Digital Signal Processing and its Applications
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Lecture 29C
Gibb's Phenomenon

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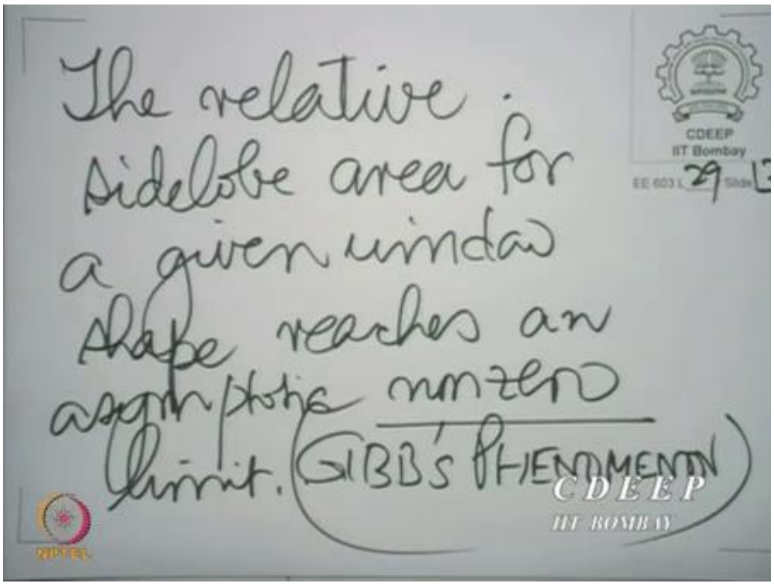

The relative side lobe area for a given window shape reaches an asymptotic nonzero limits and this is the origin of what is called the Gibb's phenomenon. Many of us have been exposed to the Gibb's phenomenon in sinusoidal series approximations. When we decompose a periodic function into its various expressions, what happens is the more you have the number of Fourier series terms that you retain in the expansion, the better and better are the discontinuities approximated.

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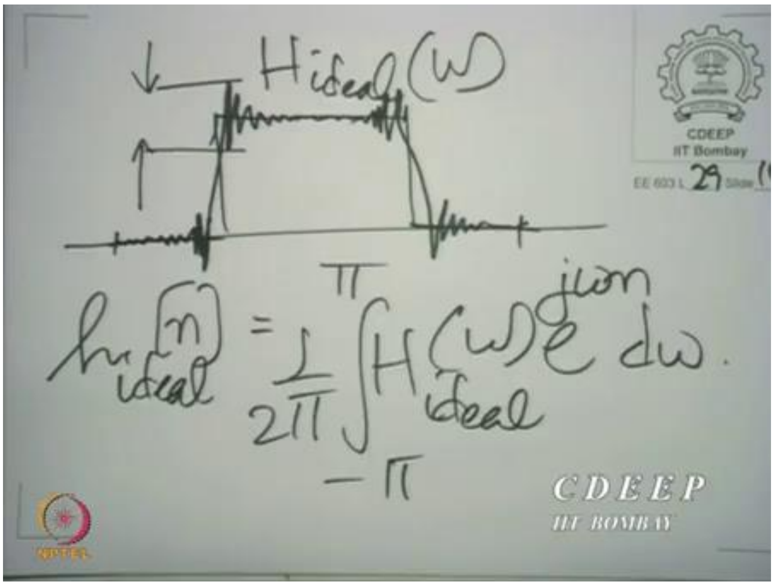

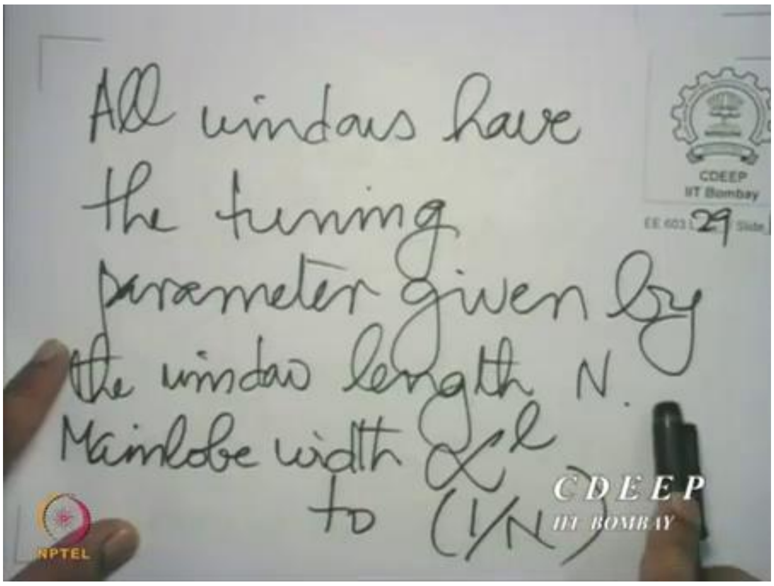

So, you see, if you recall, if you have a square wave and if you take one cycle of a square wave and after all what is an example of one cycle of a square wave? One period of the DTFT, they should take from $-\pi$ to π, the ideal DTFT and then you look at the impulse response essentially as the Fourier series coefficients of this, but that perspective is not difficult to see at all.

Think for a minute, after all, if this H ideally omega, what is the impulse response h ideal of n? h_{ideal} [n] is $1/2\pi$ integral from $-\pi$ to π H_{ideal} $[W].e^{jWm}$ dw . Effectively these are like the Fourier series coefficients of this periodic DTFT on the variable ω. That is another perspective on the impulse response, the impulse response is like a Fourier series expansion of the periodic DTFT.

Now, we are looking at it the other way. All this while, we thought of the DTFT as a consequence of the impulse response. Now, we are thinking of the impulse responses arising from the DTFT and that is not unreasonable because it is the ideal that we know in the frequency domain and we want to find out the impulse response coefficients that should give us that ideal.

And the process of finding the ideal impulse response is akin or in fact, is identical to a Fourier series expansion and when we complete this impulse response, what we are effectively doing is to truncate the Fourier series of that periodic function and that is what leads us to what is traditionally understood as a Gibb's phenomenon.

So, what happens is on the scene, you see, you have what happens is the more you know of Fourier series coefficients you retain, you have this oscillation here, you get more oscillations as you approach the discontinuity, then you get a rise, then you get the largest oscillations here and then you get smaller ones in between, you get larger and larger oscillations and then again a drop and then the largest oscillations and finally, it reaches a steady situation and Gibb's phenomenon says that this oscillation does not reduce no matter how large the length.

All that happens is that, these oscillations get confined to the boundary, you have a discontinuity, but the magnitude, the height of those oscillations reaches an asymptotic limit, you cannot go below a certain value for the rectangular window. But of course, you could go below that value if you took a different window, but of course at the cost of the transition bandwidth.

So, what I am trying to emphasize is that once again, let me say it very clearly. You see all windows have a tuning parameter given by the window length n. So all windows can give you a smaller transition band as you desire but what they cannot, once you have chosen a window, you have built the maximum deviation in the past balance top band. So the tolerance of the passband and the stop band cannot be influenced beyond the point.

You can influence the transition band but you cannot influence the tolerance and unfortunately the pass band stop band tolerances are equal or fortunately if you want to call it whatever it is. Now the Keiser window allows you one more theory parallel and that is like a shape parameter. So, you see, as you see, it is the shape which gives you the maximum deviation, relative side lobe area. So, it helps you control the shape and the shape parameter allows you a tolerance between the main lobe width and relative side lobe area.

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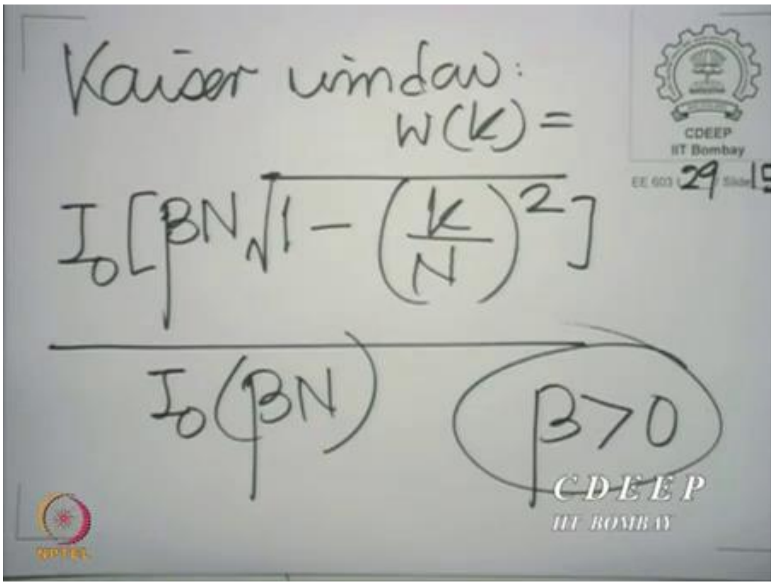

So, the Kaiser window is described by W_k is equal to I_0 beta N square root now, the complicated expression but let us not bother too much about it, square root $(1 - (k/N)^2)/ I_0\beta N$ with $\beta > 0$. So, β goes from 0 to infinity and of course, we have written the expression for I_0 before, it is the modified Bessel function of the first kind and order zero index.

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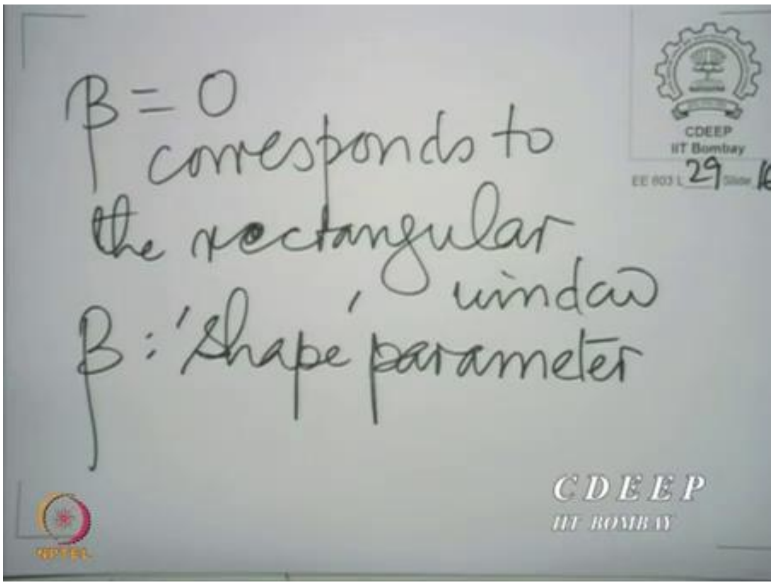

Now, of course, it is very easy to see that β = 0 corresponds to the rectangular window. β is what is called the shape parameter. The windows take different shapes of course, you must not forget that this window is described only between –N to +N. The modified Bessel functions lasts for all values of the variable but this window is defined only between –N and +N that is to be understood.

And of course, you can then draw the envelope of the window and then you could resample it to get even length width. So the Kaiser window is in some sense an optimal, optimal in this game of compromise between the main lobe width and side lobe area. It was studied in depth by the mathematician Kaiser. Typically, it is the Kaiser window which is used when one uses FIR filter design with windows.

In fact, in the assignment that all of you would do to design the FIR filter using the window approach you would use the Kaiser window. Now, how one chooses the shape parameter and the length is only by empirical experiments that have been done by Kaiser and other researchers, there is no closed form for choice and what I shall do is to put down certain guidelines, so, to speak for the choice of length.

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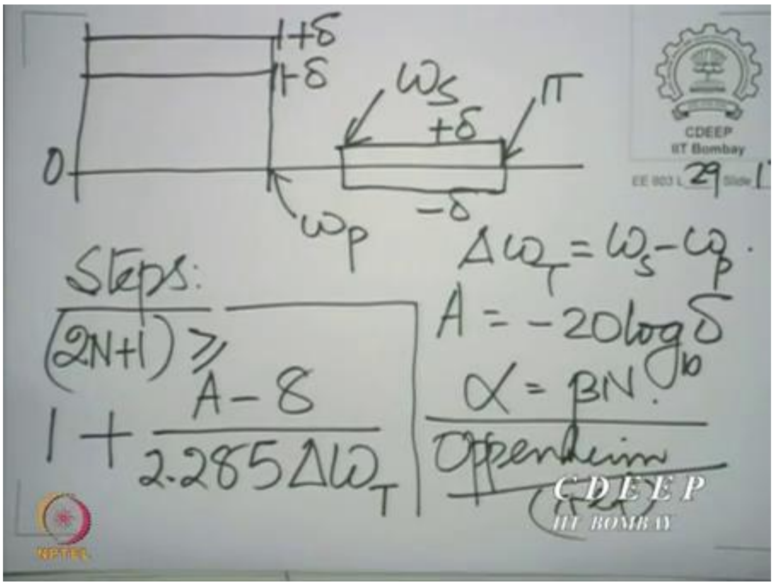

The empirical design equations for the Kaiser window: you see, what we are assuming here is that, we will take the case of the low pass filter, but a very similar kind of argument can be used for other kinds of filters. Of course, you have π here, you have ω_s there, you have ω_p. This is $1+\delta$ there and $1-\delta$ down here and δ down here $+\delta$ and if you like we will call it minus so the responses between $+\delta$ and $-\delta$ and $1-\delta$ and $1+\delta$.

And notice that the passband and the stop band tolerances are the same. We have no choice on that front. The empirical design equations are as follows. The steps are: define $\delta\ \omega_t$ or the transition bandwidth to be $\omega_s - \omega_p$. Define a to be essentially the logarithmic value of the tolerance or the DB decibel value of the tolerance. Define α to be β times N.

And therefore, the first step becomes a choice of N. So N is chosen according to 2 N+1 that is essentially the window and I am talking about an odd length window over here. So, if you want to design even length, then you can just choose one more or one less. After all these are all empirical expressions.

So, $N+1 \geq 1$ plus. Now, these numbers have been arrived at by empirical considerations $2.285\delta\omega_t$. This is the guideline for the choice of N. To see more about these steps, one could look up any standard text on DSP. For example, one could look at the text by Oppenheim.

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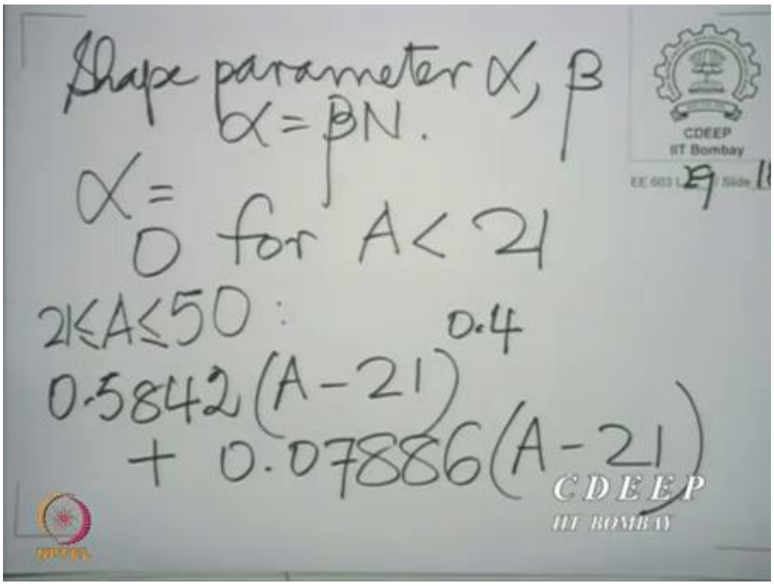

Now, of course, we also need to determine the shape parameters. For the shape parameter, which we either call alpha or beta. Alpha is of course, equal to beta times N. So, we will write an expression for alpha and then you could correspondingly write an expression for β. $\alpha= 0$ for $A<21$.

So, you notice that alpha, the choice of alpha is governed essentially only by the pass band tolerance, pass-stop band tolerance and that is to be expected because α which is β times N the shape parameter having chosen the length, then the choice of shape parameter has only to do with the tolerance.

So α can be called the shape parameter here, β times N and what this means is that as long as $A< 21$, you could as well do with a rectangular window. Because, you know $\beta= 0$ or $\alpha=0$ essentially implies rectangular window. Now if A is between 21 and 50, it does not matter, you can see less than or equal to, then we chose this according to 0.5842.

Again, all these are empirical, A- 21 to the point 0.4+ 0.07886(A- 21). So, it is the additional A above 21, which you are using in the expression. Yes, the question is when A< 21, we are effectively saying the shape parameter is 0 and we are essentially saying the rectangular window does very well. What it means is the rectangular window gives you a tolerance, which is less than 21 decibels.

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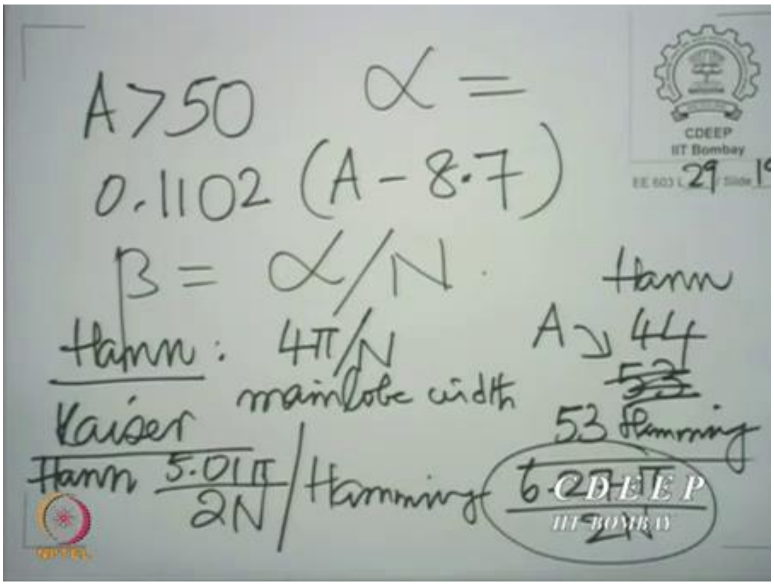

And finally, when A> 50, the expression to be used is 0.1102 (A- 8.7), α is equal to this and let us not forget the β=α/ N. Now, the beauty is that you see, for every kind of window, there is an equivalent α which gives you the same tolerance. What I mean by that is take any window shape, triangular for example. Now, the time window reaches an asymptotic limit in terms of its pass band and stop band tolerance.

What I mean by that is after a while, after you keep increasing the length of the triangular window, you cannot reduce the pass band, stop band tolerance beyond a certain point. That is called the asymptotic tolerance for that window, asymptotic δ for that window. The δ leads to an A and the A leads to an α in the Kaiser window. So for every window, there is an equivalent α in the Kaiser window which gives you a tolerance.

The beauty is for that α, you also have an N corresponding to, when you have a transition, the beauty is that, you know once you have chosen the, for the Kaiser, the beauty is the Kaiser window does better in terms of the transition bandwidth than most of the windows for that same alpha. From the, on a comparative platform, once you have chosen the α to match the tolerance in the pass band and stop band asymptotic tolerance in the pass band and stopband, for the window, the transition bandwidth is slightly reduced.

So one can construct a table for each window you can put down the way the transition band varies as a function of length and once I put down an asymptotic tolerance from the asymptotic boundaries, you can find out the equivalent alpha and then once you find the equivalent alpha, you can study how the transition bandwidth changes for the Kaiser window of that alpha.

And the transition bandwidth does better for the Kaiser window than the original window that is typically the case. That is interesting. That is why the Kaiser window is preferred. We take just one example to illustrate this. You see, for the Hann window, the peak relative side lobe amplitude or you know the effective so the peak error let us not worry about the side lobe amplitude.

The main lobe width is $4\pi/N$ and the peak error in approximation that is essentially the A that we have talked about is 53 I am sorry, this is 44 for the Hann window. I am essentially taking this data from a table. 53 actually happens to be for the Hamming window, the Hamming window and the Hann window have the same main lobe width. In fact, I can write it down here.

Hann 44, 53 for Hamming, for the Hann window, if you take the equivalent Kaiser window with the same alpha. For the Hann window, the corresponding main lobe width or the transition width becomes $5.015\pi/2N$ and for the (Hann) Hamming window it becomes $6.27\pi/2N$. Now the greater A is better, a greater A means a smaller delta, you must understand this, so greater A is better.

So in the sense of A the Hamming window is better than a Hann window. In fact that is why you see those funny numbers that were there .54, .46 instead of half and half. But of course better in the sense of tolerance and worse in the sense of the, you know well actually equal in the sense of main lobe width. But Hamming is better and can in terms of tolerance. That is why the Hamming window would be preferred over the Hann window in a sense.

The Hann Window is easy to write, but you know, it is not as good as Hamming window in terms of its tolerance. But of course, you see in the corresponding Kaiser windows, you see the compromise. The main lobe, the transition bandwidth increases little bit. So as you go from β=0 to β=infinity, it is a perpetual game. At the rectangular window, you have the very best transition width and the very worst tolerance.

And as you go towards larger and larger β, you get more and more transition bandwidth but you do better and better in terms of tolerance. This is how we design FIR filters using the Kaiser and now we are all equipped to complete your assignment, individual assignment on the design of filters with IIR and FIR with the specifications that you have been given. Yes there is a question?

Student: When we started the design, we moved from rectangular window to triangular and then window and so on. So we are trying to go more and more. In Kaiser window we are trying to approximate e^x. So why not use directly e^x?

Professor: So that is, so the question is, it seemed like when we went from the rectangular to rectangular and into the Hann, Hamming and then going further to the Keiser, you were trying to make the function smoother and smoother and he also remarked that when we looked at the Kaiser windows, it seemed to temptingly resemble e^x. So why not use an exponential itself?

The answer is that it is just a resemblance. The Taylor series is not really the expansion of an exponential, it looks like that, it looks tempting like that. It is much, it is different because you have a squared term, the x/2 the whole to the power l / l!, the whole squared. So it is not quite the exponential.

The exponential is not quite the optimal. So this modified Bessel function has been chosen by strategy. So then we conclude today's lecture about FIR filter design and some more things about FIR filters in the next lecture. Thank you.