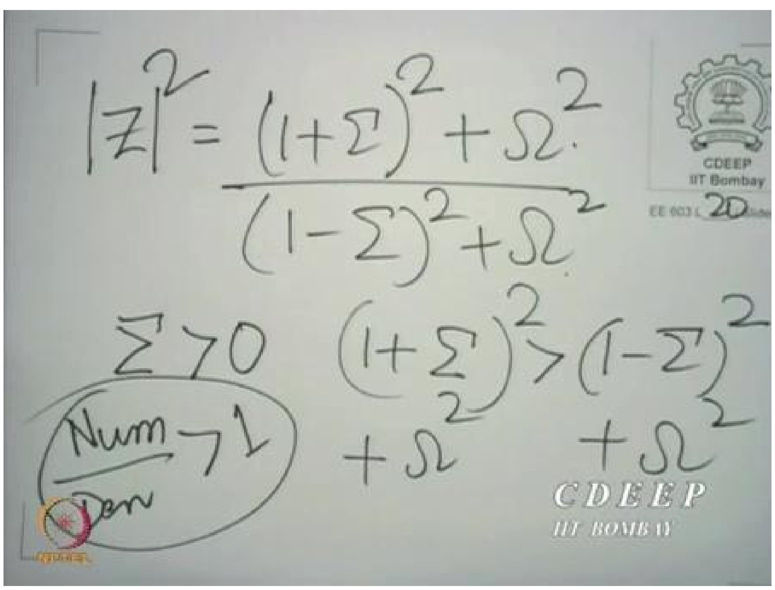

The magnitude of z is of course the magnitude of the numerator, numerator which is 1 plus in fact let us take magnitude of z^2. It is the square of the magnitude of the numerator

$(1 + \Sigma)^2 + \Omega^2$ divided by square of the magnitude of the denominator. That is easy to see. And of course, you have an Ω^2 common, so the Ω has no contribution to the magnitude, I mean it does not affect the the nature of magnitude. What what affects the, what what makes the numerator and denominator different is the factors $(1 + \Omega^2)/(1 - \Omega^2)$.

Indeed when $\Sigma > 0$ then $(1 + \Sigma)^2 > (1 - \Sigma)^2$, that is very easy to see. And nothing changes if you add Ω^2 to both sides. So, numerator is greater, so numerator by denominator is greater than 1 clearly. Is that correct? For $\Sigma > 0, (1 + \Sigma)^2 > (1 - \Sigma)^2$. In that case the numerator is strictly greater than the denominator. And therefore, $|z| > 1$.

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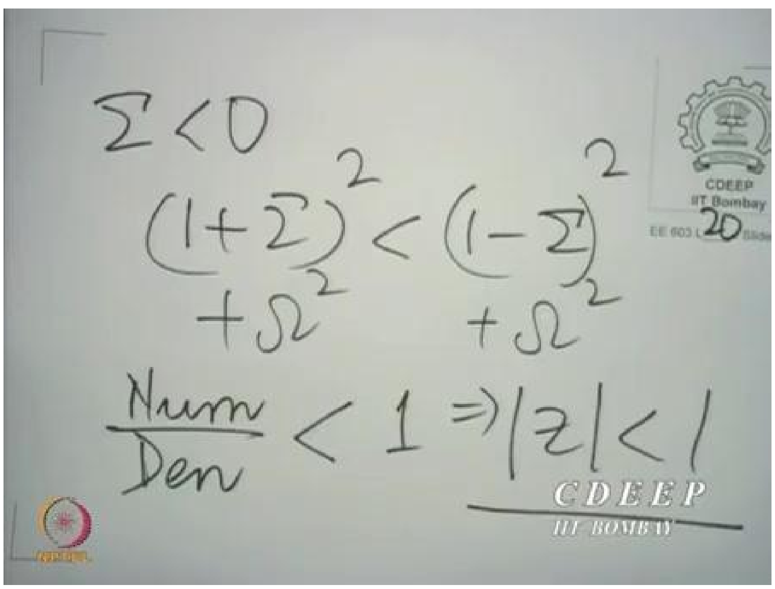

In contrast when $\Sigma < 0$ then $1 + \Sigma^2 < (1 - \Sigma)^2$. And nothing changes if you add Ω^2 to both sides. And therefore, it is very clear the numerator by the denominator is strictly less than 1. And therefore, $|z| < 1$. Is that correct? So, we have what we wanted. We have a satisfactory mapping of between the imaginary axis and the unit circle.

The outside of the unit circle corresponds to the right half of the s plane, that is the real part of s being greater than 0 and the interior of the unit circle corresponds to the left half of the z plane, is a one to one there. In fact, it is not even too difficult to see that it is one to one. You have a one-to-one mapping between s and z, not just on the imaginary axis, but everywhere. You have got an inverse, so you know you can go from s to z or from z to s uniquely. So, we have luckily arrived at the transformation that we want.

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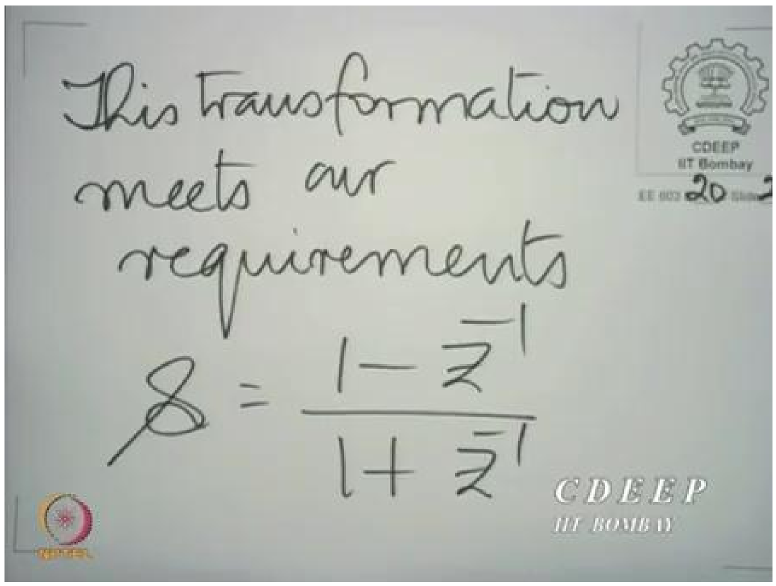

This transformation meets our requirements. s is 1 minus z inverse by 1 plus z inverse. Now as a point for further reflection I put the following question to you, how do you interpret this as an approximation of derivatives in terms of shifts? And the hint is to expand this in terms of a power series.

Of course, you have a $1 - z^{-1}$, but $1/(1 + z^{-1})$ can be expanded as a power series and that would give you some insight on what you mean on what are we trying to do in terms of approximating derivatives by shifts. It is not too simple, but it is interesting. Of course, that is besides the main point of the discussion, we have got we wanted, we are happy we have got a rational function, rational function does all that we wanted to.

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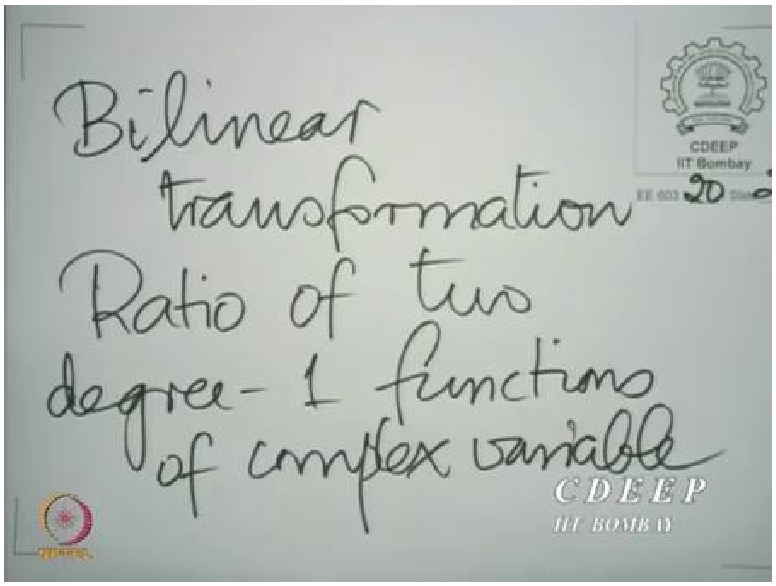

Now this transformation which we have now so satisfactorily constructed is called a bilinear transformation. It is called bilinear, because it is a ratio of two degree 1 functions of the complex variable. This is an example of a bilinear transformation. The the set of bilinear transformations has been studied in depth in complex analysis.

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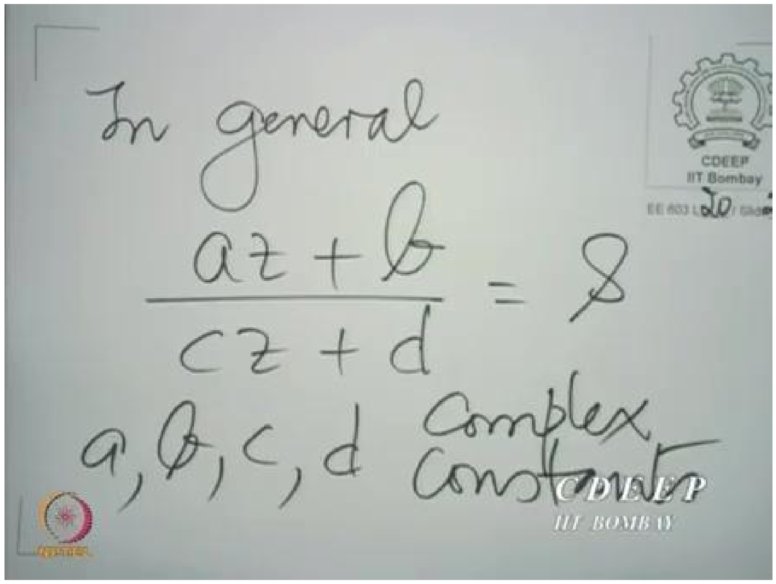

In general, a bilinear transformation looks like this, $az + b/(cz + d) = s$ is the general bilinear transformation, where a, b, c, d are complex constants in general. For those of you who wish to gain more insight into this transformation I would recommend looking at any standard text on complex analysis. Perhaps the first level engineering mathematics text and in reasonably comprehensive discussion on complex analysis which is often a part of the syllabus of an engineering mathematics subject in the first initial years.

The bilinear transformation is described in depth. One of the important properties of a bilinear transformation is that it takes straight lines in circles to straight lines in circles. That is a property which is often pointed out in the context of bilinear transforms. As you can see this transform is no exception. It takes the straight line, the imaginary axis into the unit circle and of course you can also think further to see what other straight line circle relationships there are in this bilinear transforms.

But that is typical of a bilinear transforms. There is a straight-line circle to straight line circle correspondence. Anyway, that was a remark to put you in perspective with complex analysis. But now we have the tools that we want and we have agreed that we are going to use this bilinear transformations, so now what we need to do, is to put down, what is the process that we will now follow?

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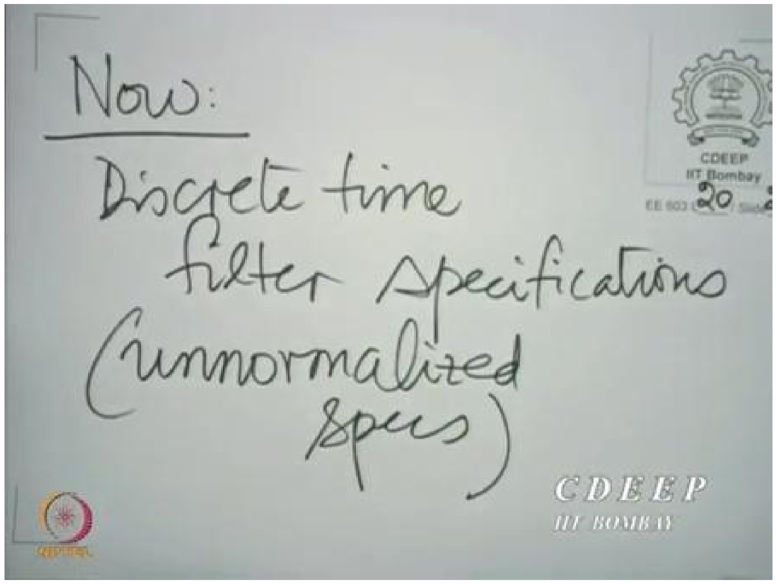

Now, we shall take the discrete time filter specifications, discrete time filter specifications. Of course, these are unnormalized, normally you would be given unnormalized specifications. What are these unnormalized specifications?

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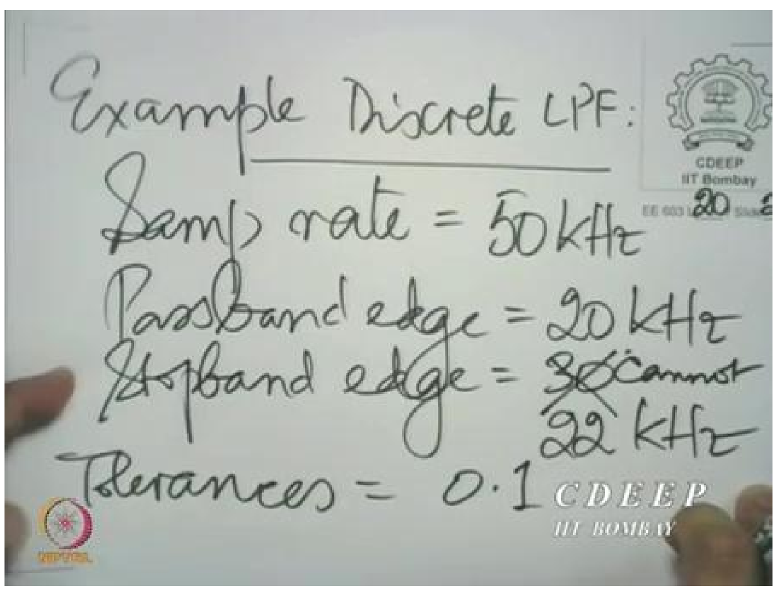

To take an example, suppose you have low pass filter, you may say sampling rate is so much, let us say it is 50 kHz just to make a point, the passband edge is 20 kHz let us say, the stopband edge is 30 kHz, can it be? No, it cannot, so this cannot be. Let us make it 22 kHz. If the sampling rate is 50 kHz at best, we can deal with the frequency of 25 kHz. So, stopband edge is 22 kHz and the tolerances of the passband and stopband are 0.1.

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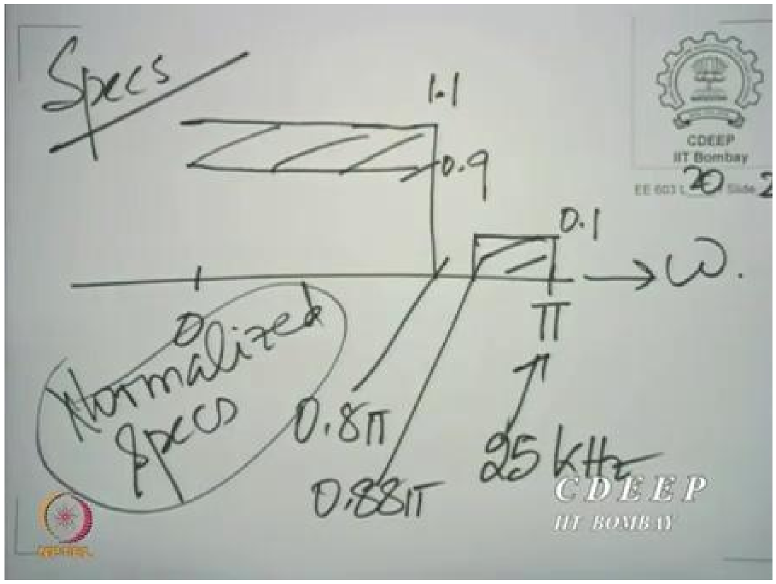

So, what we are saying in effect is that we want this kind of a frequency response, of course I am showing only the positive side. Now you seeπ corresponds to 25 kHz, since the sampling rate is 50 kHz. And therefore 20 kHz would essentially be 4 by 5 timesπ or 0.8 timesπ , this is the passband edge. And 22 kHz is 22 divide by 25 that is 0.88 π . And therefore, you have a stopband going from 0.88 to π and a passband going all the way from 0 to 0.8.

And of course, you are allowing the response to vary between 1.1 and 0.9 here and from 0 to 0.1 there. This is the response that you want to realise. So, this is how you would translate the specifications into the normalized angular frequency domain. These are called the normalized specifications on the normalized angular frequency axis. Is that clear, is that clear how we go from the unnormalized specifications to normalized specifications, very easy.

All that we are doing is really to normalize the frequency axis, is that correct? So here the changes, see it is very interesting, the s to z transformation, this process of normalization are all transformations of the independent variables, not the dependant variable. The dependant variable is not been transformed, it is the independent variable which is being transformed, and the dependant variable is being carried with the independent variable, is that right?

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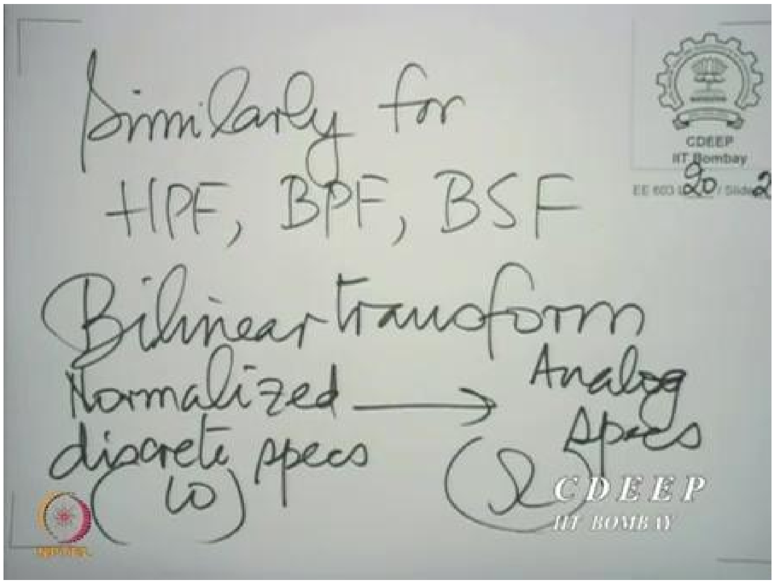

Now once we have the normalized specification, the next step, so remember this can be similarly done for band pass, similarly for high pass filter, band pass filter, band stop filter. You can go to a set of normalized specifications, and the next thing to do is to use the bilinear transform. The bilinear transform will take you from the normalized discrete specifications to analog specifications. That means you have specifications in ω and now you have here specifications in Ω . And how would we carry out this movement from discrete ω to Ω ?

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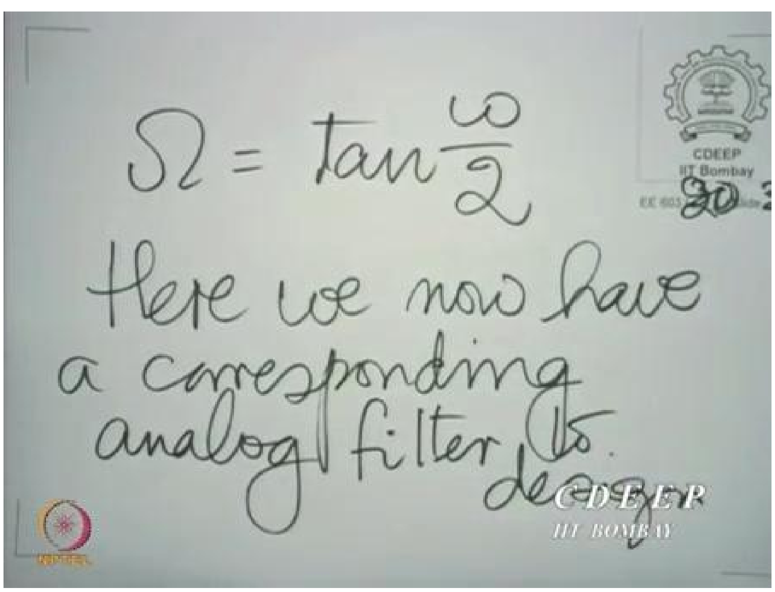

$\Omega = tan(\omega/2)$, that takes you from ω to Ω. So, the nature of the, so here we now have a corresponding analog filter to be designed. So, what would you do next? Of course, you would design that analog filter, take advantage of the known methods for analog filter design and design that analog filter.

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So, we will assume, we designed that analog filter. Let us call it H analog, and that would be a function of s, it is a rational function of s, rational, stable, causal. There are several different approaches and we will look at some of them. In fact, we will begin by looking at the low pass filter design.

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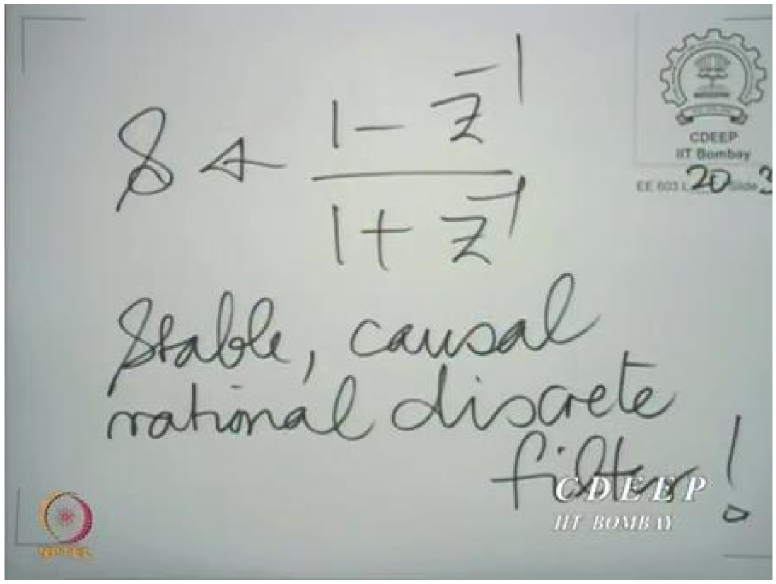

So, once we have an analog filter, we are done, because now all that you need to do is to replace s by 1 minus z inverse by 1 plus z inverse to get a stable, causal, rational discrete filter, and our job is done. So, this is the scheme of things. Therefore, now in the next lecture we need to look at how we can design analog filters, given the specifications of the analog filter, and we will begin with the low pass analog filter as a case in point. Thank you.