Digital Signal Processing and its Applications
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Lecture – 24B
Derivation of Chebyshev Filter Poles and System Response

Now, you see what we need to do is to complete the process of design; we have n with us and we have epsilon**.** And we have agreed we will choose epsilon to be the very most that it can be, and that is $\sqrt{D_1}$. If we have, we cannot unless you are willing to compromise on order; it does not make sense to choose epsilon, any different from $\sqrt{D_1}$. Having agreed to that we now want to put down the poles as usual; we need to find out the discrete system function.

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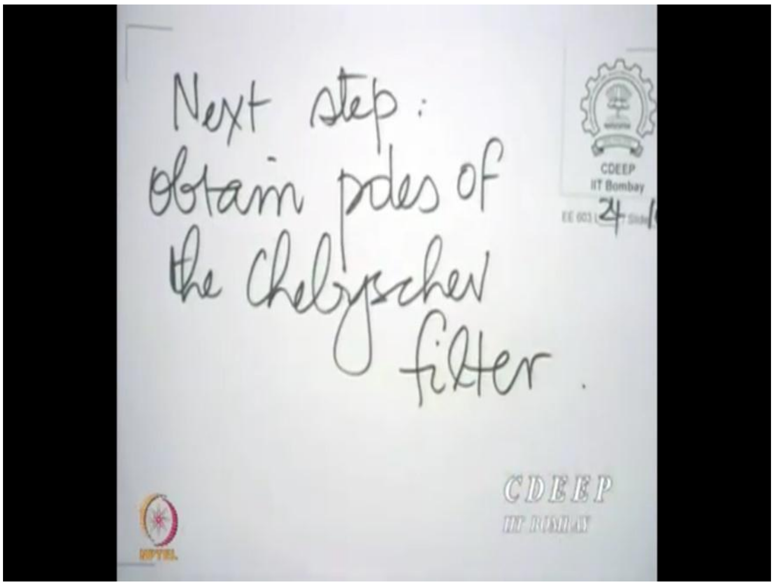

The next step we design is to obtain the poles. Now, before I proceed to this, I wish to take an observation about the transition band of filters. The transition band is characterized by no specification at all on the magnitude. Of course, it is expected that the magnitude would move smoothly from the passband to the stopband. And normally the transition band does show a monotonically decreasing character of magnitude; but, that is not specified. So, what characterizes the transition band is unspecified magnitude and not specifically desired magnitude at all.

Neither, of course we definitely we do not ask that the magnitude response be 0 all over the transition band. We cannot ask for if you are asking for; that we are in fact asking for an ideal filter or something like an ideal filter. And anyway, it serves no purpose to make the magnitude response 0 all over the transition band. Further, we are not even asking really that it be monotonically decreasing although that is how it often is.

Of course, it depends on increasing or decreasing will depend on whether the passband follows the stopband, or stopband follows the passband. So, you see whatever it be we normally do observe in most of the common designs that there is a smooth movement in monotonic fashion from the stopband to the passband; or the passband to a stopband.

But this is not specified. So, even if one comes up with a design, where it is non-monotonic; that is acceptable for the transition band. In fact, the sole characteristic of the transition band is nothing is asked either of magnitude or phase. Whatever emerges as a consequence of satisfying the passband requirements and stopband requirements is accepted in the transition band. Well, so much so then for finding the poles of the Chebyshev filter.

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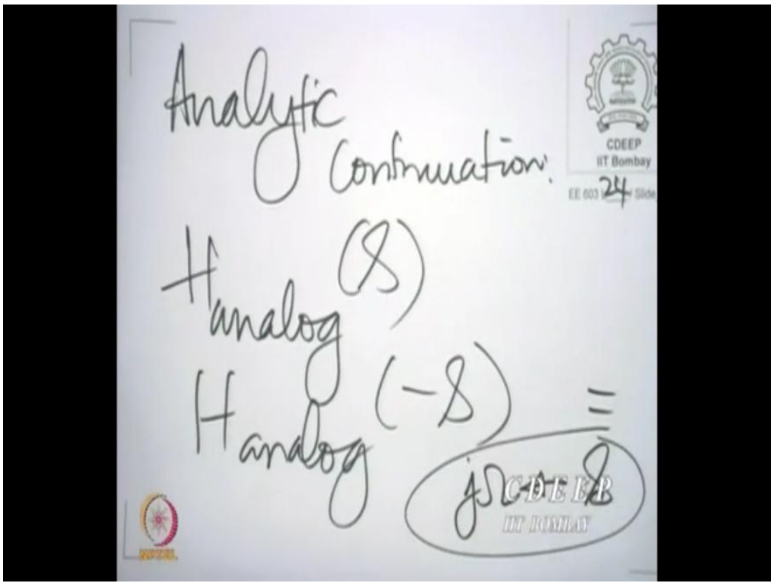

Now, how we would find the poles is to write down again the analytic continuation. And we know $H_{analog}(S)$ into $H_{analog}(-S)$ as was the case, before for the Chebyshev filter; can be obtained by replacing jΩ by S. In other words, Ω needs to be replaced by S / j.

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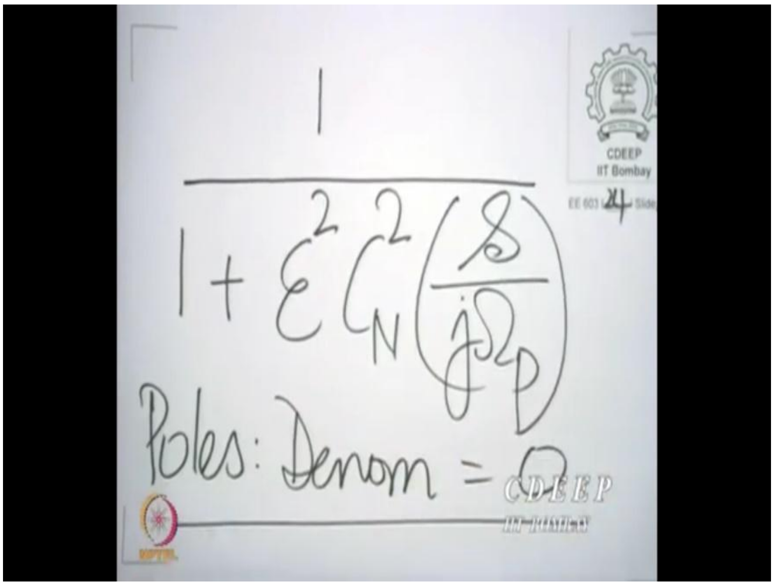

So, we have this product would be essentially the squared magnitude and analytically continued. And that is $\frac{1}{1+\varepsilon^2 C_N^2(\frac{S}{j\Omega_p})}$ and the poles are obtained by putting the denominator equal to 0 and let us solve that. Now, here you must remember that these poles are complex.

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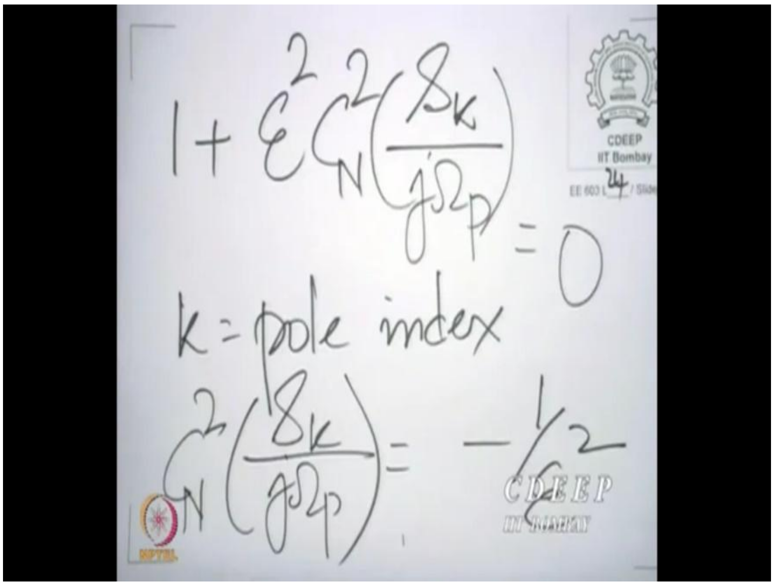

So, to solve this, $1 + \varepsilon^2 C_N^2$; now there will be several poles indexed by some integers. So, let us call that integer k index of the pole; k is the pole index, as was the case in the Butterworth filter. You run it over the set of integers. The kth pole is satisfies $1 + \varepsilon^2 C_N^2(\frac{S}{j\Omega_p})$ is equal to 0. And therefore of course, $C_N^2\left(\frac{S}{j\Omega_p}\right) = -\frac{1}{\varepsilon^2}$ .Now, we could take both the positive and the negative square root on both sides. But we will see that it is adequate to take any one of them.

Once you are going to run k over the integers; it would take care of the case of positive and negative, by running it over sufficient number of consecutive integers.

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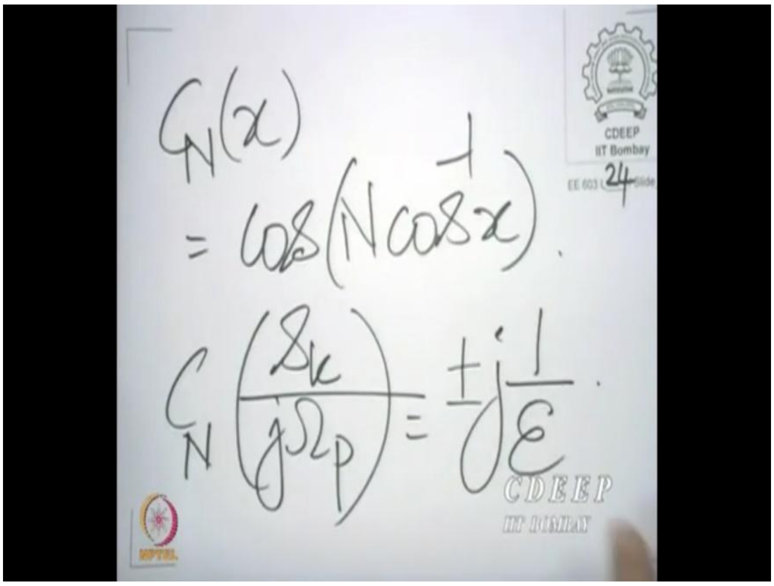

So, we will we will go back to expanding C_N. So, $C_N(x)$ if you recall essentially $\left(Ncos^{-1}x\right)$ .So, what we have for the kth pole is that $C_N\left(\frac{S_K}{j\Omega_p}\right)$ is + or -; it does not matter, but one could say 1 by epsilon. Well, +- j$\frac{1}{\varepsilon}$; so let us keep the plus minus, and later we can choose any one of them. As I said that we could we could be happy with keeping $+\ j/\varepsilon$ or $-\ j/\varepsilon$.

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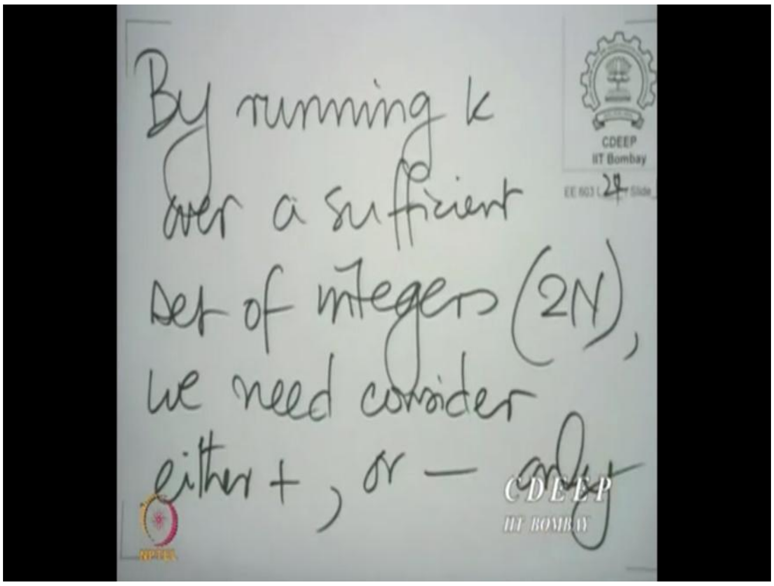

By running k over sufficient over sufficient set of integers; sufficient set means 2N of them. We need consider only one of either plus or minus only.

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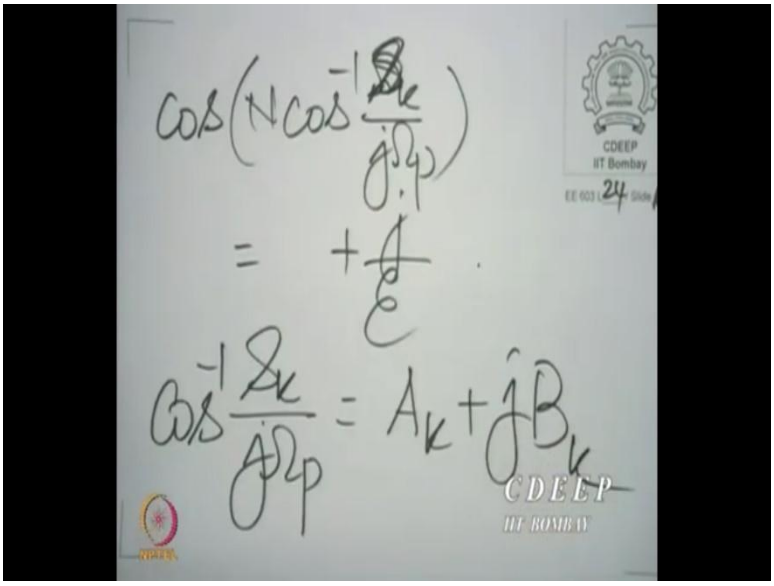

And therefore, we can now write down this equation, $\left(NCos^{-1}\frac{S_K}{j\Omega_p}\right)$, is $+\frac{j}{\varepsilon}$ .Now, let us put $Cos^{-1}\frac{S_K}{j\Omega_p} = A_K + jB_K$. So, remember this is a complex argument and therefore we need to have complex solutions to it. Now, we are working with entirely complex cosine, sine and everything; so, we can now take the cosine of both sides and solve this.

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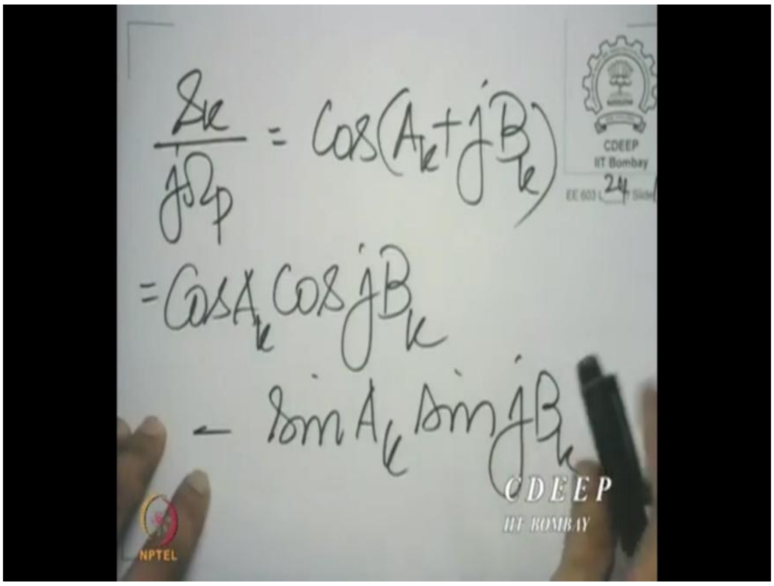

So, we have $\frac{S_K}{j\Omega_p}$ is $Cos(A_K + jB_K)$. And that can be expanded in the standard way in which we expand trigonometric functions; this is $Cos(A_K + jB_K)$ - $Sin(A_K + jB_K)$. But we recall that $Cos(jB_K)$ is nothing but Cos hyperbolic of B_K. Remember A_K and B_K are now real; so cos jB_K = $Cosh(B_K)$; and sin jB_K = $-\ j\ Sinh(B_K)$,is that right. So, there we go.

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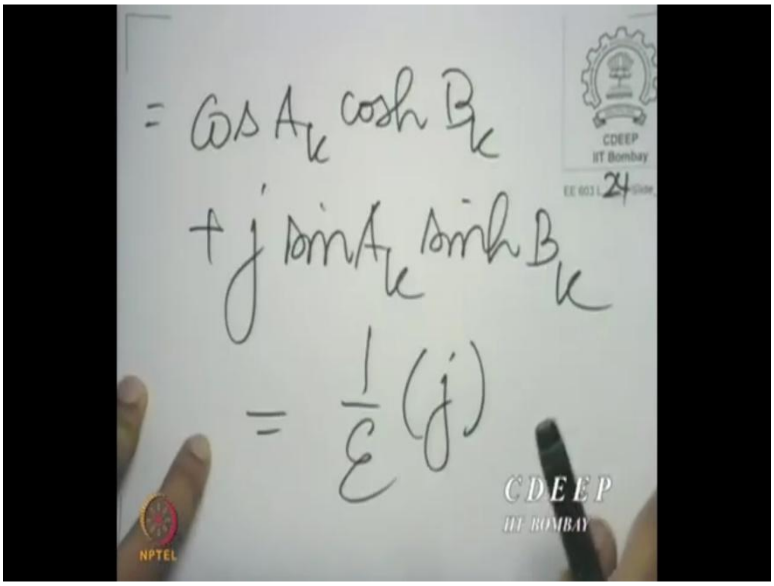

We have this is equal to $Cos(A_K).Cosh(B_K) + jSin(A_K).Sinh(B_K)$; and we equate this to $\frac{1}{\varepsilon}(j)$. From where we can equate the real part and the imaginary parts separately? So, we equate the real part of cos to 0 here, and the imaginary part to $\frac{1}{\varepsilon}$.

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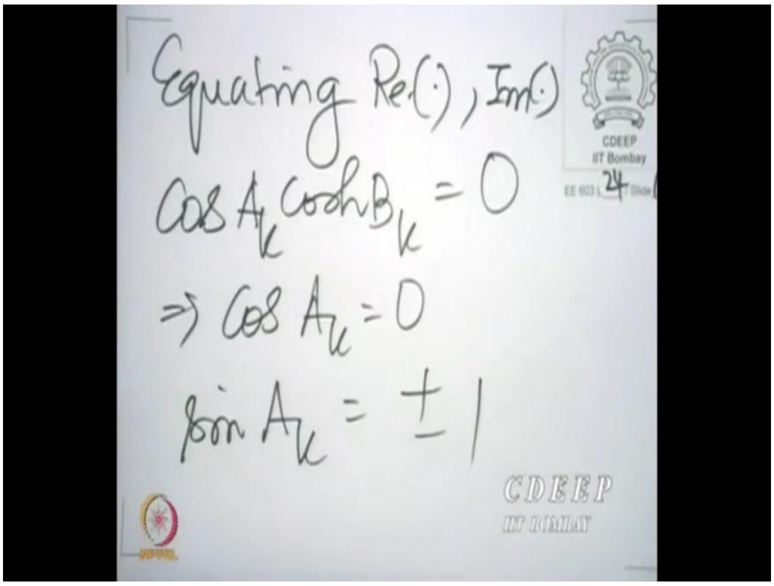

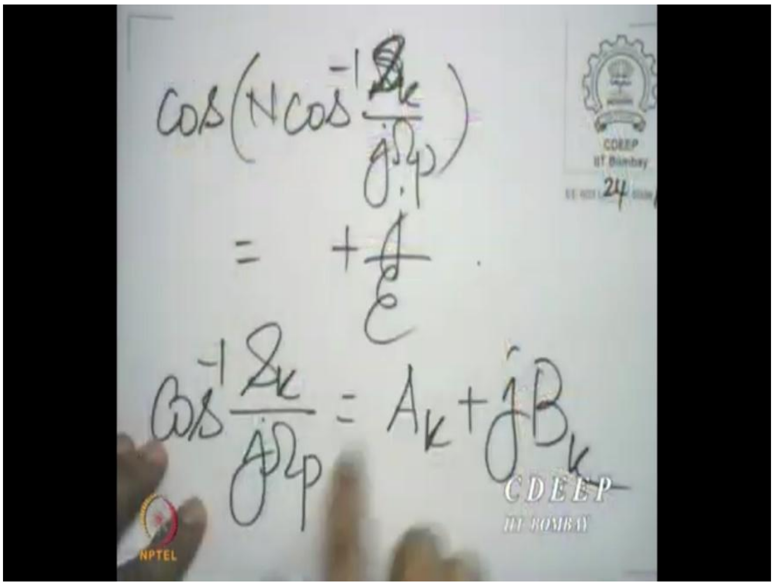

$Cos(A_K).Cosh(B_K) = 0$, now $Cosh(B_K)$ cannot possibly be 0; so, the only possibility is that $Cos(A_K)$ is 0. $Cosh(B_K)$ as we have seen before must be greater than 1, so therefore $Cos(A_K)$ is 0; and if $Cos(A_K)$, now A_K is of course real argument, so there is no problem. If $Cos(A_K)$ 0 then $Sin(A_K)$ is = +-1 that is very clear. Yes, please.

Student: (()) (10:59)

Professor: Yes, yes yes yes, he is absolutely correct; there is a question, that is that is correct. I need to make a correction, I am very glad that somebody pointed this out. So, you see we have taken, we need to write down n times; so we need to write down n times, this is absolutely. So, I need to make a correction here; that is correct. So, you see $\frac{S}{j\Omega_p}$, well let us let us therefore make a correction here. Let us put back this argument; so you have $A_K + jB_K$ here; so let me repeat that step.

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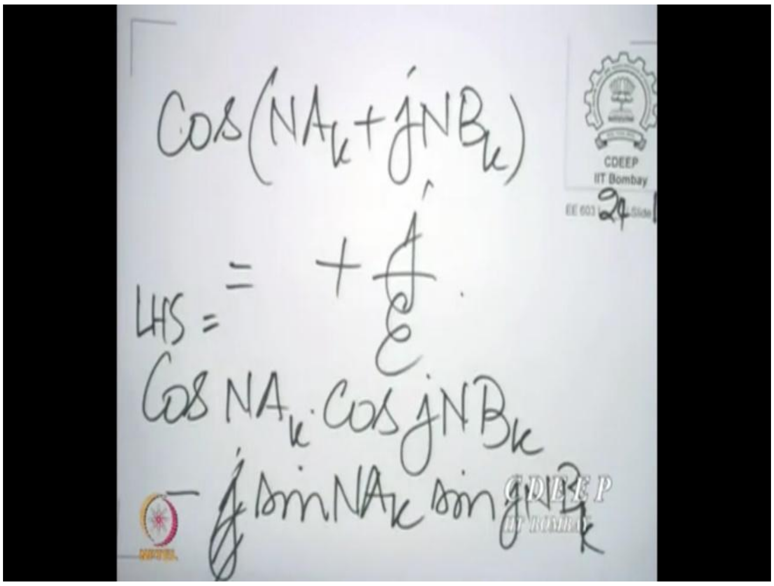

In fact, let us, so we have $\left(NA_k + jNB_K\right)$ = + $\frac{j}{\varepsilon}$; that is correct. So right, so we need to so let us complete this working. Let us expand this on the left-hand side we have $\left(Cos(NA_k).Cos(jNB_K)\right) +$ or rather - $jSin(NA_k) + Sin(jNB_K)$; or well I am skipping a step here, but maybe I will write it down first. So, $jSin(NA_k) + Sin(jNB_K)$, this is the left-hand side. Is that all right? So, we need to correct it, yes.

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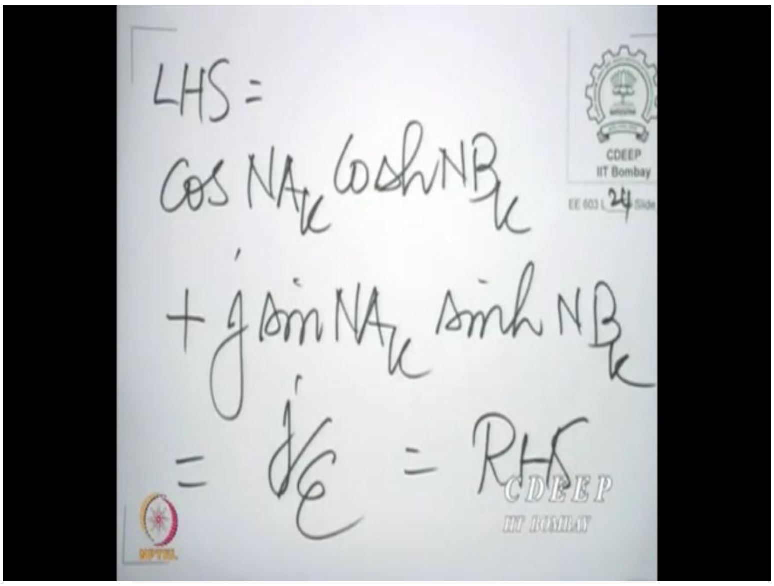

And LHS will therefore evaluate to $Cos\left(NA_K\right)$ as usual $Cosh\left(NB_K\right)+$ $jSin\left(NA_K\right).Sinh\left(NB_K\right)$; that is correct. And this is equal to $\frac{j}{\varepsilon}$ which is the RHS; is this fine? So, we do not need to introduce a correction there; I am glad that was point out. Is that clear? So, we need we now proceed to equate the left-hand side and the right-hand side; and therefore the real and imaginary parts of the left and right side.

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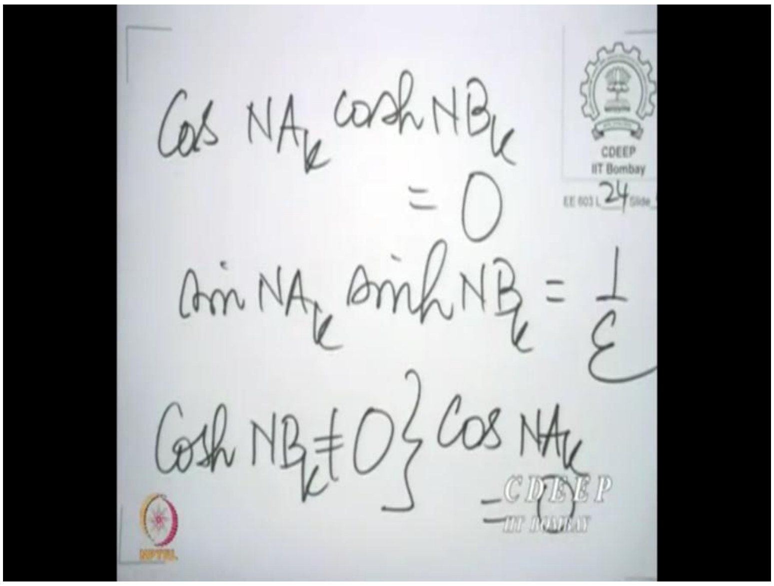

So, once again we would get $Cos\left(NA_K\right).Cosh\left(NB_K\right)=0$; and $Sin\left(NA_K\right).Sinh\left(NB_K\right)$ is $\frac{1}{\varepsilon}$. And before we observe that $Cosh\left(NB_K\right)$ cannot be 0; and that means that $Cos\left(NA_K\right)$ is 0. Is that correct? Yes, $Cos(NA_K) = 0$; now if $Cos(NA_K)$ is 0, then clearly $Sin(NA_K)$ has no choice. But to be either + or - 1, $Sin^2(NA_K)$ needs to be 1; because $Cos^2(NA_K) + Sin^2(NA_K)$ = 1.

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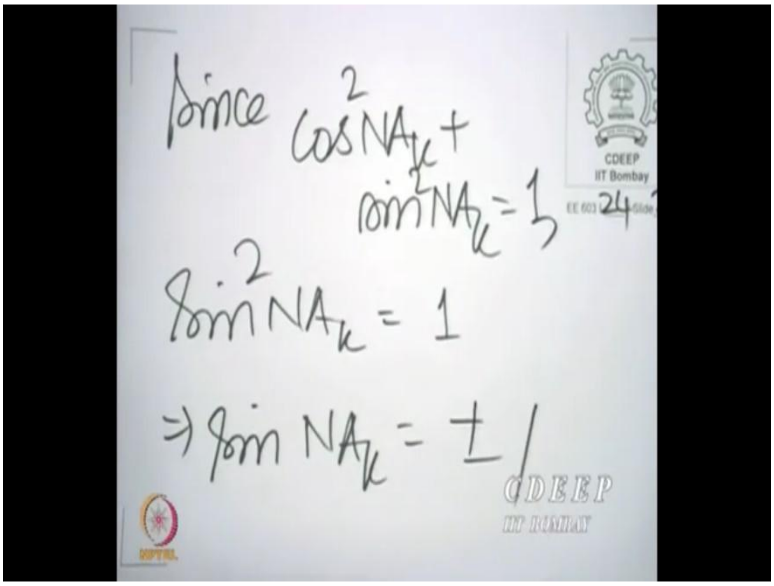

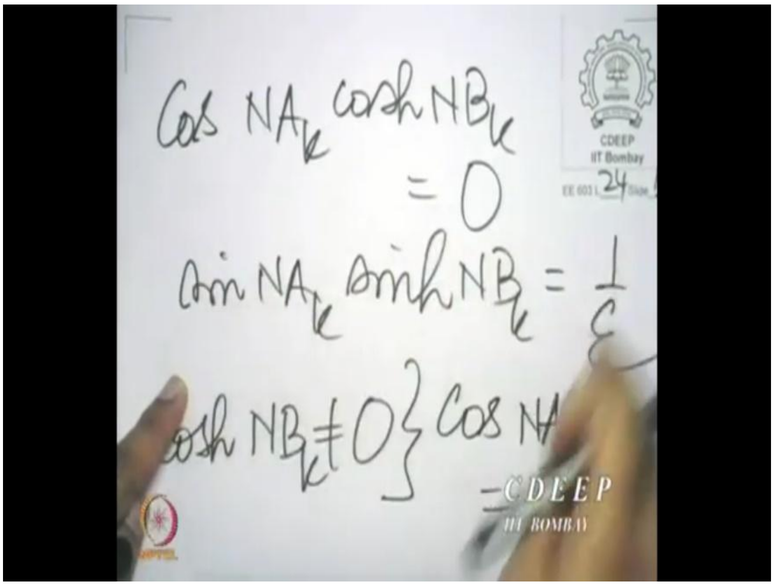

So, of course you have $Sin(NA_K)$ again is either + or - 1; and here too we might take either the positive sign or the negative sign. And all that will happen, the only change that will take place is that we need to run k over all the integers once again; all the required integers to cover both the positive and negative sign. So, here too we can be satisfied with taking one of them, and run k over a sufficient number of integers. So anyway, what we have is, well let me put, because this we do not need to refer to it again and again. So, here the situation is that this is either +1 or -1; let us take it to be +1, in which case this becomes $\frac{1}{\varepsilon}$, is that right?

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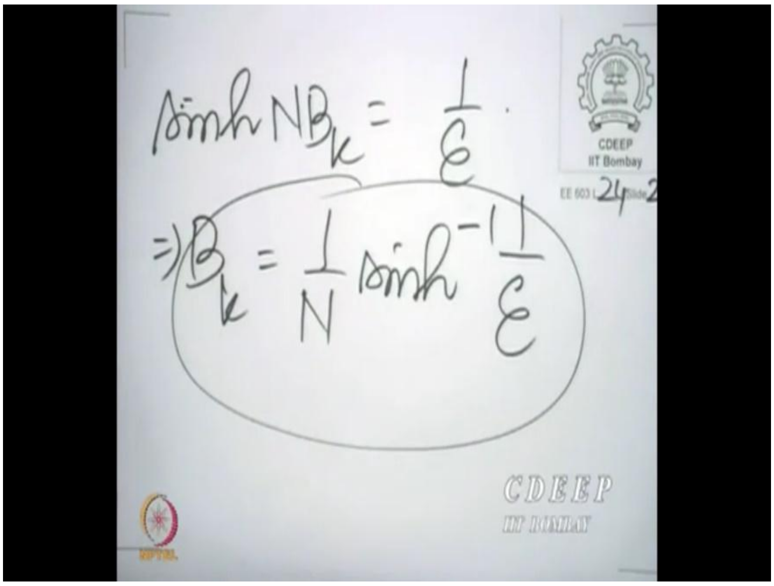

So, if $Sinh\left(NB_K\right)$ is $\frac{1}{\varepsilon}$, which means B_K is now clearly $\frac{1}{N}$, $Sinh^{-1}\left(\frac{1}{\varepsilon}\right)$ inverse of $\frac{1}{\varepsilon}$, that is interesting. And of course, B_K has as you can see has nothing to do with k so that is interesting; so, B_K is not indexed by the integers at all. So, the index, the integer index is going to act on A_K not on B_K . How will it act on A_K? And yes, so here it is very good that that student corrected; because had we not made that correction on route; we would have had trouble now in indexing A_K. So, it was very appropriate that that student interjected and made a correction on N.

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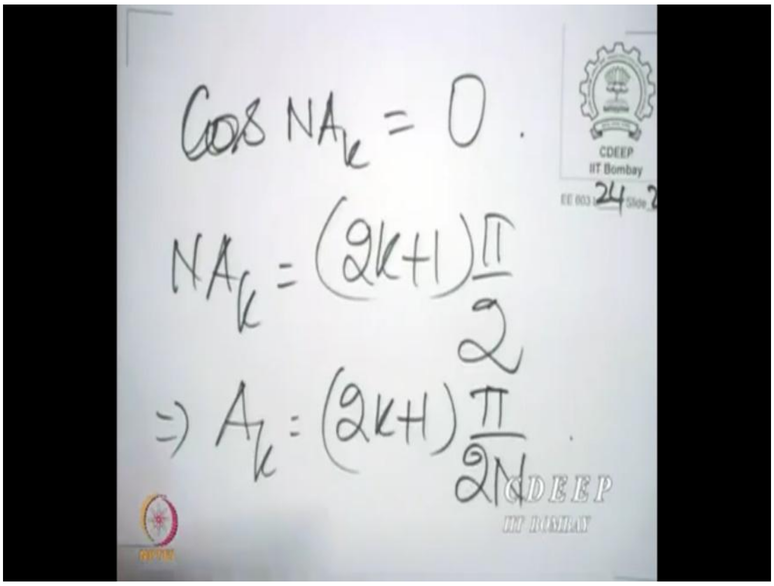

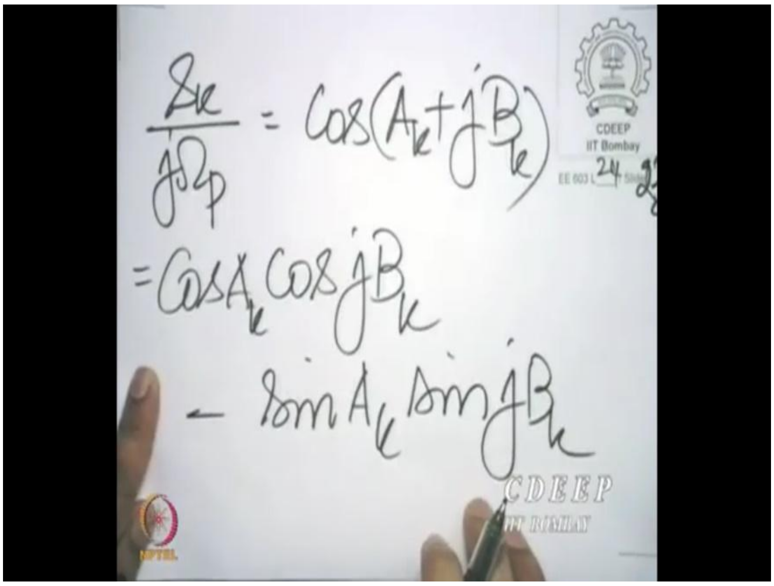

So, we have $Cos\left(NA_K\right)$ = 0; which means NA_K must clearly be an odd multiple of $\frac{\pi}{2}$. And that tells us that A_K must be of the form $(2K + 1)\frac{\pi}{2N}$; and now we know the poles because we know again A_K and B_K . Now, emphasize in fact maybe it is a good idea, maybe it is good that you know it was serendipity that we made that mistake; because it is very important to see that we do need a dependence on N, when we satisfy the equation for the cosine part. It is a dependence on N which allows you to create multiple poles; the dependence on k on the integer index is a consequence of cos NA_K being 0, not just cos A_K anyway. So, coming back to this, we now have an expression for the poles.

And that we have done partly before; let us put back that transparency. So, you will recall that we had written down $\frac{S_K}{j\Omega_p}$ is cos of. You see cos inverse of this was equated to $A_K + jB_K$; so $\frac{S_K}{j\Omega_p}$ is $Cos(A_K + jB_K)$. And we have expanded this. So, I will just renumber this; I will give this the number 23. So, now we know where the poles lie; S_K can now be calculated.

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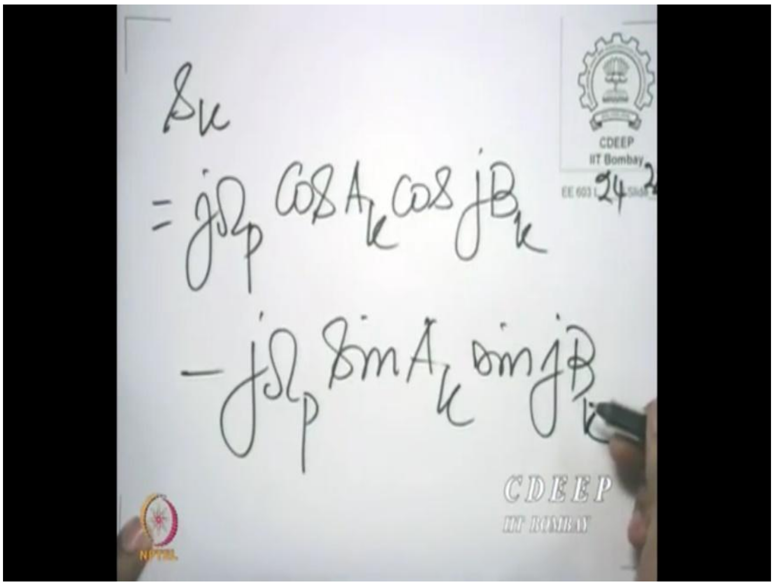

It $j\Omega_p.Cos\left(A_K\right).Cos\left(jB_K\right)$ - $j\Omega_p.Sin\left(A_K\right).Sin(jB_K)$. And now we can use the standard strategy of putting $Cos\left(jB_K\right) = Cosh\left(B_K\right)$, and $Sin\left(jB_K\right) = \ - jSinh\left(B_K\right)$.

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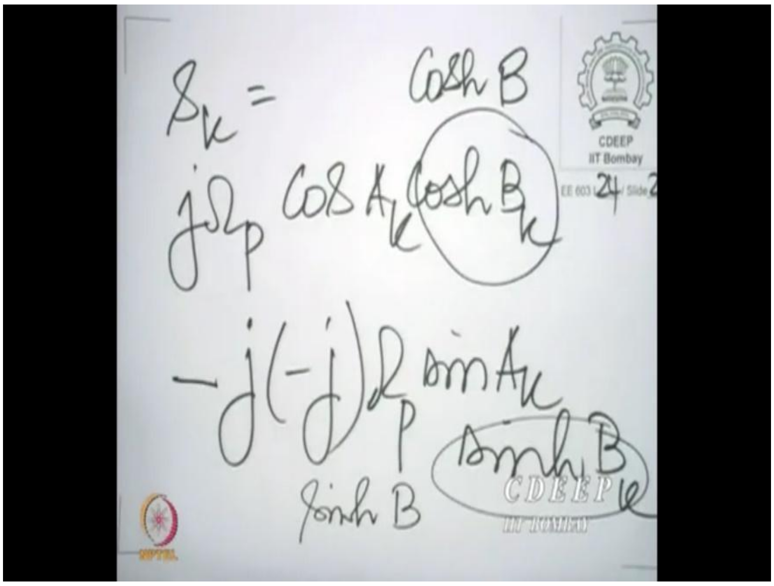

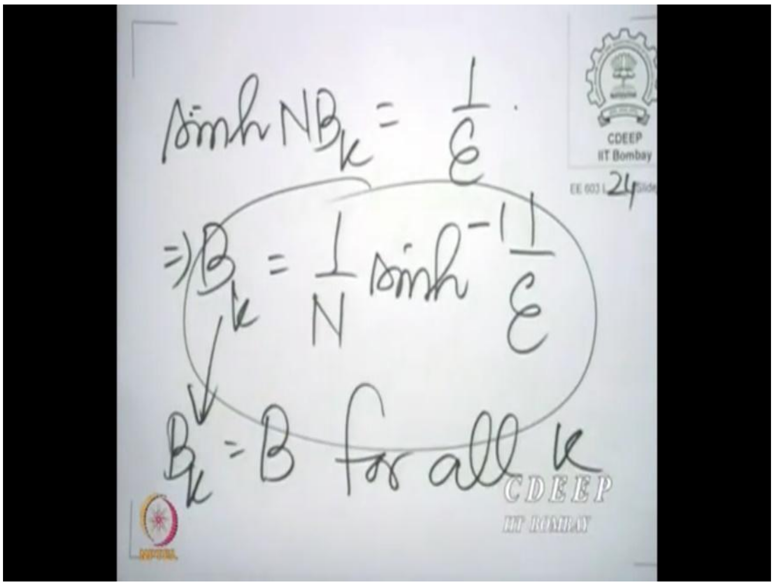

And therefore, we have S_K is $j\Omega_p . Cos(A_K) . Cosh(B_K)$, -j times, you see -j x-j, Ω_p, $Sin(A_K) . Sinh(B_K)$. And once again remember neither $Sinh(B_K)$ nor $Cosh(B_K)$ have anything to do with k. $Sinh(B_K)$ is essentially $\frac{1}{\varepsilon}$ or B_K, or rather if you if you look at it before we written an expression for B_K here.

So, $B_K = \frac{1}{N} Sinh^{-1} \frac{1}{\varepsilon}$; so neither sinh B_K nor $Cosh(B_K)$ have anything to do with k. So, you might as well just write $Cosh(B)$ and $Sinh(B)$ there. So, we will just write B_K is equal to B, for all k. So, we can just call this cosh B and we can call this $Sinh(B)$. And clearly this is the real part and this is the imaginary part here. This is the imaginary part and this is the real part of the pole.

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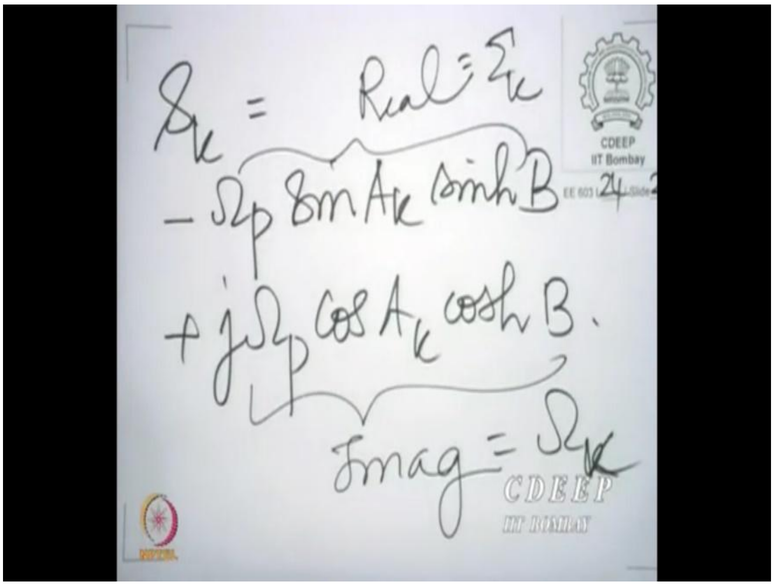

So, S_K is the real part, so $-\ \Omega_p . Sin\left(A_K\right). Sinh(B) + j\Omega_p . Cos\left(A_K\right). Cosh(B)$. And now what we need to do, you see we want to find out this is the real part, this is the imaginary part. And the real part and we can call the real part sigma k as we did, and the imaginary part capital Ω_K. And we can now write down an equation that relates Σ_K and Ω_K. We are trying to find a contour, a curve in the imaginary complex plane, on which these poles lie.

So, where is that contour? Well, how do you obtain the contour, the different poles are indexed by the k's. So, if we eliminate k, we get the contour; and to eliminate it all that we need to do is to note that $Sin^2\left(A_K\right) + Cos^2\left(A_K\right) = 1$.

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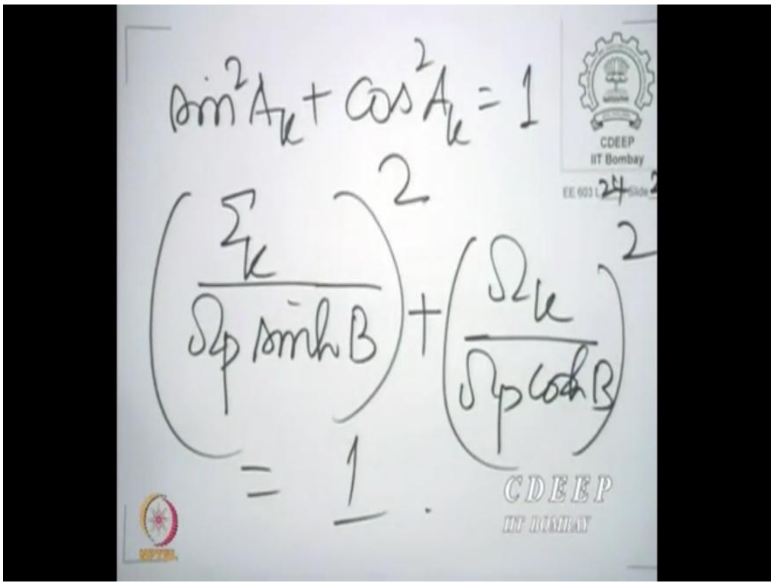

So, therefore $Sin^2\left(A_K\right) + Cos^2\left(A_K\right) = 1$. And that means $\left(\frac{\Sigma_K}{\Omega_p Sinh(B)}\right)^2 + \left(\frac{\Omega_K}{\Omega_p Cosh(B)}\right)^2 = 1$. And we know what contour this is, if the two arguments with the real, if these coefficients have been equal. If this had been equal to this, we would have landed up with a circle. Because these are unequal, we get an ellipse. Further, which is the major and which is the minor part axis of the ellipse? Now, this ellipse is aligned with the axis.

In other words, the major and minor axis are coincident with the vertical and horizontal here; it is not inclined ellipse. The question is which is the major axis and which is the minor axis? And to answer that question we need to decide which is greater. Is $Cosh(B)$ greater or is $Sinh(B)$ greater? Which one would be greater? It is the cosh hyperbolic which is always greater, for a real argument. Because $Cosh^2(B) = 1 + Sinh^2(B)$; and therefore, $Cosh^2(B)$ is always going to be greater than $Sinh^2(B)$. And therefore, in this it is very clear that the major axis is on the imaginary; and the minor axis is on the real. Is that clear to everybody?

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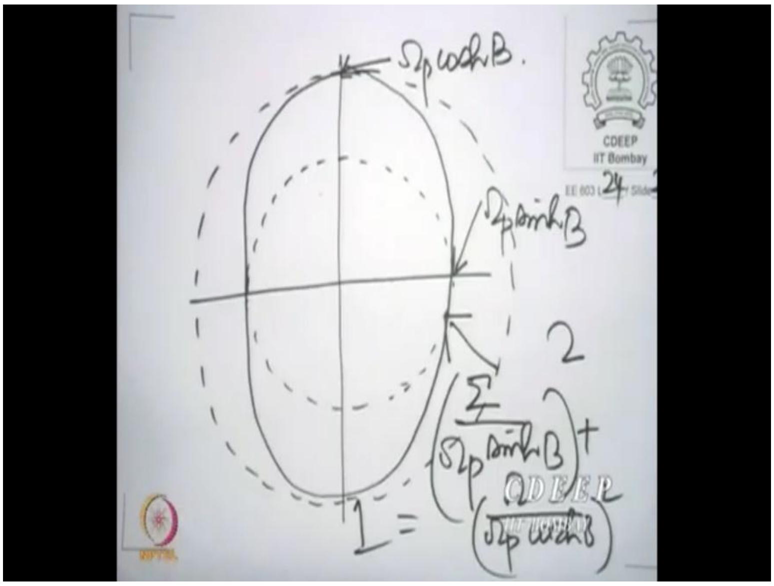

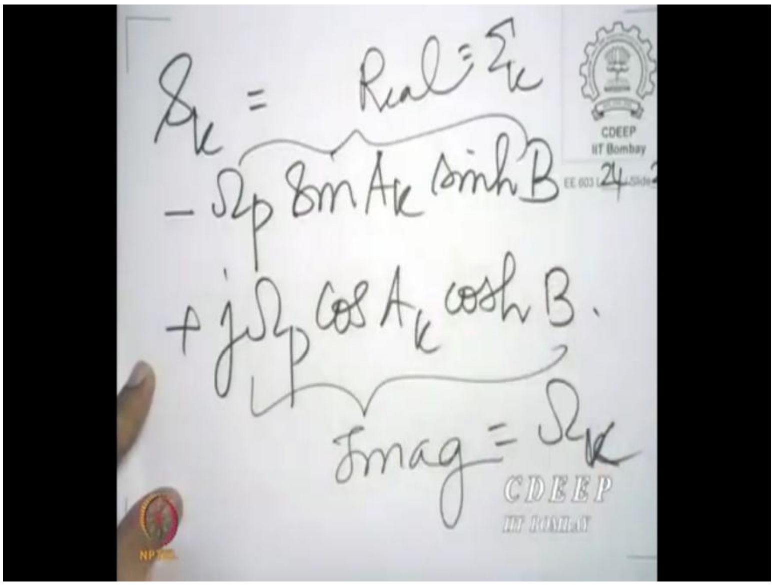

So, we get an ellipse, this is the contour that we land up with. This is the contour $\left(\frac{\Sigma}{\Omega_p Sinh(B)}\right)^2 + \left(\frac{\Omega}{\Omega_p Cosh(B)}\right)^2 = 1$; this is the contour. And of course, this point would be $\Omega_p Sinh(B)$, and this would be $\Omega_p Cosh(B)$, and the other points can be determined; of course, one must mark the specific poles. So, you know A_K and $Sin\left(A_K\right)$ and $Cos\left(A_K\right)$ now need to be determined; or you need to find out A_K and then find out where these poles will lie precisely. Now, the easiest thing to do is to how do you mark these poles? The easiest thing to do is to draw two circles. One with radius $\Omega_p . Sinh(B)$, and the other with radius $\Omega_p . Cosh(B)$.

What I am saying is it will be easiest for us to draw two circles like this. The real part can be marked by taking the inner circle here; all that you need to do is to see where this angle. The angle is $(2K + 1)\frac{\pi}{2N}$. All that you need to do is to draw and draw a radial line making an angle of $(2K + 1)\frac{\pi}{2N}$. And see where it intersects the circle and that gives you the real part. On the other hand, the imaginary part can be obtained by using the larger circle. And on the larger circle one takes the same radial line; but then the imaginary part is obtained by the measurement coming from the larger circle.

So, one has to be careful in marking the poles; the one, I must emphasize that when you draw radial here with angle A_K. A_K is $(2K + 1)\frac{\pi}{2N}$, where it intersects this circle will give you the real part; where it intersects this circle gives you the imaginary part. But, one must not straightaway take the intersection of this arc with the ellipse to find the location of the pole; no, that is not what it is. One must take the measurement of real imaginary part and mark it; and it of course would lie on the ellipse. So, I am just giving you a strategy to measure the real imaginary part.

But one must not use the radial line with angle A_K to intersect with the ellipse and mark the pole there; that is not correct. Anyway, I leave it to you as an exercise; this is an exercise actually mark the poles.

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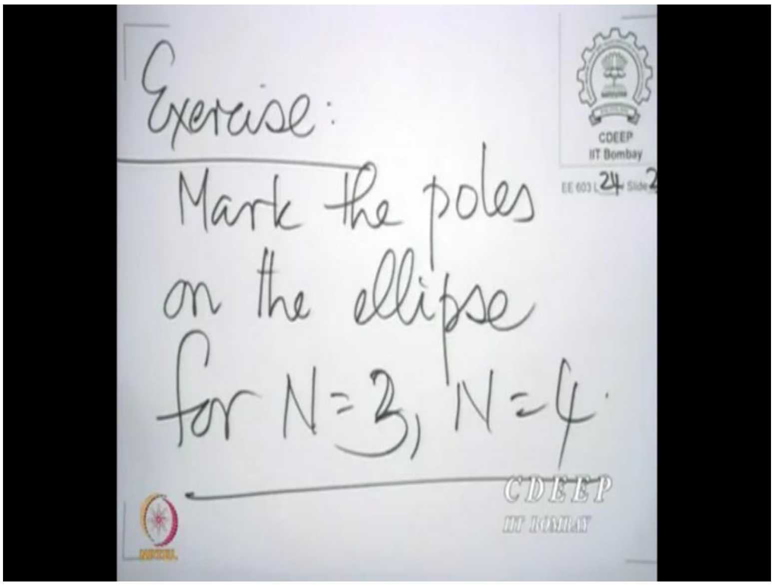

Mark the poles on the ellipse, for n equal to 2; I am sorry for n equal to 3 and n equal to 4. And you will get a feel of how the poles are located; you would also observe that there are 2N values of k to be taken, as was the case with the Butterworth filter. You can take any consecutive 2n values; so you could start with k equal to 0 and run all the way up to k equal to 2N -1. Or you could start with 1 and run up to 2N, it does not matter. Whatever it be after you mark all the 2N poles on the ellipse; the poles in the left half plane would give you the poles corresponding to $H_{analog}(S)$. So, let us write that down. The remaining steps are identical.

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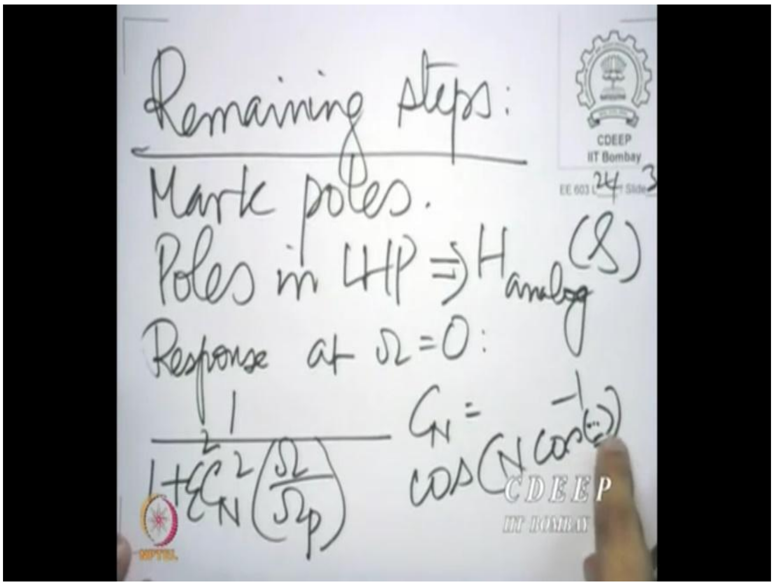

Mark poles, poles in LHP give you $H_{analog}(S)$. Now, there is one important observation here, which does not happen in the Butterworth filter. And that is what do you want the numerator to be in $H_{analog}(S)$? In other words, what is the magnitude response when omega equal to 0. Now, there you have to be careful, because the response for omega equal to 0 is not 1 identically here; it depends on whether N is odd or even. So, you recall the response at Ω equal to 0; how would you determine it? Well, we had $\frac{1}{1+\varepsilon^2 C_N^2(\frac{\Omega}{\Omega_p})}$, and $(NCos(...))^{-1}$.

Now, let us look at the two situations. When you have $Cos^{-1}(0)$, $Cos^{-1}(0)$; of course can be taken to be $\frac{\pi}{2}$. However, when N is even, this becomes cos of an even multiple of $\frac{\pi}{2}$; which is either +1 or -1. So, when N is even, this evaluates to +-1; and therefore, this evaluates to 1.

So, this response is $\frac{1}{1+\varepsilon^2}$. On the other hand, when N is odd; then you have an odd multiple of $\frac{\pi}{2}$, so this evaluates to 0. So, this response is just $\frac{1}{1+(...)}$ that is 1. So, you have to distinguish between N odd and N even to put down the response at omega equal to 0.

And therefore, when you put down the numerator in $H_{analog}(S)$; you must at S equal to 0, equate it to the expected response for omega equal to 0, and not identically 1. If N is odd, it evaluates to 1; if N is even, its square would evaluate to $\frac{1}{1+\varepsilon^2}$. And therefore, the magnitude itself would evaluate to $\frac{1}{1+\varepsilon^{\sqrt{2}}}$, is that right? That care needs to be taken when specifying the Butterworth, the Chebyshev filter, unlike the Butterworth filter. And finally, once you have $H_{analog}(S)$, the remaining process is common to the Butterworth filter.

Replace S using the bi-linear transform and get the discrete time system function; that completes the design of the Chebyshev lowpass filter. And now we are well equipped to proceed to see how we could design other kinds of filters; either with the Butterworth approximation or the Chebyshev approximation. By using what are called analog frequency transformation.