Digital Signal Processing and its Applications
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Lecture 30 A
Digital Differentiator

A warm welcome to the thirtieth lecture, on the subject of digital signal processing and applications. We are now at a position where we have seen some approaches to discrete time filter design. Let us take a minute or two to recapitulate; we have seen the design of IIR filters where we have essentially taken advantage of the known methods in analog filter design.

So, what we did there was to use the known designs like the Butterworth design, the Chebyshev design and we did not talk about it, but one could also use the Jacobi or the Elliptic design and then a generic transformation that converts the S variable to the Z variable or the bilinear transformation can then be employed.

The, we have seen what properties that transformation needs to satisfy and therefore, we evolved the whole process of design by taking advantage of known analog designs. Now, what we really did was to design low pass filters, we designed analog low pass filters and then we evolved what is called a frequency transformation in the analog domain to design other kinds of analog filters.

We then saw that there is one important limitation of analog filters, namely that they can never give linear phase. To demand linear phase we then need to move to finite impulse response filter design and we did that. In the approaches, there are several different approaches that can be used to finite impulse response or FIR filter design. However, we have looked at one of the approaches, which is in some sense the simplest, namely the approach based on Windows.

What we do in a windowing approach is essentially, as the name suggests, by the way at this point, it is a good idea to take stock of the meaning of the word window in this context. Why the word window is used? A window as we understand it in common conversation, common parlance means an opening to the external world against the wall. So, if there were no, if there were no wall, then the entire scenery outside be visible.

But because we have walls that separate us from the outside world, in a residence, we also provide windows which give us a limited view of the scenery outside. Now window can also be shaped or it can be (taped) tinted glass on a window can be tinted and if we use such tinted or colour glasses on the window, then the scene outside is appropriately modified. Perhaps, to our liking, or may not be to our liking, whatever it be.

Now, the same principle have been applied in this FIR filter design context. What we have done here is to use the window, which shows us a part of the impulse response, not the entire impulse response and it also modifies the impulse response in a manner that we wish or in a manner that we find advantageous. So this is the explanation of the word window.

We have seen different windows for FIR filter design last time. In fact, we have come to the conclusion that there is a basic conflict that we can never quite resolve, but we can optimize a conflict between the main lobe width and the side lobe relative side lobe area and the main lobe which contributes to transition bandwidth, the relative side lobe area to the maximum deviation in the pass band and the stop band.

Now among the windows, the optimal window so to speak is what is called the Kaiser window. Named after the mathematician who proposed that window, the Kaiser window has the ability to change both shape and length. Unlike other windows where the shape is fixed, but the length can change. Changing the length would in general reduce the transition bandwidth.

According to a certain rate of fall off, different windows have a different rate of fall off the transition bandwidth as a function of length. But no matter how large you make the window, you can never do away with the pass band and stop band tolerance and for a given window shape, that tolerance reaches an asymptotic limit. An asymptotic limit can only be changed if we change the window shape.

And that is why the Kaiser window which is optimal in the sense of the kind of transition bandwidth that you get and the kind of tolerance that you get for a given shape. Now, we also saw finally that one could always associate a given shape with a tolerance.

So, once you have a tolerance, then you can find out the Kaiser window parameter corresponding the Kaiser window shape parameter corresponding to the tolerance and for that shaped parameter, you will have a certain way in which the transition bandwidth changes with length and the Kaiser window always does better than the corresponding window of known shape which gives the same pass band and stop band tolerance.

So, these are some of the things that we discussed in the previous lecture. Now, we are quite well equipped to complete a design on FIR filters using windows. Now, one important observation here is that all our discussion is relevant when the filters that we are trying to design a piecewise constant.

In fact, all the discussion about passband tolerance stop band tolerance and so on and transition band are meaningful when there are piecewise constant responses which we are trying to realize. All the standard responses that we try to realize are indeed piecewise constant. But let me now give you an example of a system that we sometimes want to realize but which is not piece wise constant in its magnitude response.

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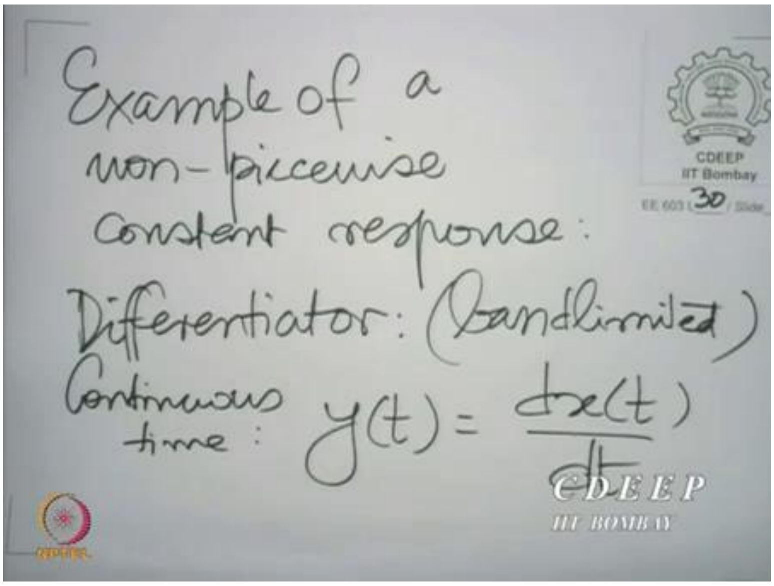

So, an example of a non-piecewise constant and the simplest example is what is called a differentiator or a band limited differentiator to be more precise. So, a differentiator is described in continuous time by y(t) = dx(t)/ dt, where x(t) is the input and y(t) the output. In the frequency domain, this translates, now incidentally, a differentiator is a linear shift invariant system.

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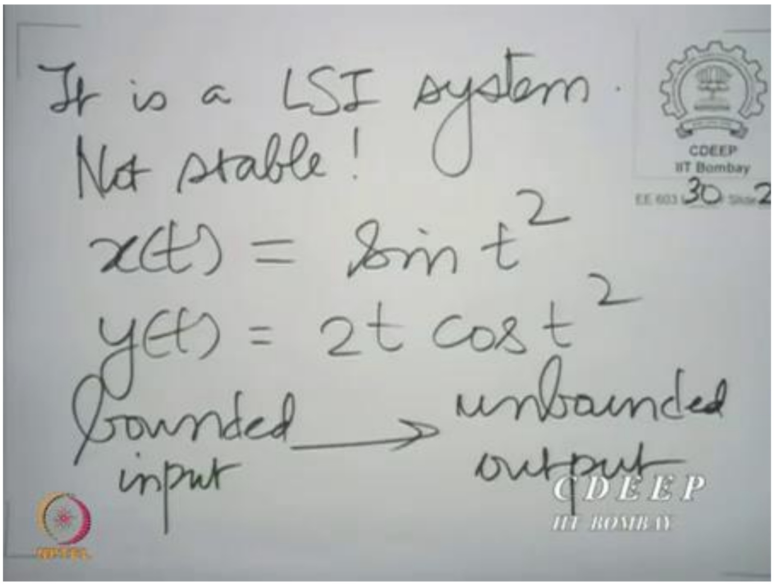

The only problem is it is not stable. That is very easy to see, all that we need to do is to give $x(t) = (\text{sine}(t))^2$ to the system. Now, obviously this input is bound and the output then turns out to be $2t(\cos t)^2$ which is unbounded, bounded input leading to unbounded out. Therefore, the system is not stable. However, we can associate with it a frequency response, meaning we can see what happens in general when we give a sinusoidal input.

So, beauty is not this is a beautiful illustration, where you can have a response a bounded response to a sinusoidal input. Therefore, you can talk about a frequency response but you cannot quite immediately conclude the system is stable. So, having a frequency response does not necessarily mean the system is stable. That is because if the output might be bounded for given sinusoidal inputs, but it may not be bounded for all bounded inputs.

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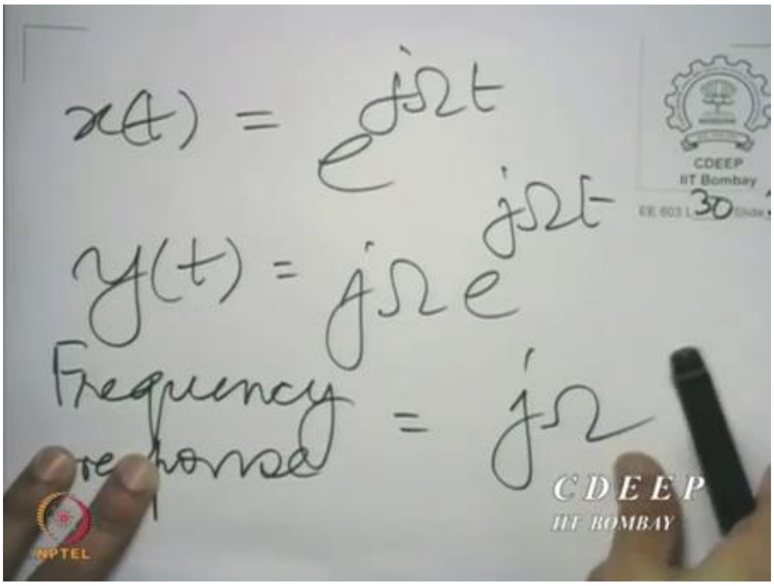

And indeed, if we do happen to give it the complex exponential, if x(t) happens to be the complex exponential $e^{j\Omega t}$ then y t clearly becomes $j\Omega e^{j\Omega t}$ and therefore, the frequency response of the differentiator is j omega. So, in spite of the system being unstable, it does have a frequency response and the physical meaning of this is that the magnitude varies linearly as a function of **Ω** and the phase is +pi/2, when **Ω** is positive and –pi/2 when **Ω** is negative. So, we can draw the magnitude and phase response.

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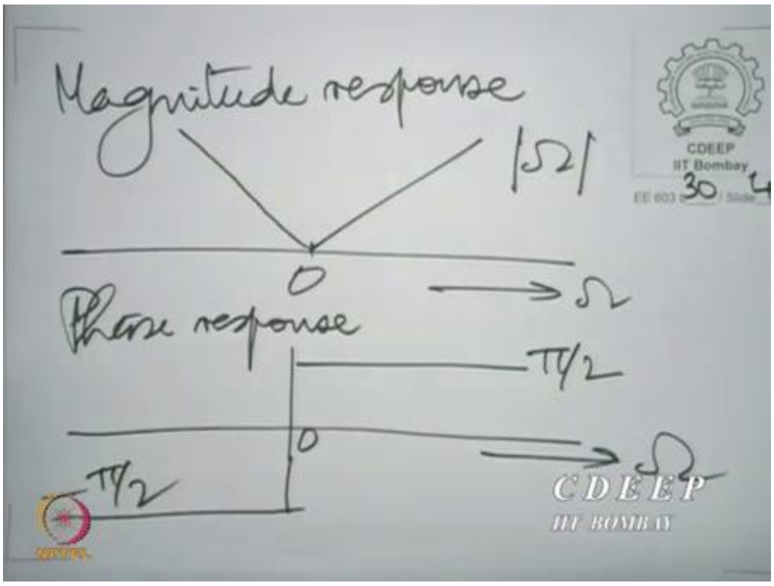

Magnitude is linear, |**Ω**| is the magnitude and the phase response is equally easy to draw. The phase response is +pi/2 for **Ω** greater than 0 and –pi/2 for **Ω** less than 0 that is easy to see. Incidentally if we dissociate this magnitude response, but retain this phase response that means, we keep this phase response, but make the magnitude response different, we make it 1 everywhere. A system which has a magnitude response of 1 everywhere, but this phase response is called the Hilbert transformer.

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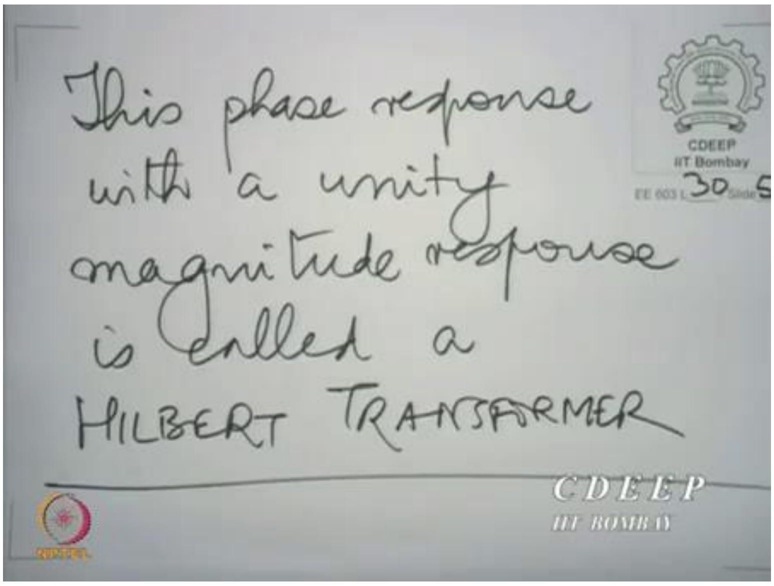

This phase response with a unit, unity magnitude response is called the Hilbert transformer. The physical significance of a Hilbert transformer is that, it adds a phase of pi / 2 or a phase of 90 degrees independent of frequency. So, in a way informal language you could say it converts cosine to sine or sine to cosine. I mentioned the Hilbert transformer because it is very useful in communication.

The idea of a Hilbert transformer has been employed in understanding, unlock communication particularly amplitude modulation and some people use it indirectly in phase modulation too. Anyway, that was a point besides, but you see the reason why I mentioned these is that one can find the ideal impulse response, of a band limited differentiate. So, now you could restrict this differentiator to operate only between 0 and half the sampling frequency.

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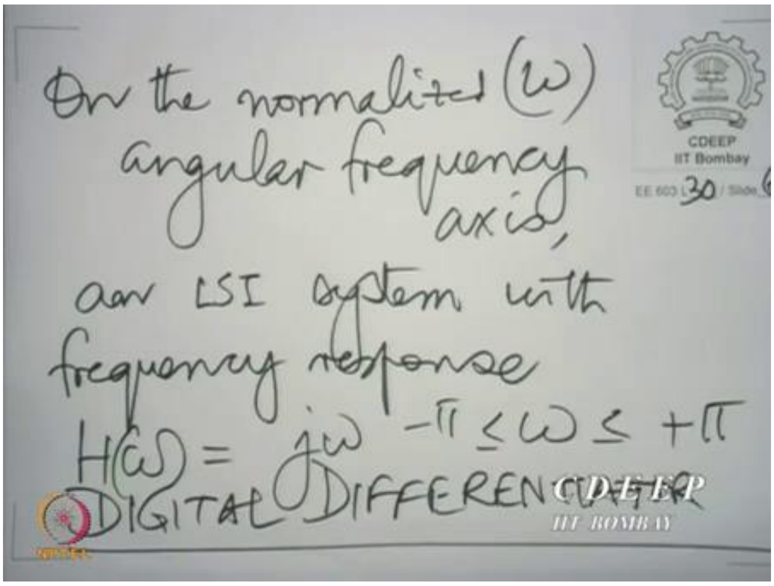

So, you could then describe on the normalized angular frequency axis, you could describe a system, an LSI system with frequency response, so we will use small omega to denote the normalized angular frequency axis. An LSI system with frequency response, H of Ω given by jΩ, omega between -pi and +pi is called a digitally differentiator or discrete time differentiate.

This is an example of a non-piecewise constant frequency response which is useful. What is the physical significance of a digital differentiator? It essentially differentiates the underlined continuous signal and then we assume that we have sampled that signal which has been differentiated. So, you could think of the discrete sequence which came from a band limited sequence, there is an underlying continuous signal there.

An underlying continuous signal has been differentiated with respect to time and it has then been resampled at the same points. That is the physical interpretation of the action of a differentiator, digital differentiator. Now, the digital differentiator, as I said is an example of a system which is not piecewise constant in its magnitude response and therefore, we could in principle, do the same thing that we did in windowing. Namely, we could find the ideal impulse response. In fact, let me give this to you as an exercise.

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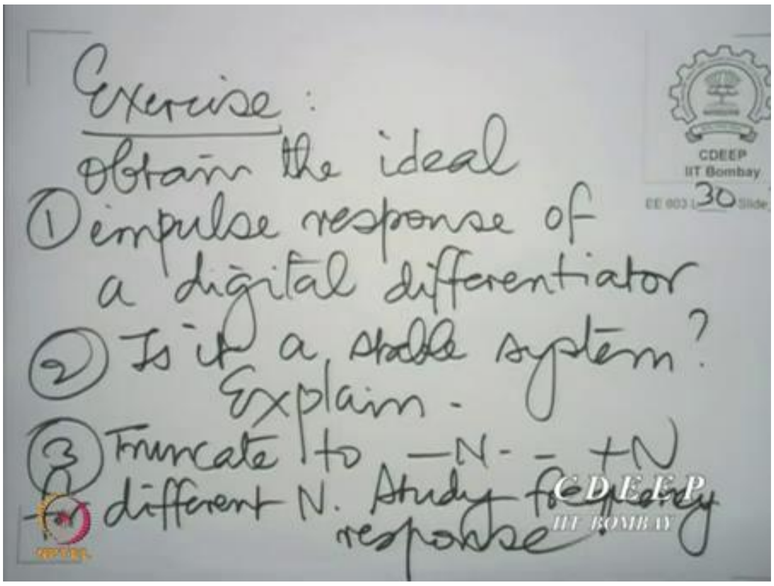

Exercise: obtain the ideal impulse response of a digitally differentiator. That is very easy to do. Find the inverse DTFT essential. Second, is it a stable system? Explain. If we take a cue from the analog domain, then we kind of expect the answer to be no, but we should find out independently in the discrete domain. It is not obvious that it should be unstable in the discrete domain, but perhaps we do expect that because it was unstable in the analog case, but one should you know investigate independently.

Third, truncate this to -N two +N for different N. Study the frequency response. This would have to be done with some software. So the question is, what do you expect? It has no easy answer, we do not know what will happen when you truncate this between -N and +N and find the frequency response. It is not going to be easy to explain. Of course, you can say how to find the, you can find it by convolving this ideal impulse, the ideal frequency response with the windows spectrum.

But what that convolution will yield is not easy to say because here, we cannot use the argument of main lobe coming in and going out and so on. There is no ideal pass band. But it should therefore be interesting to see what happens and this is left to you as an exercise to study with. Anyway, it turns out I mean, I might give you just a part of the answer. It turns out that build good windows, quote unquote, “good windows” do work reasonably well, even for such responses, though it is more difficult to explain why they do.

So that is one observation. The good thing about FIR filter design, the way we study the FIR filter design based on window functions is that you can also use it to design non-piecewise constant responses and you leave the actual degraded response to nature. You cannot say too much about how the response will get degraded but experience shows that it is acceptable and therefore, unlike the analog filter design approaches, which you cannot use for non-piecewise constant responses, the discrete the FIR filter design approach can be used for non-discrete non-piecewise constant responses. So, the bilinear transform for example, cannot be employed for non-piecewise constant responses.