Digital Signal Processing and its Applications
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Lecture 25 C
High Pass Transformation

And now we would like to look at the corresponding transformation for band pass filters. Now for band pass filters, we need to think a little more. You know, for band pass filters, it is very clear that you have multiple stop bands. Even though there is only one pass band, there are multiple stop bands. How on earth could you get multiple stop bands from a single stop band? You cannot do it by a single element.

In fact, here we can probably take a cue from what we do in *LC* networks again. Remember we are going to choose an impedance of an *LC* network. Now, what simplest form of an *LC* network and give us an impedance, which has multiple mappings? That is, you want multiple frequencies to have the same magnitude; that is the kind of mapping you want.

You want the stop band to go to multiple places. How on earth can that happen? That can only happen if you have a second order system, at least, it can be less than second order. But we can probably make do with second order. Now, what we mean by second order is you need an inductance and a capacitance to come together. So, let us consider a series *LC* network and consider its impedance.

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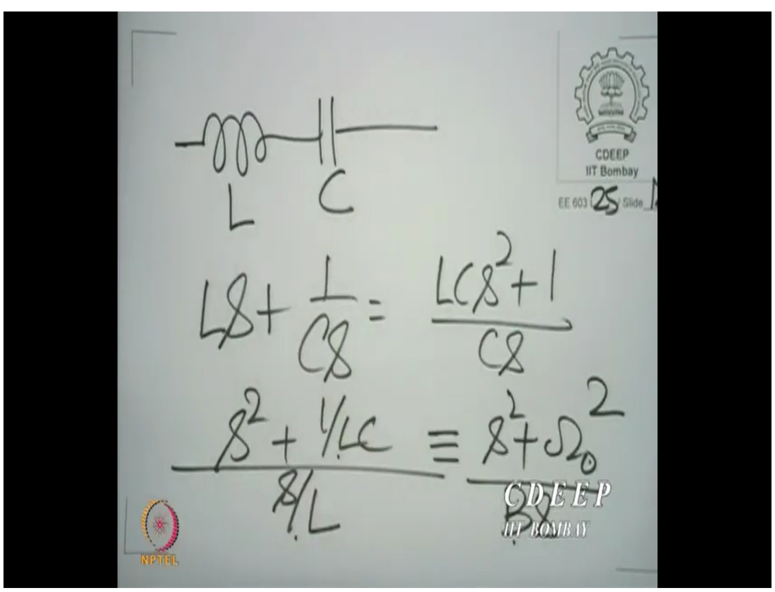

We have an L and a C. And of course, we have $Ls + \frac{1}{Cs}$. So, that gives us an $\frac{LCs^2+1}{Cs}$. Now, we can divide by LCin the numerator and denominator, and that gives us $\frac{s}{L}$, which we will write as a $\frac{s^2+\Omega_0^2}{Bs}$, where Ω_0^2 is $\frac{1}{LC}$. And B is a positive quantity equal to $\frac{1}{L}$.

So, let us consider this transformation. Now, Ω_0^2, which is $\frac{1}{LC}$ has a very important physical significance. In fact, all of us would probably be aware that for an LC network, that corresponds to what is called the resonant frequency of the network, the resonant angular frequency of the network. In fact, it is that frequency at which this impedance vanishes.

So, you see the interesting thing is that B also has a significance, but that significance, we may not all be aware of, and let us not be too worried about it for the moment. We will see it in the long run. At the moment, we will just regard B as a positive quantity. It is the reciprocal of *1* of L or $\frac{1}{L}$.

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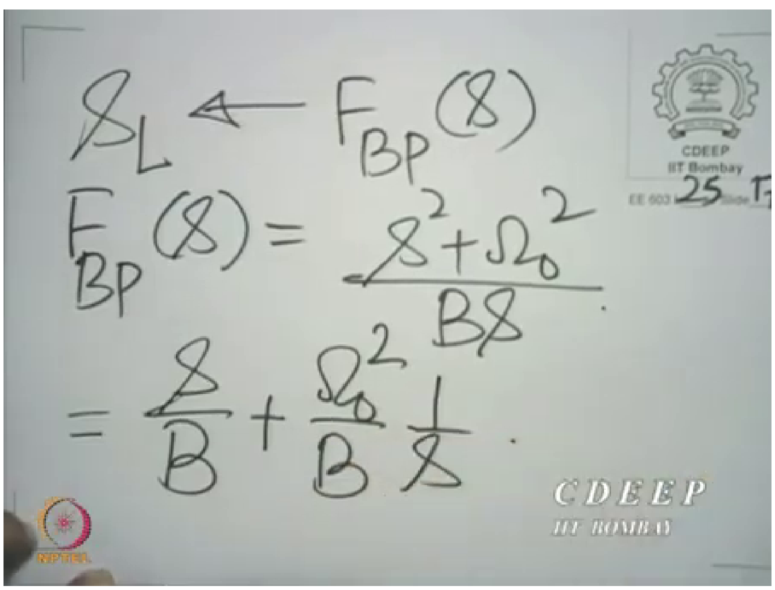

So, let us consider the candidate s_L to be replaced by $F_{BP}(s)$, where $F_{BP}(s)$ is $\frac{s^2+\Omega_0^2}{Bs}$. And let us go through the exercise first, of checking for the, now, we will do it the other way. We will first check property one, then property two and then we will look at property three.

So, of course, property one is obvious again, this is rational transformation, rationality is going to be maintained. So, if the original low-pass filter is rational, this has, the transformed filter has no choice but to be rational because you are replacing s_L by a rational function of s. So, rationality is maintained anyway, there is no problem with that.

Now, we need to look at the second property - stability. So, all we need to do is to substitute s equal to, in fact, you know, we can do something simpler. Let us divide this into two parts. Let us write this as $\frac{s}{B} + \frac{\Omega_0^2}{B}\frac{1}{s}$. And now let us substitute $s = \Sigma + j\Omega$.

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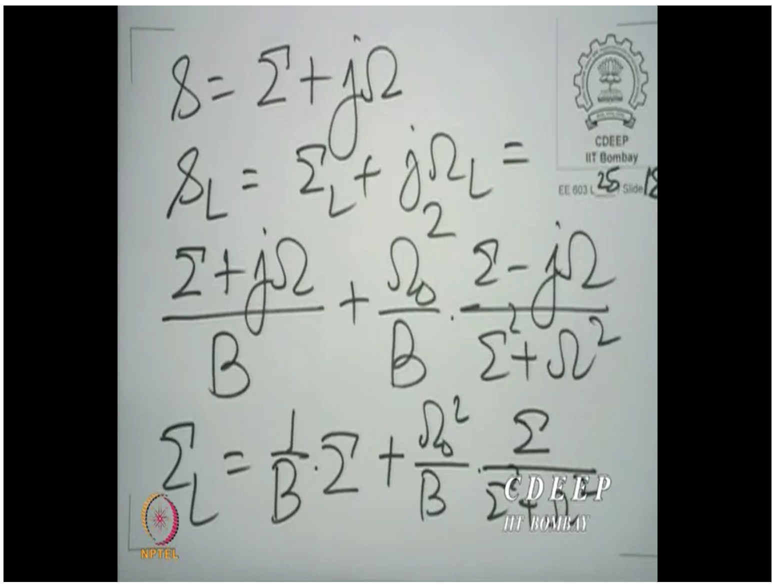

And then we have s_L becomes $\frac{\Sigma + j\Omega}{B} + \frac{\Omega_0^2}{B}\frac{\Sigma - j\Omega}{\Sigma^2+\Omega^2}$, if you write s_L as $\Sigma_L + j\Omega_L$, then Σ_L is $\frac{1}{B}\Sigma + \frac{\Omega_0^2}{B}\frac{\Sigma}{\Sigma^2+\Omega^2}$.

The real part, we are taking the real part out. This is the real part of this quantity, and this is the real part of these quantities, we have extracted them. Now, you see it is obvious. If Σ is positive, then this has no choice but to be positive and neither does this because $\frac{1}{B}$ is positive, $\frac{\Omega_0^2}{B}$ is positive. If Σ is positive, $\Sigma^2 + \Omega^2$ cannot possibly be *0*. It has to be positive anyway.

And therefore, this is bound to be positive and so is this and therefore Σ_L is positive. On the other hand, if Σ is negative, then this is bound to be negative. And this is definitely bound to be non-zero because $\Sigma^2 + \Omega^2$ cannot possibly be *0* if Σ is negative, it is bound to be positive.

So, this is positive, this is positive, so this whole thing would become negative. So, negative quantity plus a negative quantity would give a negative quantity. And therefore, Σ and Σ_L necessarily have same sign, strictly. So, if Σ_L is positive, if Σ is positive, then Σ_L is positive. Σ is negative, Σ_L is negative.

And in fact, right from here we can also see that if Σ is *0*, which means if you are going to be, imaginary axis then both of these quantities are *0* and therefore Σ_L is *0* as well. And therefore, we also know what to expect on the imaginary axis. The imaginary axis goes to the imaginary axis, which is not a surprise because anyway, you are dealing with an *LC* impedance.

So, stability has been obeyed and therefore we are guaranteed that if you have a stable analog filter designed, that would continue to remain stable when you go to the band pass domain. Now, the question is, why should we call this band pass? And for that, we need to look at the frequency, sinusoidal frequency transformation.

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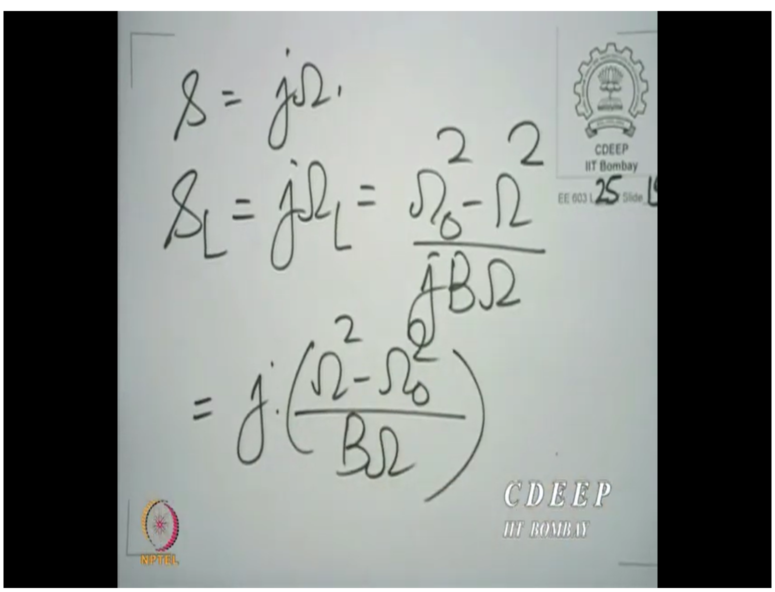

So, now let us look at what happens when $s = j\Omega$only, $s = j\Omega$ only. So, of course, that gives us $s_L$which is equal, it will give us $j\Omega_L$and $j\Omega_L$is going to be $\frac{\Omega_0^{\ 2}-\Omega^2}{jB\Omega}$. That is $j\left(\frac{\Omega^2-\Omega_0^{\ 2}}{B\Omega}\right)$, and this is very interesting.

Let us see how this behaves. Now, let us consider some critical points on the frequencies. Which critical points we need to look at? Well, let us take a band pass.

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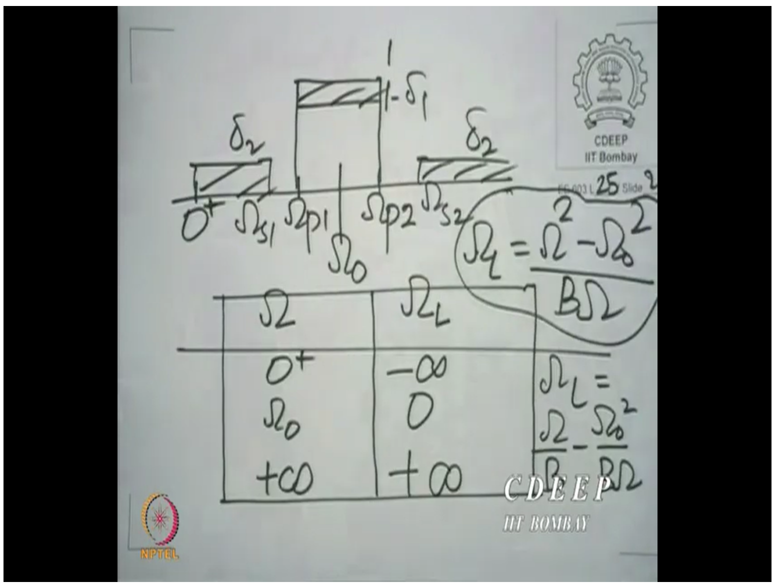

So, a typical band filter would have a characteristic like this, it would have two edges of the pass band. Let us call them $\Omega_{p1}$and Ω_{p2}. Yes please, there is a question. So, the question is, what are we practically doing when we make this transformation? Now, the answer is, all this is still in the phase of design, so we are not realizing it as yet.

So, all this is still design and we are computing the filter points, we have not yet reached a point where we have obtained the filter coefficients. We are yet designing on paper. After we have completed the design on paper, then we would translate it into realization. So, as yet, all this design is in calculation, right? So, we have specifications for the discrete time filter. We have converted them into specifications for the desired kind of analog filter.

Now, we are going into the specifications of the desired kind of low pass filter, we will see in a minute. We will design that low pass filter, we will then convert *s* by using *s* equal to, by using the bilinear transformation and there we get a discrete time filter that we would try to realize. So, realization is after completing the design.

So, there we have a band pass filter. And we assume as usual that we have tolerances. Now, we are quite satisfied with letting the past band be between 1and $1 - \delta_1$and the stop band being not more than δ_2 in magnitude. And of course, you know, the nature can be specified. Now, you see, you will have to decide on the nature of the stop bands here.

There are two stop bands, you cannot have different natures for the two stop bands; either both of them must be monotonic, or both of them must be equiripple. We do not at the moment, have a facility to allow different kinds of natures for the two stop bands. Anyway, we do have a facility to allow different tolerances in the two stop bands. All that we need to do is if the tolerances are different, consider the stronger of the tolerances.

So, for example, if one of the stop bands is a tolerance, which forces it to be less than *0.1*, and the other one must be less than *0.09*, then it is *0.09* that you must choose in your design, choose the stronger or the more stringent one. And then you can put that one, if you are satisfying *.09*, you are of course, satisfying *0.1*. So, you will have to do that. Take the most stringent one.

Having done that, we can agree on a set of specifications like this. Let us put $\Omega_0$somewhere in between here. We expect $\Omega_0$to be somewhere in between here. And then let us see where this maps, you see, let us make a mapping of what are called some critical frequencies. So, we have Ω mapping to Ω_L.

The critical frequencies are actually *0*, $\Omega_0$and $+\infty$. Or rather 0^+, 0^+means a very small, positive frequency which goes almost to *0*. Now, it is very clear, now the mapping of course you know Ω_L is $\frac{\Omega^2 - \Omega_0^{\ 2}}{B\Omega}$, this is the mapping. Now it is very easy to see that Ω equal to 0^+, you see, this is a little tricky. Now, if it is 0^+, then this of course *0*, this is $\frac{-\Omega_0^{\ 2}}{B\times 0^+}$. So it is $-\infty$ because you have a quantity in the denominator, which is very small and positive.

Here, you have a fixed negative quantity, or it is a fixed negative quantity divided by a very small positive quantity, gives you an infinite negative quantity. What about Ω_0? That is very easy to see, Ω_0 maps to *0*, that is very easy to see. And as you go towards $+\ \infty$, now, let us see what happens. You see, Ω_L can also be written as $\frac{\Omega}{B} - \frac{\Omega_0^2}{B\Omega}$.

So, as Ω goes to $+\ \infty$, this quantity goes to *0*. And this quantity goes to $+\ \infty$. So, therefore $\Omega =\ +\infty$ goes to $+\ \infty$. Now, what happens in between? We have only taken the critical points, what happens in between, we need to see that too. So, for that, let us write down this expression here. Ω_L is $\frac{\Omega}{B} - \frac{\Omega_0^2}{B\Omega}$.

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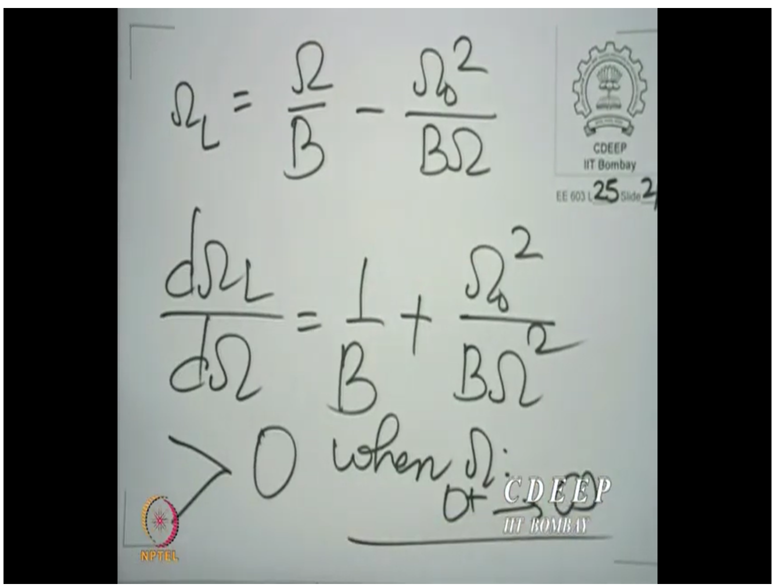

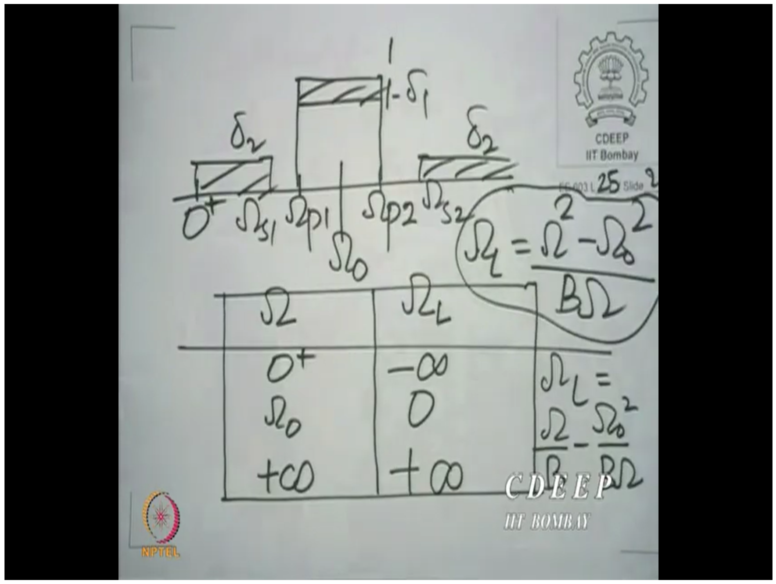

So, Ω_L is $\frac{\Omega}{B} - \frac{\Omega_0^2}{B\Omega}$. So, $\partial\Omega_L$, see this $\partial\Omega$, is $\frac{1}{B} + \frac{\Omega_0^2}{B\Omega^2}$. So, it is strictly greater than *0* when Ω runs from 0^+ to ∞. That is obvious because when Ω is positive, then this is definitely positive.

What it means is that as you increase Ω from 0^+ towards ∞, you expect Ω also to monotonically strictly increase. And therefore, going back to the previous table here, as you go from 0^+ towards $+\infty$ here, $0^+ - \infty$ on Ω_L, Ω_0 is at *0*. And $+\infty$ goes to $+\infty$.

So, as you move Ω from 0^+ towards $+\infty$, Ω_L would move monotonically from $-\infty$ to $+\infty$. And Ω_0 would come to *0*. Now, you need to, you see, the situation is like this. 0^+ would go to $-\infty$. Somewhere in between, you will have a mapping of this Ω_{s1}. Then you would have a mapping of Ω_{p1}.

Then you would have *0* here. Then you would have a mapping of Ω_{p2} and then you would have a mapping of Ω_{s2} and finally $+\infty$. It is going to follow the same sequence. Now, you see, we have full control on choosing Ω_0^2 and B to meet the specifications that we want. What are the specifications?

The first thing is that you want a symmetric magnitude response. So, this is going to map some point on the negative axis. And this is going to map to some point on the positive axis. You want them to be equal and opposite.

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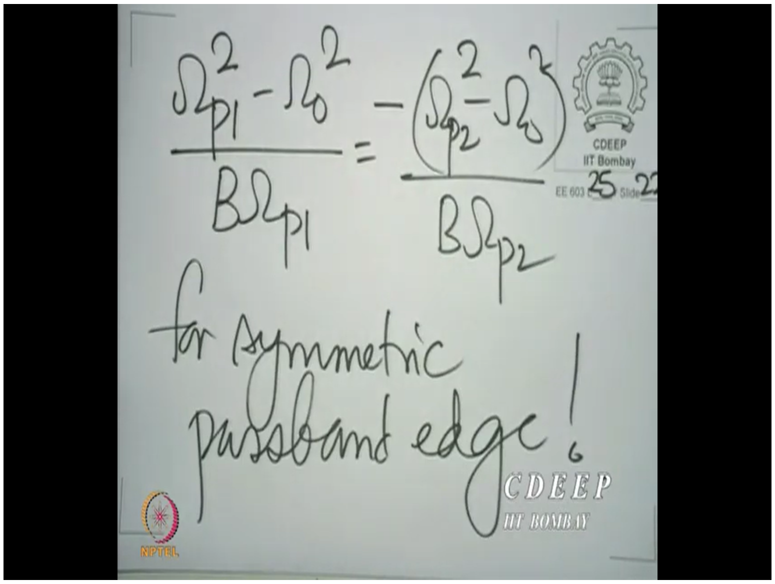

So, what you want first, is that $\frac{\Omega_{p1}^{2}-\Omega_{0}^{2}}{B\Omega_{p1}}=\frac{-(\Omega_{p2}^{2}-\Omega_{0}^{2})}{B\Omega_{p2}}$ for magnitude symmetry, for symmetric pass band edge. Otherwise, the low pass filter would have one negative pass band edge and a different positives pass band edge, we can’t allow that. Now, we can solve this, it is very easy to solve this.

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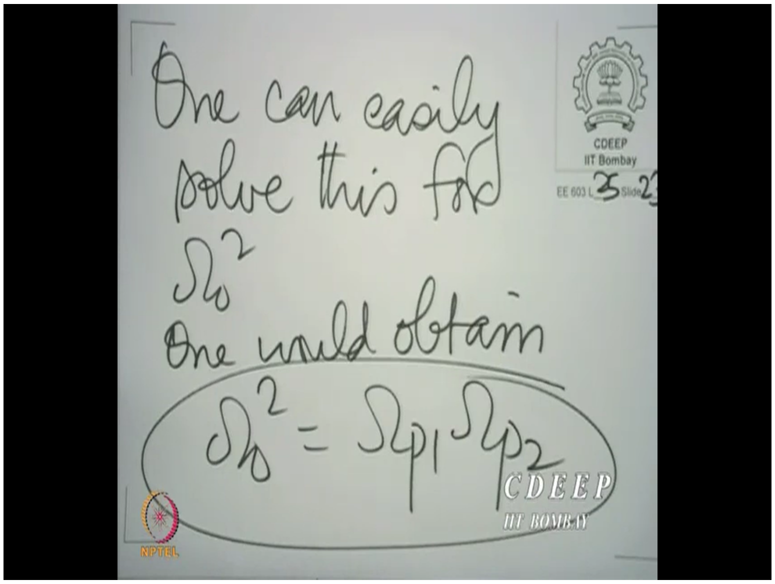

One can easily solve. And in fact, one would obtain, $\Omega_0^2 = \Omega_{p1}\Omega_{p2}$, which is very interesting. Now, this is not at all a difficult equation to solve, it is a simple linear equation in Ω_0^2, so you can easily solve it and you would obtain Ω_0^2 is Ω_{p1} times Ω_{p2}, this is very interesting.

What it says is that the so called center frequency or the resonant frequency, as we chose it at the beginning is the geometric mean of the pass band edges. And this is not unfamiliar. In fact, if we take, you know, a typically *LC* network with a resistance added, by the way, then, it is indeed true that the edges of the pass by the points of $\frac{1}{2}$ power, as known, do turn out to have a geometric mean equal to the center frequency or the resonant frequency, this is not surprising.

This is indeed a property of a band pass filter as known with *LCR* networks. So, this is surprising, but true in this case as well. So, Ω_0^2 is determined. Now, what is it that would determine *B*? Actually nothing at all, but we can put something down to make our whole life easier.

You remember, in the high pass filter, we had put down the low pass, pass band edge as *1*. Now, let us standardize that to make life easy. So, we could put down the pass band edge as *1*, not just, you see what we have said so far is that the pass band edge on the positive side and on the negative side must be the negative of one another, that is all right. But let us make that 1 and -1, that makes life easier.

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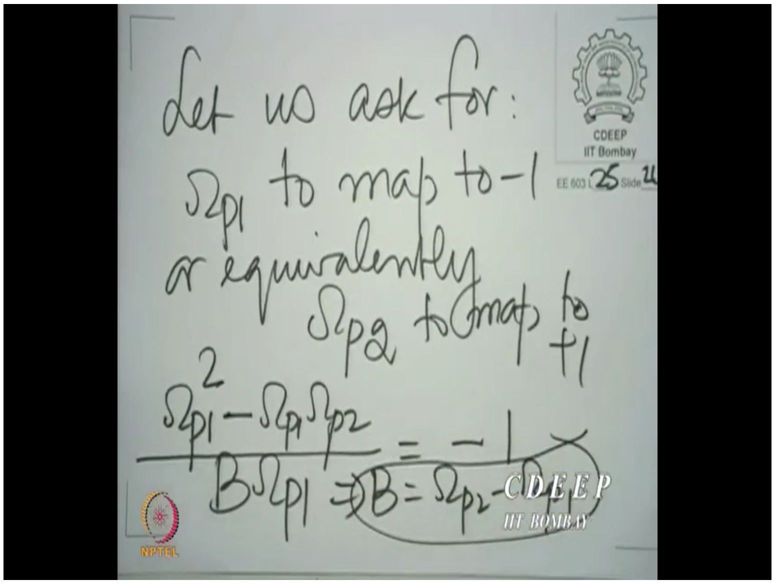

So, let us put down, let us ask for, this is our choice, let us ask for Ω_{p1} to map to $-\ 1$, or equivalently, $\Omega_{p1}$to map to $+\ 1$. That means, $\Omega_{p1}^{\ 2} - \Omega_{p1}\Omega_{p2}$, because now we have agreed that $\Omega_0^{\ 2}$ must be $\Omega_{p1}\Omega_{p2}$, divided by $B\Omega_{p1}$, should be equal to $-\ 1$.

That would, of course, very clearly give us B is $\Omega_{p2} - \Omega_{p1}$. And, of course the same thing would follow if you put the conditional Ω_{p2}. So, now we have an explicit value for B and for Ω_0. So, in fact, our band pass transform is complete and we are now in a position to design band pass filters too. What do we need to do? Let us just put down the steps.

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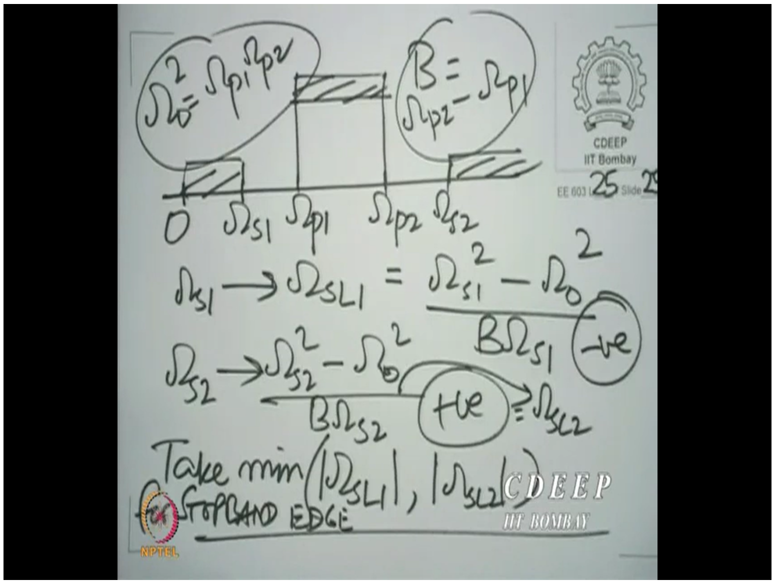

We have this band pass filter. First step, take $\Omega_0^2 = \Omega_{p1}\Omega_{p2}$, $B = \Omega_{p2} - \Omega_{p1}$. And now we know why we call this *B*; *B* stands for bandwidth. So, in a way, *B* is the length of the pass band. Having taken that, the next step is to design a low pass filter with the following specs. You see, Ω_{s1} would transform to Ω_{sL1}, Ω_{sL1} is $\frac{\Omega_{s1}^2 - \Omega_0^2}{B\Omega_{s1}}$.

And similarly, Ω_{s2} would translate to $\frac{\Omega_{s2}^2 - \Omega_0^2}{B\Omega_{s2}}$. Now, Ω_{s1} would definitely be negative and this would be positive. Take the smaller of $|\Omega_{sL1}|$ and $|\Omega_{sL2}|$. This is $\Omega_{sL2}$for stop band edge. You see, what we are saying in effect is that the pass band edge is of course *1*, but this would give us one stop band edge here and this would give us another stop band edge.

Which of them should be taken? They may not be equal. So, naturally you must take the stringent, the more, the tougher condition. Which is a tougher condition? The one which is closer, that is how you have taken a minimum here. So, take the minimum of the $|\Omega_{sL1}|$ and the $|\Omega_{sL2}|$, whichever is less, that means it is close, of course, both of them will be more than *1* in magnitude, that is for sure.

But the one which has a smaller magnitude should be chosen as the stop band edge. So, of course now, you know, the pass band edge of the low pass filter, you know the nature of the pass band, you know the stop band edge of the low pass filter, the more stringent of the two, you know the nature of the stop band, you know the tolerance in the pass band, you know the tolerance in the stop band.

So, you can design the low pass filter using either the Butterworth or the Chebyshev or inverse Chebyshev or elliptical approximation. And then transform the low pass variable s_L using $\frac{s^2+\Omega_0^2}{Bs}$, where you know what Ω_0^2 and B are. With that then you would, yes, please. There is a question.

Student: ...

Professor: So, the question is what happens to the $-\infty$ to *0* interval, where the $-\infty$ to *0* interval, oh, you see, it is very clear that $-\infty$. Yes, that is a good question. So let us just look at, let us answer that.

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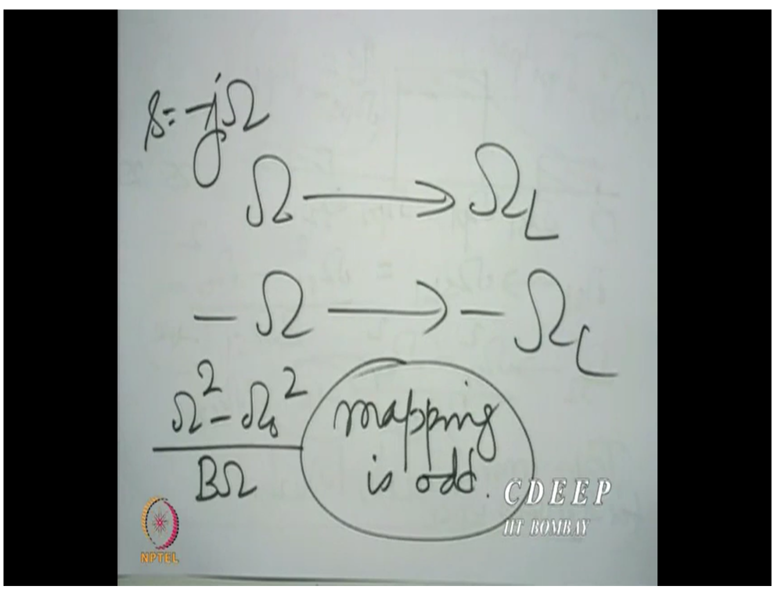

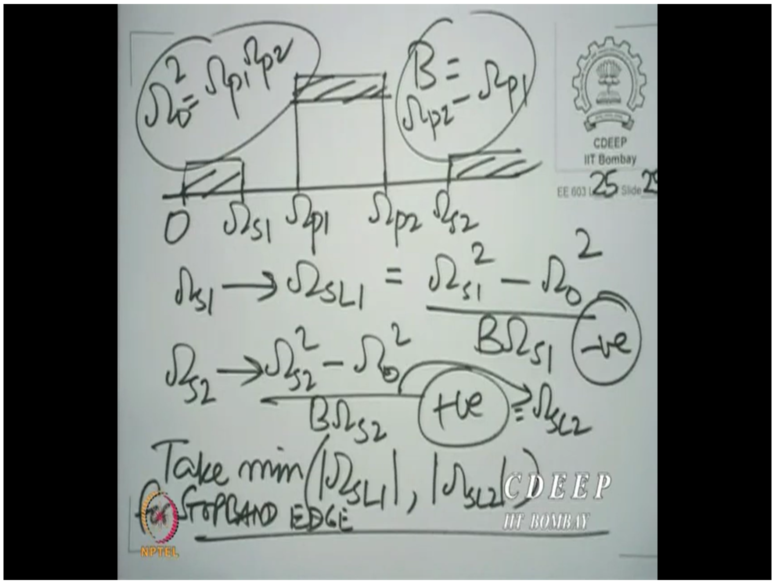

You see, one thing that you need to check is what happens when you put $s = \; - j\Omega$? Here, you see. So, we will see very quickly that this mapping must be odd. You see? The mapping, the frequency transformation is odd. So, s equal to, so, if Ω goes to Ω_L, then $-\;\Omega$ would go to $-\;\Omega_L$.

That is very easy to see, the mapping must be odd. In fact, you can see it here. $\frac{\Omega^2 - \Omega_0^2}{B\Omega}$, when you replace Ω by $-\;\Omega$, this whole thing is negated. The mapping is odd. Is it not? So, therefore, whatever you are doing on the positive side of the frequency axis is mirrored on the negative side. And that anyway is required because you want magnitude symmetry. Is that clear?

So, with that, then we have completed our design of the band pass filter as well. And now all of us are in a position to, of course, once you have designed the band pass analog filter, you can transform it using $s = \frac{1+z^{-1}}{1-z^{-1}}$. Using the bilinear transform, you have the discrete time filter with you, and then you can realize the discrete time filter as you desire.

So, now we have completed the design of band pass filters and now all of us are well equipped to complete our assignments on the design of the band pass filter as well. And then we shall, in the next lecture, look at how we might design band stop filters and with that, we shall be equipped to complete the assignment given to us on filter design.