Digital Signal Processing and its Applications

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Lecture 29B

Types of Window Functions

Anyway with that background we must now see some examples of window functions, we saw 2 in the previous lecture, but we must see more in this one. So, we now look at a very common window called the Hann window.

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We shall now take one by one expressions for different windows. So, in general, we shall specify a window as follows; we will specify a window to lie between k= -N to +N and we specify the window by the expression W_k. So, for example, the rectangular window is specified by W_k equal to 1 and the Hann window is specified by W_k equal to 1/2(1 + (cos 2πk)/2N) and we can easily see that at k =-N, we have cos π, so this is 0.

And so also +N, + or - N. This becomes 0 and of course, what we are really doing is to place one whole cycle, you see, if you look at it, when k goes over an entire interval from - N to + N, you are running over a whole cycle of the cosine, taking you from an argument of - π to an argument of +π and you are putting that whole cycle of a cosine on a constant and therefore, it will always be non-negative, but it will follow a course in a faulty pattern.

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So we can sketch the Hann window for our convenience. As, before the best way to sketch is to draw the corresponding continuous function and then discretize it. So you see we have this cosinusoidal function here, bell shaped, if you want to call it that, the Hann envelope. So for each window, we have an envelope and the window is obtained by sampling the envelope.

Now, this idea of the envelope is important, you see how you get the envelope is very easy, you get the envelope by replacing k by t, t becomes continuous. So in this expression, you can think of replacing k by t, it gives you a (cos 2πt)/2N. For t = -N, you can say that anyway, of course, it takes the value 0 as before. At +N also it takes the value of 0.

And then of course at k = zero, it takes the value 1/2. So it rises and then so, I am sorry, at k = 0, it takes the value 1. I mean 1/ 2 *1+1. Now this envelope of the window is important for all the windows because when we are looking for an even length FIR filter, we need to do a bit of work on this,you see. Instead of for an odd link window, of course, you would include t = zero among the samples.

But when the window is of even length, you must include +1/ 2 and -1/ 2 among the samples and then displace the other samples by 1. So you will get an equal number of samples on the positive side of t and on the negative side of t. So you know, when you have an even length window, you need to draw the envelope and then sample it starting from +1/ 2 and -1/ 2 and then displace by the integers.

And the same is true for the ideal impulse response. We need to take the ideal impulse response, draw the envelope of the ideal impulse response, and then resample it, starting from plus half and minus half and then displace by one. For even length FIR filter design we need to take this care. That's why the envelope is important. Now let us look at a few more envelopes.

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The Hamming window, by the way, Hann and Hamming are names of researchers in this field, who suggested this window. The Hamming window is described by W_k = 0.54 + 0.46 (cos 2πk)/2N and now of course you will wonder what sanctity these numbers have 0.54 and 0.46, except for the fact that they add to 1 they do not seem to have any other sanctity, they really do not.

You see, in general, one can talk about a generalized Hamming window where you give this value alpha and therefore this becomes 1 minus alpha. That is the generalized Hamming window and you see the whole game in this Hamming window is to try Hann window, the hamming window is to try and optimize between main lobe width and side lobe area. So at alpha is like a tuning parameter, it allows you to tune main lobe width and side lobe area.

And of course, you cannot take it arbitrarily, you cannot use any value of alpha between 0 and 1. No, you would want in fact, you would normally want it to remain at least reasonably positive all over the interval. But whatever it be, I mean, it is really a game of compromise between the main width and the relative side lobe area.

And again, the game is that when you go to a smoother function, now in general, the Hann or the Hamming windows do better than the triangular window in general, in terms of their main lobe width cum side lobe relative side lobe area performance, and the secret of this better performance is the fact that they have essentially a sinusoid or you could say 0 frequency plus a first fundamental term. Now, you can introduce one more term and perhaps introduce some more compromise that is what leads you to what is called a Blackman window

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Again Blackman is the name of another researcher. In the Blackman window W_k is described by 0.42 + 0.5(cos 2πk)/2N + 0.08 (cos 4πk)/2N. So, essentially one has introduced a second harmonic term and just as a good dish requires just the right proportions of different ingredients, a good window requires just the right proportion of different harmonics to achieve a good main lobe area and main lobe width and side lobe area compromise.

And of course, you could then conceive of introducing a little bit of a third harmonic and tuning it and so on. So, window design is as much of an art as a science. See, wherever there is no one unique answer and different answers involve different kinds of compromises, the subject also acquires the form of an art because there are beauties of various kinds. There is some kind of beauty in some window and some in some other.

But a masterpiece among all these windows, there are several other windows; a masterpiece among these windows is what is called the Kaiser window.

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The Kaiser window is based on what are called the modified Bessel functions of the first kind, and order 0. So let us denote it by I_{0x}. An I_{0x} is described, essentially by its Taylor series, it is difficult to describe it in closed form. It looks temptingly like an expansion of e ^ x, but it is not. This is a Taylor series expansion of the modified Bessel function of first kind in order 0.

Now, just a little bit of background, you will recall that Bessel functions arise as a solution to what is called the Bessel differential equation. It is a very important equation in the whole field of differential equations. In fact, the Bessel functions occur in many different contexts, seemingly unconnected. There are different kinds of Bessel functions again. They occur in the context of communication in the description of frequency modulation.

And they also occur in other disciplines of engineering. Here, of course, you see the current window design; we have some very interesting properties. Now, Kaiser studied this modified Bessel function, the Bessel function for its spectral properties and came out with the observation that it seemed to offer an excellent compromise between main lobe width and side lobe area.

What I mean by that is, of course, there is a fundamental limit, you cannot do arbitrarily well on both, but you can probably do better on both fronts and that also can be taken to a certain degree of achievement and the Kaiser window is probably known to give some of the best results in terms of window design and the beauty about the Kaiser window is that it has two tuning parameters. Of course, all windows have one tuning parameter. That is the length. So let us write that down.

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All windows have the tuning parameter given by the window length. In fact, the main width is inversely proportional to N. So the more the length the less the main lobe width; we saw that in the case of the rectangular and the triangular windows, but that is true of all windows. So you see, it is not correct to say that you cannot get a small transition band as you want you can get a smaller transition band as you desire. The only problem is what is not affected by the window length is the relative side lobe area and that is the tragedy of series approximation, sinusoidal series approximations.