Digital Signal Processing and its Applications
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Lecture 28 B
Convolution of the Window with the Ideal Filter and its Impact

So, we saw so far, so we understand, we can multiply the original ideal impulse response by a window function V[n]. That window function could be rectangular, the window function could be triangular, the window function could be cosinusoidal and could have several other shapes. Now, the question is, why should we have other shapes? We were beginning to answer the question in the previous lecture but now we will complete the answer.

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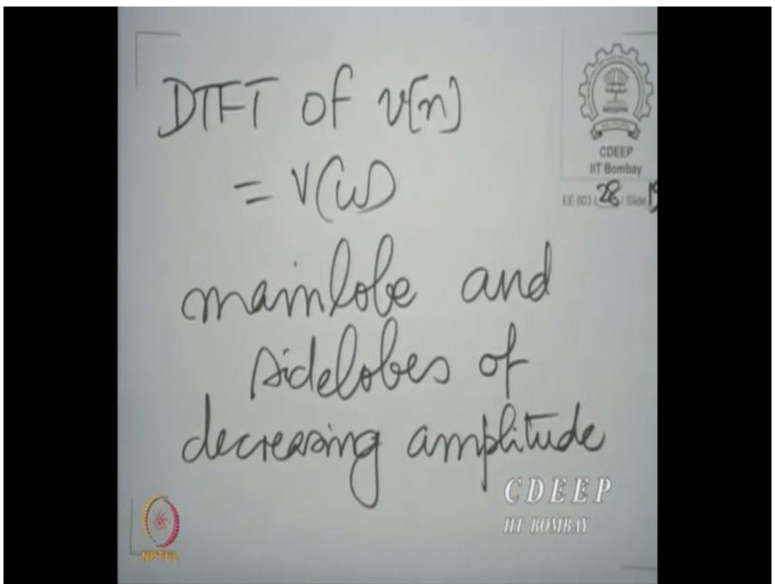

Now, we saw, for the case of the rectangular window and we now, also make the statement and this is true for most windows, that there is, in the DTFT of the window, DTFFT of V[n], which we will call V(w), typically, a main lobe and several side lobes. This is typical, side lobes of decreasing amplitude. In fact, just to convince ourselves, let us see how you could obtain the discreet time Fourier transform of that triangular window.

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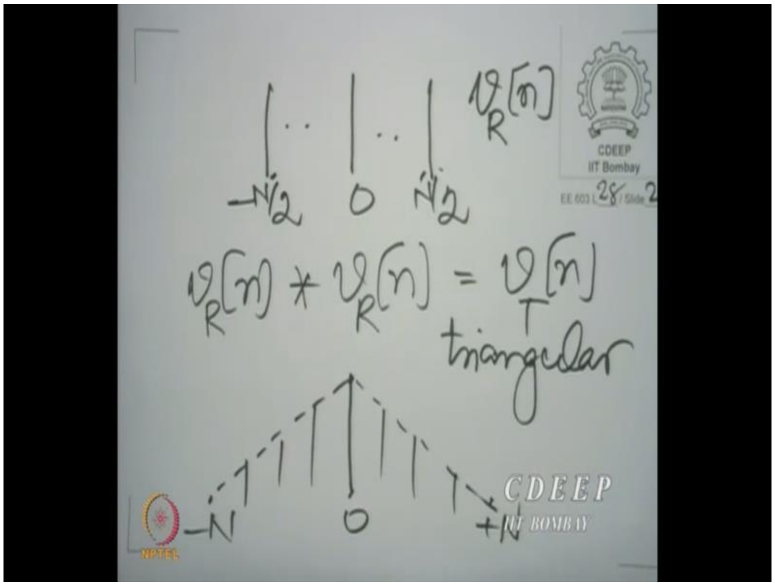

Now, the triangular window can be obtained by convolving the rectangular window with itself. So, if you have a rectangular window between minus N/2 and +N/2. Let us call this $V_R[n]$, R for rectangular. And if you convolve $V_R[n]$ with itself, we get $V_T[n]$ or the triangular window which looks like this,-N to +N. It goes maybe to the highest value in a triangular fashion at 0 and then drops on either side as you go to N.

Needless to say here we assume that N is even to make matters simple. Though that is not Necessary , you could always conceive of a triangular window even if N is odd but here we will make matters simple to understand how the discreet time Fourier transform looks. Now, when you convolve 2 sequences, their discreet time Fourier transforms are multiplied.

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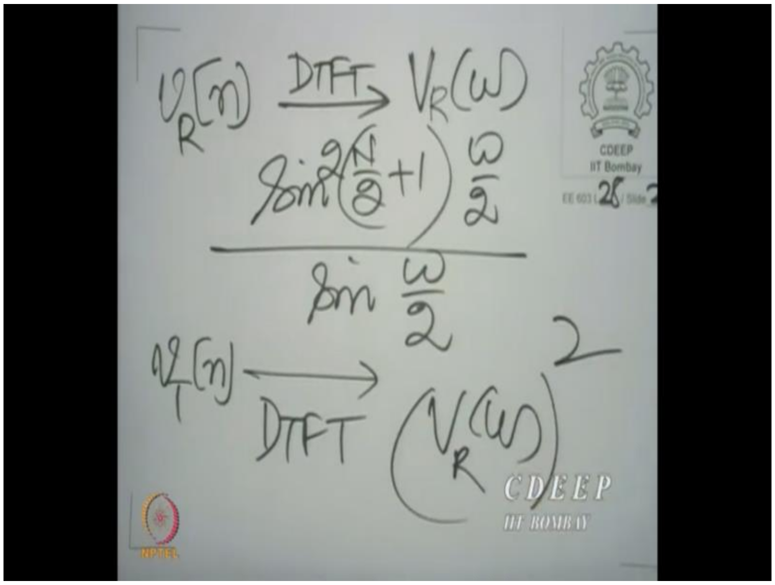

So, we know what the discret time Fourier transform of $V_R[n]$ is, we have calculated it the last time. Its DTFT is sin (N/2)+1, w /2. In fact, [sin(2(N/2)+1(w)/2] / sin(w)/2, we have seen that the last time. All that we need to do is last time, we had calculated for a length of N and now N has been replaced by N/2. So, you see the triangular window is therefore going to have the DTFT squared.

Let us call this capital $V_R(w)$. So, obviously $V_T[n]$ is going to have the DTFT V_R (w) squared and we can sketch that.

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So, V_R[n] omega had an appearance like this, we saw it the last time. Whereupon, V_T (w) is going to have an appearance something like this. Now, the V_T(w) is going to be always non-negative. What is more, if you treat the amplitude of this as 1, if you treat the, I mean if you, of course, you remember that this was 2N+1 where N is the length, or here it would be 2*N/2 + 1, but let us call this height H, whatever it would be.

And let us call this h or you know, to distinguish, let us call it H_1 and h_1. So, you see, this is going to be $(H_1)^2$ here and this is going to be small $(h_1)^2$. And therefore, one thing that you see is that that drop of height from the main lobe to the first or the principle side lobe is going to get squared when you go from the rectangular to the triangular window.

So, for example, just to take an example, suppose you happen to find that the height of the side lobe is some percentage of the main lobe height, then treating that percentage or fraction, if that percentage is 20%, I mean that is too small, but anyway suppose that percentage is let us say 30%, then you treat it as 0.3.

So, now you are going to have 0.3 the whole square, that is going to be lower, that means the side lobes are going to be of squared lower height. That means the side lobes in some sense have got suppressed but again, not without a cost. You see, we also see where the main lobe ends. This main lobe ends at the 0.2 π/N, is it not, 2π by this width is inversely proportional to the length of the window.

So, you know, when, now what is going to happen, is that when the rectangular window having length of N and when it has a length of N/2, now this width is going to be more. And of course, this width and this width are equal because it is just the square. So, therefore now the width has got doubled. Although the side lobes have been suppressed, the width has got doubled for the same length.

So, when you go from the rectangular triangular window of the same length, the side lobe, the first, the principle side lobe and therefore, all the other side lobes are suppressed to the square of the original rectangular window. But the width has got doubled. So, there is a compromise as you see, between the width of the main lobe and the strength of the side lobes. So, we note this.

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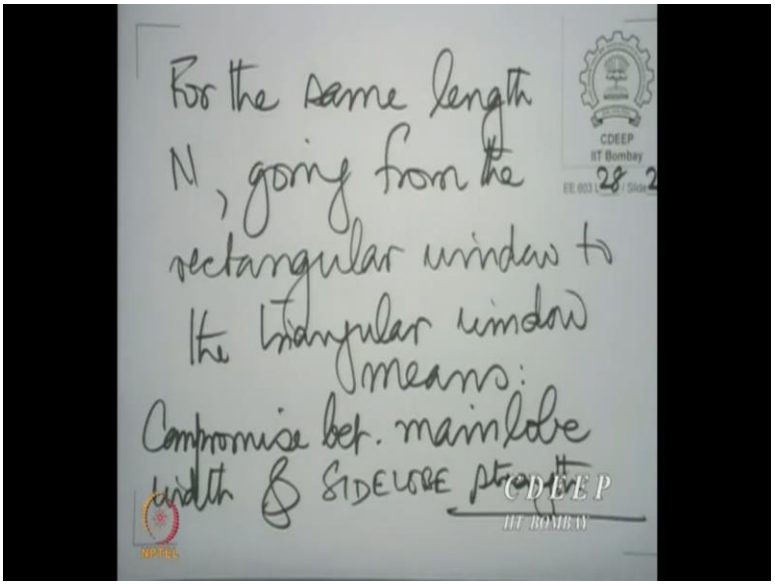

For the same length, n going from the rectangular window to the triangular window, means compromise between main lobe width and side lobe strength. You can understand strength in different ways. We will see which specific way we should use later, but one way to understand strength is the relative height of the side lobe as compared to the main lobe.

Now, you see, why are we interested in this compromise? We now need to go back to what we were doing the last time. We need to see what exactly these main lobes and side lobes do in degrading the frequency response. And we will now go back to the drawing that we had created the last time, we had tried to analyze the effects, so we said...

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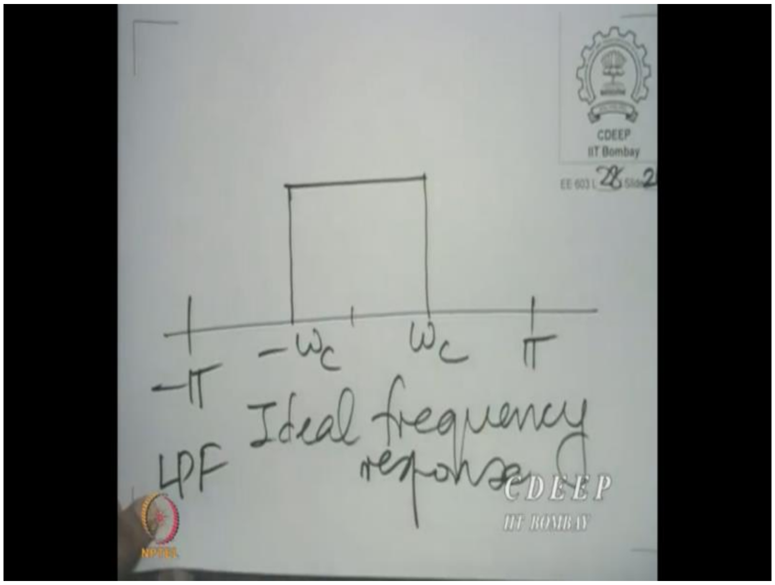

Now, by the way, I would like to mention a very interesting innovation which one of the students in the class suggested, Vivek Kumar. You see, last time we had written, we had of course observed that we need to convolve; we need to make a periodic convolution. So, we had this ideal impulse response here, w_c and $-w_c$.

And we need to convolve with, so this is the ideal frequency response of the lowpass filter, we were trying to observe what happens in the case of the lowpass filter.

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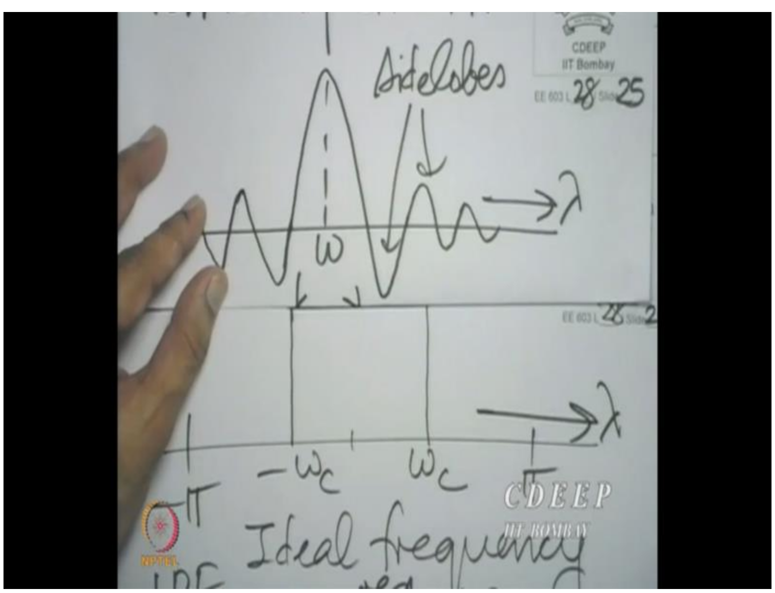

And we also had, I need to move it so I am going to draw this on a separate sheet of paper, we also had this window spectrum, V(w). And now we have agreed, we have seen two examples and for the moment, we will say this is always true, that you have main lobe and side lobes. So, the window spectrum.

And we also had, I need to move it so I am going to draw this on a separate sheet of paper, we also had this window spectrum, V(w). And now we have agreed, we have seen two examples and for the moment, we will say this is always true, that you have main lobe and side lobes. So, the window spectrum.

Now, that amounts to truncating one of them to one period. So, we could either choose to truncate the ideal filter response to one period and retain the periodicity of the window spectrum or we could choose to truncate the window spectrum to one period and retain the periodicity of the ideal filter. So, essentially a periodic convolution means to retain the periodicity of one of those periodic functions and to restrict the other only to one period and then convolve.

Now, that was an interesting observation by one of the students and we will do that now. So, last time, we had of course started justifying that we do not, but here all that we need to do is to... let us essentially restrict the window spectrum to one period and we will keep the periodicity of the ideal filter. You could do it the other way also.

So, now let us see again what happens when you convolve. So, I will start moving this as we did the last time. So, there we are. We saw that there are three regions that we need to deal with. One region is when the main, now here we are moving the window spectrum and we have agreed that variable should be called lambda here and this variable is also lambda. And this is at the point lambda equal to omega here.

So, when omega is far enough, so that it is essentially some of the weak side lobes that fall into the pass band, then what do we observe? As omega goes from that point and allows these side lobes to move into the pass band, then the resultant frequency response is the integral of the part of this window response that falls within the pass band as your very omega.

So, each omega you need to calculate an integral, an integral of that part of the side lobe which falls into the passband. And as you can see, this area is going to oscillate weakly because there is going to be sometimes a negative contribution and then sometimes a positive contribution and sometimes a negative contribution, so there is going to be a smooth movement from negative to positive to negative.

And what is more, is that as you come, as omega comes closer to the passband edge here, as omega comes closer to the passband edge, the stronger side lobes come into the passband and therefore, that oscillation grows. And it grows to a point where the principle sidelobe has entered the passband as is here. Afterwards, there is only going to be a growth upwards because it is the main lobe which is going to enter the passband.

The main lobe contributes a huge area in comparison with all the other side lobes. So, once the main lobe begins to enter, there is a steady upward growth of the area, of the frequency response. And this growth continues all the way up to where the main lobe is entering the passband. So, right from here, where you know, main lobe has just begun to enter, up to the point where the main lobe has completely entered, you have a steady growth of area, so that the frequency response rises at that point.

After it has thus risen to a sufficiently high level, then we have again, just the side lobes playing their game. So, the principle side lobe, of course, first plays its game and then the weaker side lobes start entering and while the principle side lobe enters, some of the weaker side lobes are leaving from the passband and so on so forth. So, what do we see as an overall consequence of this movement of the window function over the ideal passband?

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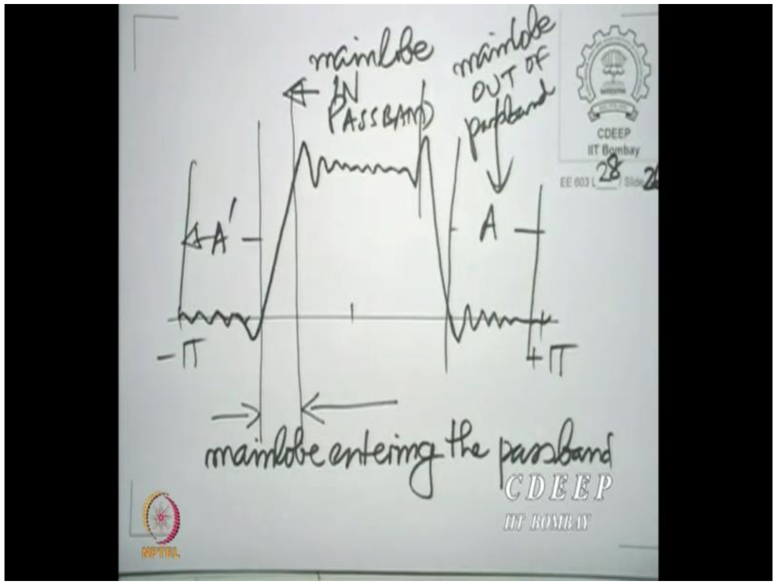

We see the following nature of the degraded frequency response. Far away, around pi, you would have some weak oscillations. Then you would have a strong movement upwards and then again, a strong oscillation and weaker oscillations as you go to the center and then a strong oscillation and then a downward movement and then weaker oscillations again.

So, this is the part where the main lobe is entering the passband. This is the part where the main lobe is in the passband and these are the parts, part A and A dash, where the main lobe is out of the passband. Is that clear to everybody? What is more, is that these oscillations are going to be stronger just as the main lobe has entered and just as the main lobe is leaving, because the strongest side lobes are either entering or leaving there.

And they become weaker towards the center. The strongest oscillations are just around this point of entry and the weaker oscillations are away, is that clear to everybody? Yes, everybody understands this? And now, we can also see what results in each of these quantities. So, how long would this region last? How long would this region last? This region would last as long as the main lobe needs to enter. And that means the width of the main lobe plays a role in this region. How would these oscillations be?

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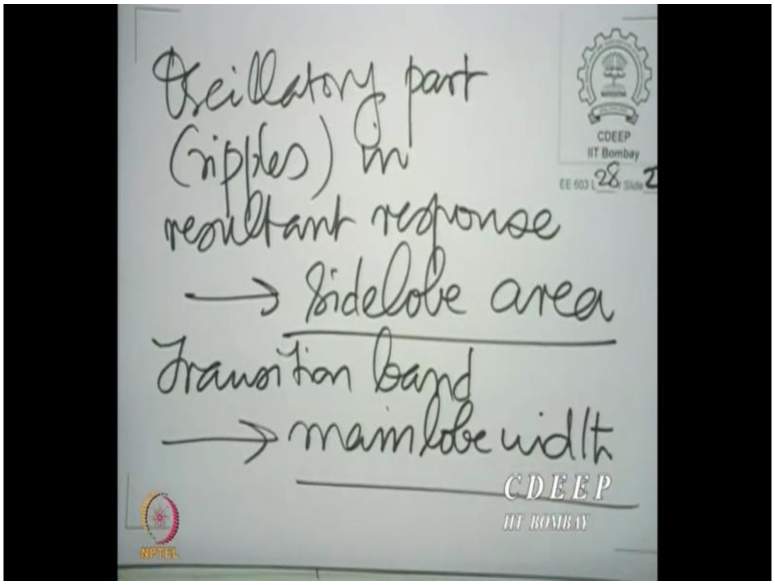

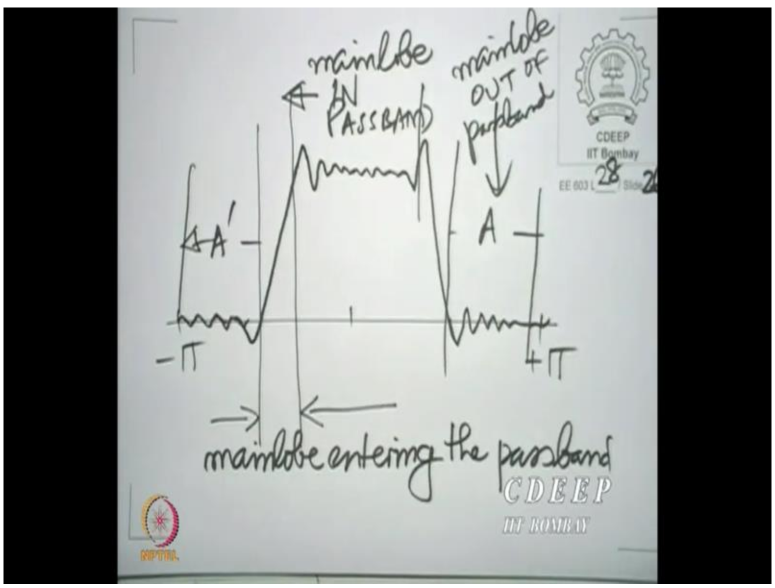

These oscillations would be as high as the area under the principle side lobes. So, the oscillatory part, what we can see very clearly, is the oscillatory part or the ripples in the resultant frequency response are governed by the side lobe area. And the transition band, now we will give it that name, the transition band is governed by the main lobe width.

Please note, the transition band is the part where you move from, what is effectively the stopband here to the passband here, the effective passband. And the movement from stopband to passband is governed by the main lobe width. The oscillations in the passband or in the stopband are governed by the side lobe area. The main lobe width and side lobe area is what plays a role in the quality of the response.

Now, we know why we have to choose between windows, we have the rectangular window, we could choose the triangular window, we could choose a cosine window, we could choose several other shapes and the whole game is a compromise between main lobe width and side lobe area. In fact, we have seen that right in the case of the rectangular and triangular window.

We can at least see the compromise of main lobe width, to see the change of side lobe area requires a little more calculation but I, in fact, put it as an exercise for you. Approximate these side lobes, the principle side lobe area for the triangular window in comparison with the rectangular window for the same length n and show that when you move from a rectangular window to a triangular window of the same length n, you are actually making a compromise between main lobe width and side lobe area.

The triangular window is going to have a longer width, a larger width, main lobe width, but the side lobe area would come down. And in fact, different shapes would have a compromise. Now, as I said, it is not always a compromise of inconvenience, people have designed windows strategically and window design is as much of an art as a science, because what you would like to do ultimately is to gain both in terms of main lobe width and side lobe area.

Can you have less main lobe width and less side lobe area? Well, you cannot do too much in that direction but you can do a little bit. And we shall see in the next lecture one systematic approach to designing windows which, in some sense, offer a good compromise between main lobe width and side lobe area. Thank you.