Digital Signal Processing & Its Applications

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Lecture 33a

Recursion in system function and FIR system realisation

So, warm welcome to the 33rd lecture on the subject of Digital Signal Processing and its applications. We have begun a discussion on realization in the last few lectures and in the previous lecture, we had also begun the specific theme of design of lattice structures we are also motivated to the reason for the lattice structure design, the lattice structure is important because it is of course, a uniformly repeated structure.

But more importantly it gives some insights into the stability or otherwise of a system when the parameters are calculated and while the direct form or the cascade forms also have their own modularity, the lattice structure has some interesting properties that we shall now slowly develop.

The lattice structure takes a while to understand, it is a little difficult to understand for the beginner. So, it is worth reviewing a few ideas before we go further, even though that might be some amount of repetition. So, let us look at a few ideas that we have discussed in the previous lecture. Once again, briefly, that we put our discussion in perspective.

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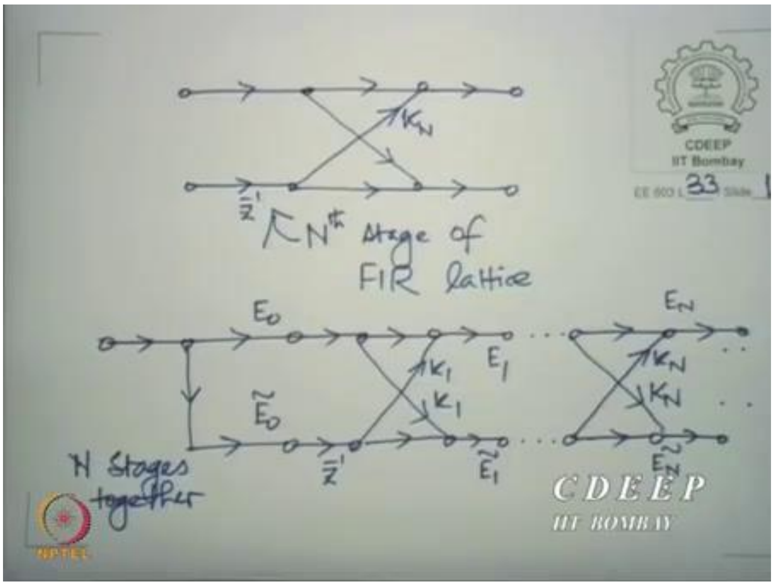

We have said that we would take a typical stage of the following form in the lattice structure. We said we take the N^{th} stage, what I am doing here the signal for graph just for the N^{th} stage and the overall FIR lattice appeared like this, there was essentially an input point we will call it E_0 and $\tilde{E}_0$ if you like.

You had the first stage and then we have the N^{th}, for the first stage output you will have $\tilde{E}_1$ and E_1. At the N^{th} stage output you would you have $\tilde{E}_N$ and E_N itself and this can of course continue, this is the N^{th} stage, this is how to N^{th} stage looks and this is all N^{th} stages together. Now, what we are trying to do in the previous lecture was to obtain a recursive relation between the system function from E_0 or $\tilde{E}_0$ equivalently to E_N and $\tilde{E}_N$ and the recursion was on the system function itself. That means we expressed the system function at the N+1 stage in terms of the system function at the N^{th} stage. That was what we did last time and the motivation for doing that is to see certain properties that evolve, evolve these system functions.

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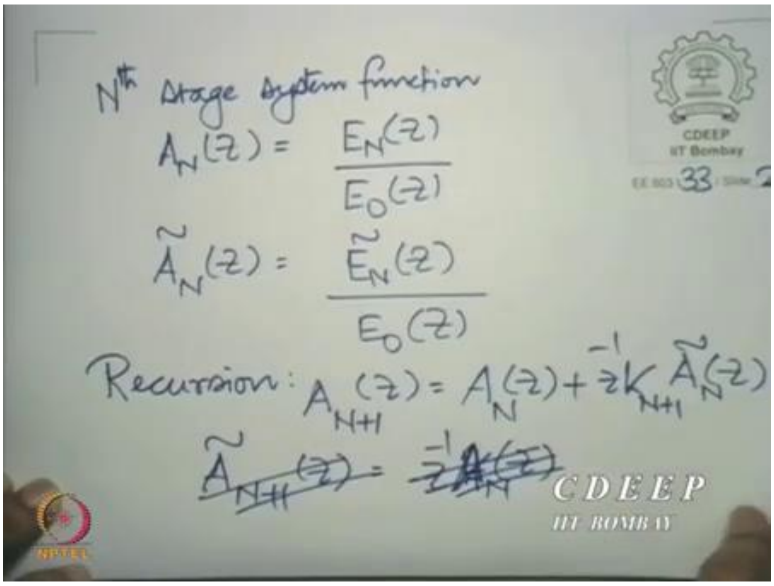

So, what we had seen in the previous lecture was that if we took the N^{th} stage system function, $A_N(Z)$ defined by $E_N(Z)$ divided by $E_0(Z)$ and $\tilde{A}_N(Z)$ defined by $\tilde{E}_N(Z)$ divided by $E_0(Z)$. In the recursion on the system function is as follows. $A_{N+1}(Z) = A_N(Z) + Z^{-1} K_{N+1} \tilde{A}_N(Z)$ and $\tilde{A}_{N+1}(Z)$ is $Z^{-1} K_N$ plus, I am sorry $Z^{-1} A_N(Z)$. Let us write that down. Let us rewrite this.

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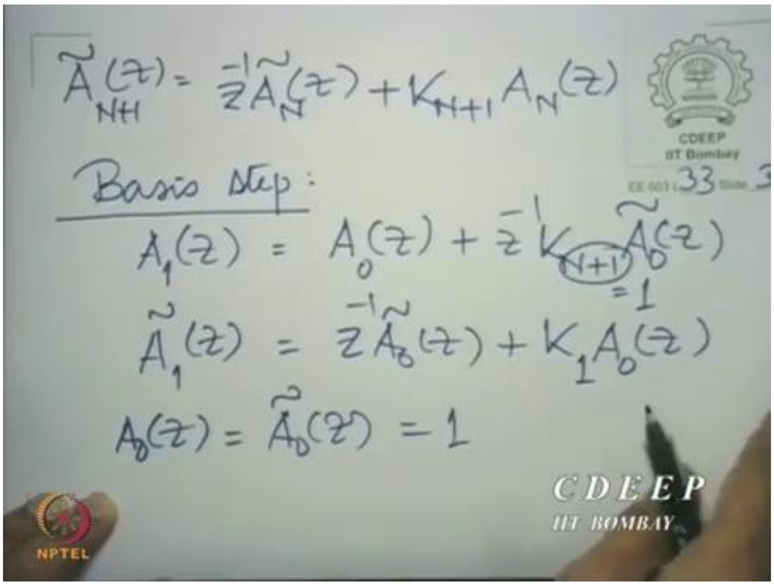

$\tilde{A}_{N+1}(Z) = Z^{-1}\,\tilde{A}_N(Z) + K_{N+1}\,A_N(Z)$. Now, what we saw is that if we take the basis step here, Then $A_1(Z)$ is $A_0(Z)$ which is 1 of course plus $Z^{-1}\,K_{N+1}\,\tilde{A}_0(Z)$ and $\tilde{A}_1(Z)$ is $Z^{-1}\,\tilde{A}_0(Z)$ plus Yes so of course, N+1 = 1 here. So, $K_1\,A_0(Z)$ and of course, $A_0(Z)$ is equal to $\tilde{A}_0(Z)$ obviously it is equal to 1 and therefore we have $A_1(Z)$ is simply 1 plus $K_1\,Z^{-1}$.

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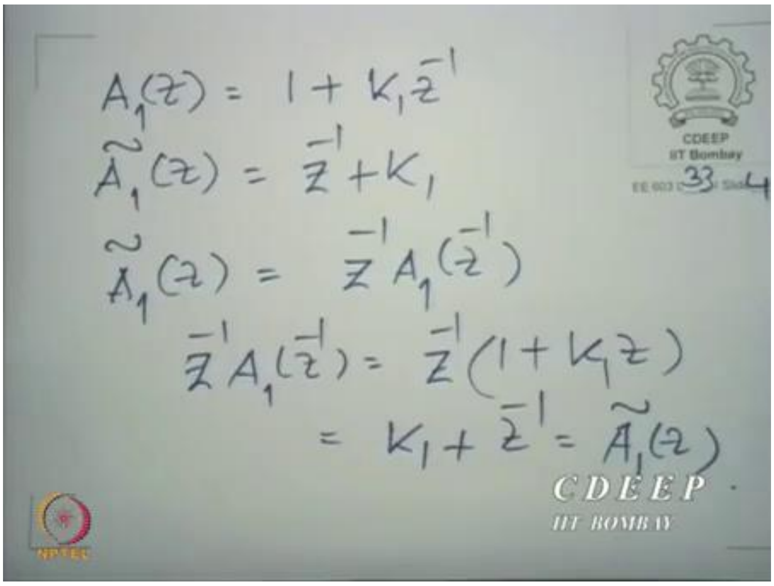

We had seen most of this the last time and $\tilde{A}_1(Z)$ is $Z^{-1} + K_1$ and therefore, we observed that $\tilde{A}_1(Z) = Z^{-1}\,A_1(Z^{-1})$ which is easily verified because $Z^{-1}\,A_1(Z^{-1})$ is essentially $Z^{-1}\,(1+ K_1 Z)$ which is indeed $K_1 + Z^{-1}$ which is $\tilde{A}_1(Z)$.

We have verified that, essentially there is a reversal of the order of coefficients. The coefficients in order of powers of Z^{-1} are 1 and K1 in A_1 of Z, in $\tilde{A}_1$ there are K_1 and 1. So, essentially, we are saying that the order of the coefficients in terms of powers of Z^{-1} reverses when you go from A_N to $\tilde{A}_N$ and we want to prove this by mathematical induction. Now, we had of course gone through the details of that mathematical induction in the previous lecture, I shall go through all of it again, but we will just look at a few points to put our discussion in perspective because it is a very important topic and perhaps requires multiple attempts to comprehend completely.

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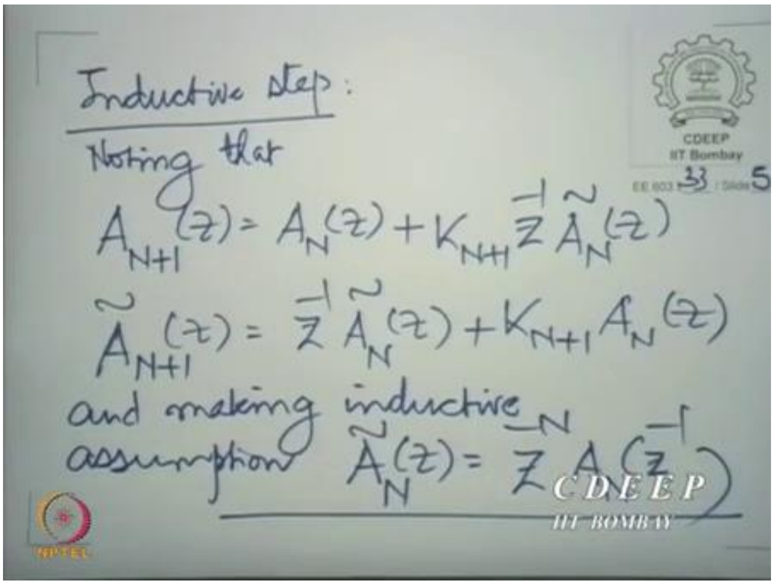

So, what we saw was that the inductive step could be carried out as follows. Noting that $A_{N+1}(Z) = A_N(Z) + K_{N+1} Z^{-1} \tilde{A}_N(Z)$ and $\tilde{A}_{N+1}(Z) = Z^{-1} \tilde{A}_N(Z) + K_{N+1} A_N(Z)$ and making the inductive assumption, $\tilde{A}_N(Z) = Z^{-N} A_N(Z^{-1})$. Now, what is this inductive assumption, this inductive assumption essentially is saying that you can obtain the coefficients of $\tilde{A}_N$ by writing the coefficients of A_N in reverse order In terms of powers of their inverse.

This is a mathematical way of saying this. Towards the end of the previous lecture, we also explained why this is a mathematical way of saying that essentially you are reversing the order of the coefficients when you go from A_N to $\tilde{A}_N$. For making this inductive assumption, and then using two recursive steps, we can then prove.

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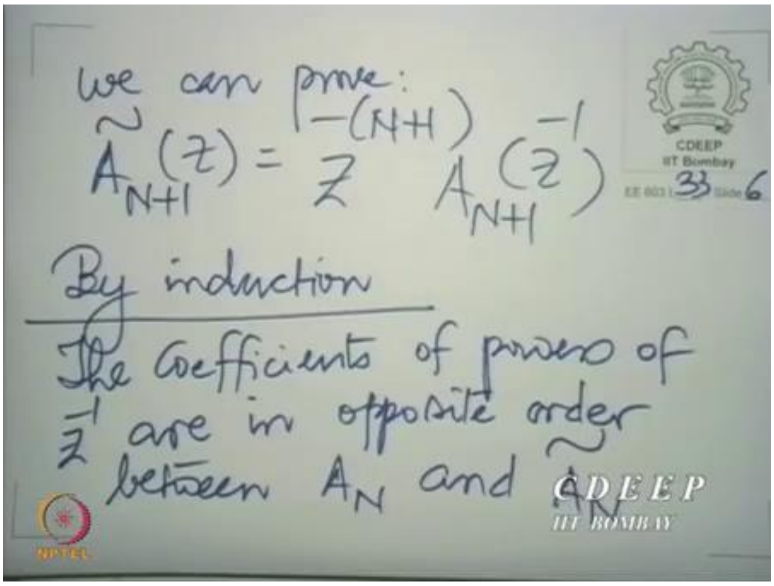

We can prove, $\tilde{A}_{N+1}(Z)$ is $Z^{-(N+1)} A_{N+1}(Z^{-1})$ which means that by inductive proof the coefficients of powers of Z^{-1} are in opposite orders or reverse orders between A_N and $\tilde{A}_N$, we proved this. Now, the reason why we proved this is because, if you know A_N you should be able to write down A_N tilde by a very clear algorithm here, we have a very clear algorithm. Once you know A_N, A_N tilde is known. Now, what we want to do next is to realize an FIR function.

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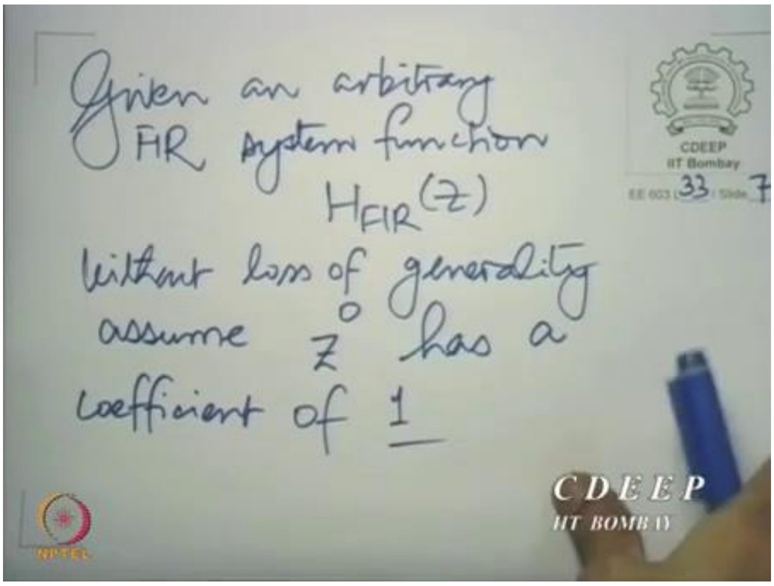

So, suppose we have given an arbitrary FIR function, H(Z), $H_{FIR}(Z)$. Now, we shall assume without loss of generality, assume that Z^0 has a coefficient of 1. I will explain why it is without loss of generality. You see suppose, for example you have the following system function which violates this requirement.

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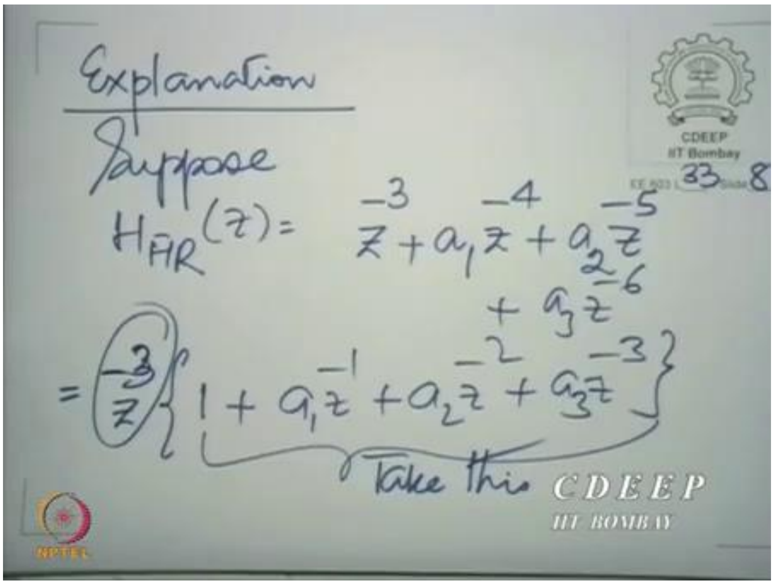

So, suppose H_{FIR} (Z) is of the form $Z^{-3} + a_1 Z^{-4} + a_2 Z^{-5} + a_3 Z^{-6}$, I can always extract Z^{-3} common from here and write $1 + a_1 Z^{-1} + a_2 Z^{-2} + a_3 Z^{-3}$ and this is to be then taken, take this and this essentially three delays in cascade, Z^{-3}. For 3 means you have 3 stages of that where all the Ks are lattice 0. You see, you can always write this the first three steps.

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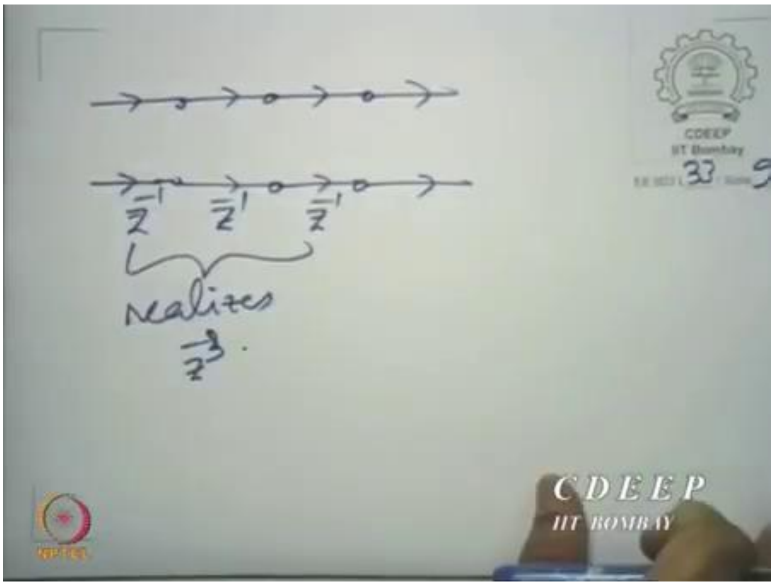

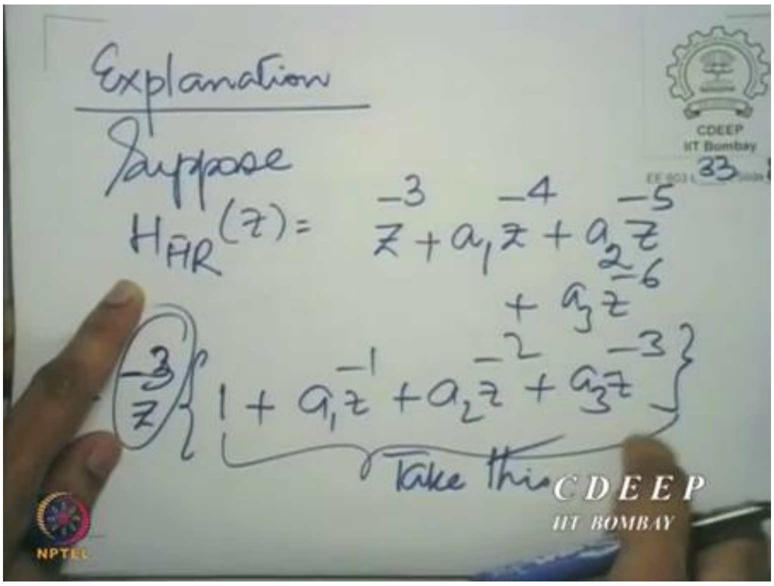

As Z^{-1} with zero case, you see you do not need to put the Ks at all. See, you can always put a cascade of three delays. In fact, I do not even need to leave any blanks in between, so this realizes Z^{-3}. So, you can realize a few delays separately and the rest of it is then realized using the lattice structure, in any case the lattice structure is meant for giving insights into the part of a system function which we cannot obviously see, what we can obviously see does not require to be used in the lattice structure.

So, it is the rest of one plus, this part is what we are interested in after all, we are trying to see what properties this part has $1 + a_1 Z^{-1} + a_2 Z^{-2} + a_3$, it is this part which is of interest. So, at that part can be realized with the lattice structure, what is anyway trivial can be put separately.

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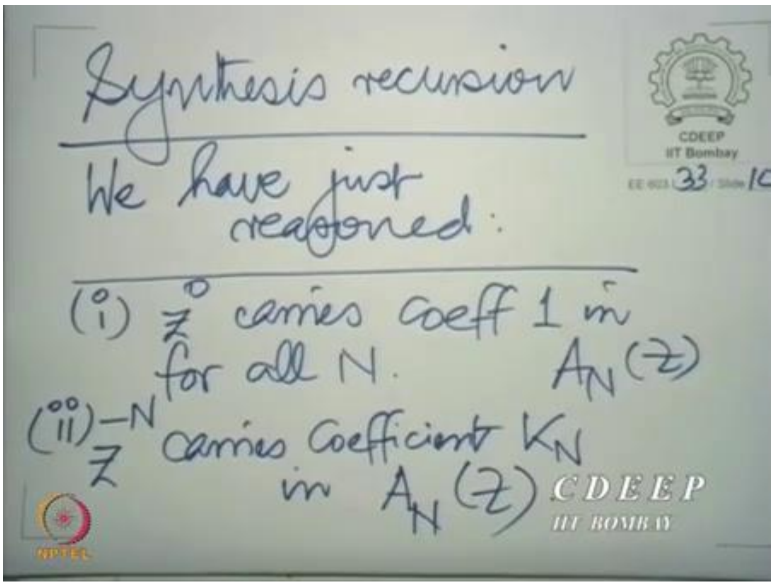

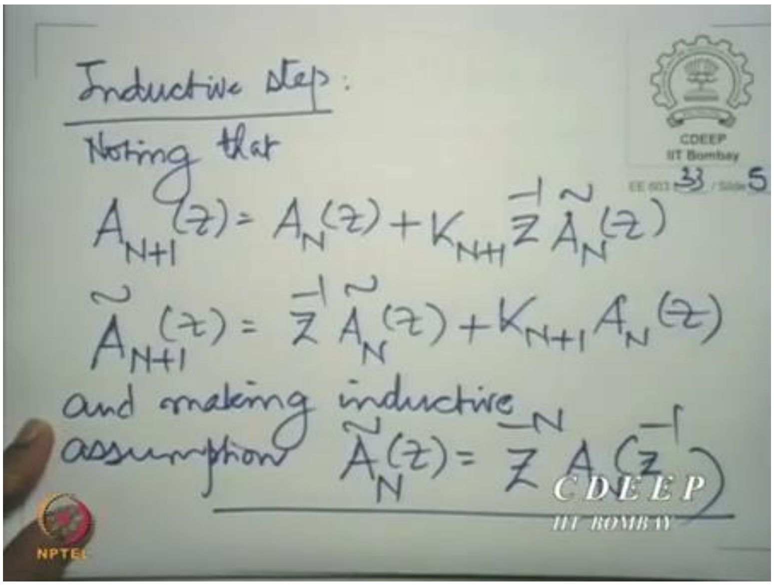

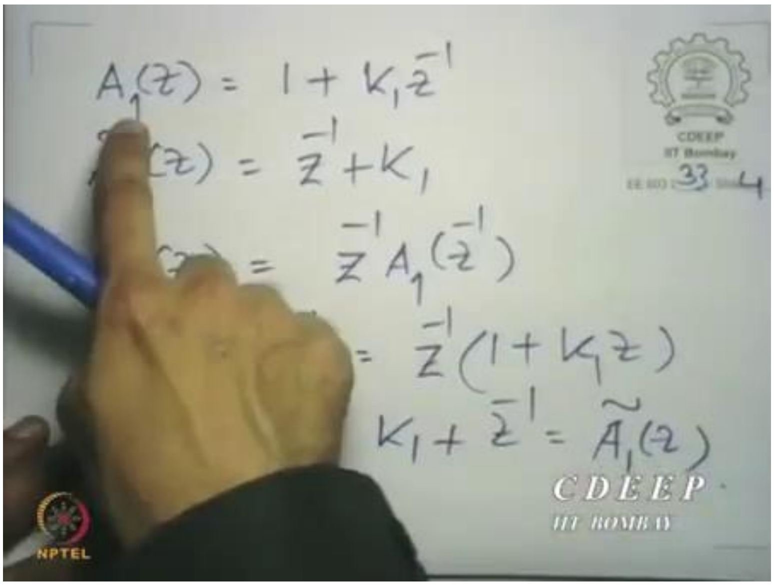

So, now the next step is to obtain what is called a synthetic or synthesis recursion. You see what we did so far was an analysis recursion. Analysis means you are moving forward from a_0 to a_1 to a_2 to a_3 that is analysis, you have to understand how to build a certain A_N. But now we want to build the other way. You see, you are given the K_Ns and you want to build the A_N, that is analysis, you are given the final A_N and you want to build the K_Ns that is synthesis.

So, you want to construct the lattice structure that realizes a certain system function that is synthesis, which means you are at the end now. So, let us assume, now it is very clear that every time you add one lattice stage, you are increasing the degree of A_N and $\tilde{A}_N$ by 1. In fact, that is very easy to see, let us go back to the expression. You see if we look at it, in the recursive step, it is very clear that A_1 and $\tilde{A}_1$ of course, have a degree 1.

So, what happens to A, suppose we will assume that A_N and $\tilde{A}_N$ have a degree N, I do not need to write down this proof I am just going over it orally. So, you see, let us assume by the inductive step, inductive assumption that A_N and $\tilde{A}_N$ have a degree N, now this has a degree N this has a degree N, multiplying by Z^{-1} makes it degree N+1. So, you have a, this sum would be of degree N+1.

So, by inductive assumption by inductive step, this has a degree N plus one. Similarly, this has a degree N, so this one have a degree N plus one and degree N plus one plus degree N would give you degree N+1 and therefore, both A_{N+1} and $\tilde{A}_{N+1}$, are going to be of degree N+1. That is quite clear. What is also clear is that the leading coefficient that means the coefficient of Z^0, in the coefficient of Z^0 in A_N is 1. Now, we can see that again from the basis step, in the basis step, A_1 has a coefficient of one for Z^0.

So, let us assume the inductive step that A_N has for its leading coefficient, that means the coefficient of Z^0 equal to 1. Now, of course you will notice that this term does not contribute any Z^0 at all. Because there is Z^{-1} multiplying it, so all the terms have at least a power of one in Z^{-1} and therefore the contribution to the Z^0 term comes only for this.

So therefore, the Z^0 term is carried as it is from A_N to A_{N+1} and therefore Z^0 term is 1 always in all the A_Ns. Now, as a consequence, it is very interesting to see that if you look at the coefficient of Z^N, Z^{-N+1}, the highest power of Z or Z^{-1} rather. So, if you look at the highest power of Z^{-1} in A_{N+1}, what would it be? You see.
Student: N plus one

Professor: You see it is very simple, in $\tilde{A}_{N+1}$ the linear term is K_{N+1}. Because this has no contribution to the constant term, the constant term comes only from here and the constant term is one in $A_N(Z)$ and therefore it is going to be K_{N+1} in $\tilde{A}_{N+1}(Z)$. So, in $\tilde{A}_{N+1}(Z)$, the constant term is K_{N+1}. Now, if the constant term is K_{N+1} in $\tilde{A}_{N+1}(Z)$, then the highest power of Z^{-1} will carry the coefficient K_{N+1} here, because the coefficients are in reverse order.

You see the coefficients of A_{N+1} and $\tilde{A}_{N+1}$ are in reverse order. So, the constant term in $\tilde{A}_{N+1}$ is the term carrying $Z^{-(N+1)}$ or the highest power of Z^{-1} in A_{N+1}. Now, what is a constant term in $\tilde{A}_{N+1}$, this has no contribution to the constant term. So, it is only this which has a contribution to the constant term and the constant term here is a constant term here multiplied by K_{N+1} and the constant term here is 1.

And therefore, the constant term in $\tilde{A}_{N+1}$ is K_{N+1} and therefore, the term or the coefficient associated with the highest power of Z^{-1} in N+1 is K_{N+1}. Is that clear to everybody? Let us write down this the very important observation that we have made.

We have just proved Z^0 carries a coefficient of 1 in A_N (Z), for all N. Z^{-N} carries coefficient K_N in A_N (Z). In fact, at the end of the synthesis that means, suppose you indeed have ensured that the leading coefficient that is the power, the Z^0 carries the coefficient 1, then the coefficient associated with the highest power of Z^{-1} is automatically the last lattice coefficient.

So, at least one part of the job is done for you. There is one thing that we have to keep in mind here. You see, when we go backwards what we will need to do is to express A_N and $\tilde{A}_N$ in terms of A_{N+1} and $\tilde{A}_{N+1}$. So, we would need K_{N+1} to be known so far, you know, actually K_{N+1} is a part of the synthesis, now let us be clear.

So, all this while we have assumed that we knew the K_Ns and knowing the K_Ns we are calculating the A_Ns, now we are going the other way. We know the last of the A_Ns, we know the final system function. we want to obtain the K_Ns, which realize that system function, so we have to begin with the last one, not the first, Right. So, let us do that. Let us begin with the last.

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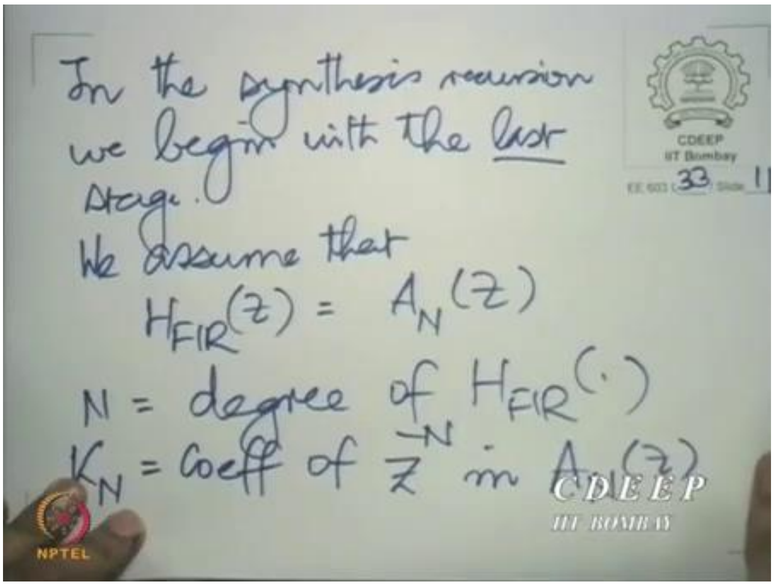

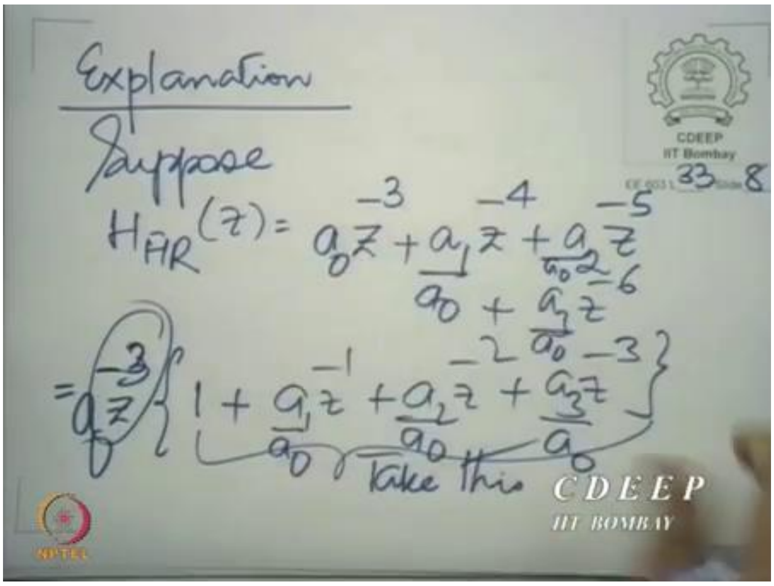

So, in the synthesis recursion, we begin with the last stage.

Student: Sir?

Professor: Yes.

Student: Sir the last A_N will give us the system function?

Professor: So the question is, would the last A_N give us the system function? Yes, indeed.

Student: Sir, the last A_N tilde?

Professor: So, the question is what happens to the last $\tilde{A}_N$, the last $\tilde{A}_N$ is an auxiliary function that we are going to obtain for some other purpose. So, what we are going to do is we are going to use A_N explicitly and we are going to use $\tilde{A}_N$ implicitly in a recursion for some other reason. So, it is an auxiliary quantity, which we are getting for some other work.

We begin with the last stage. So, we assume that $H_{FIR}(Z) = A_N(Z)$ and A_N is of course equal to the degree of H_{FIR} that is obvious. Now, incidentally again that issue about loss of generality see, in this again, we have perhaps, not so justifiably assume the coefficient of this to be one. But even if it is not, suppose the coefficient a_0 here, you can always extract a_0 here and divide everything by a_0.

That is not really a problem. So, even if this coefficient is not one, you can always extract it common. So, it is indeed without loss of generality that we can take H_{FIR} to have the coefficient of Z^0 is equal to 1 and in that case, the degree of the two, the degree N is equal to the degree of this FIR system function.

And therefore, K_N is simply the coefficient of Z^{-N} in $A_N(Z)$ which we know very well. So, this is essentially what we called the basis step in the synthesis, because we have to begin with something. So, here we have a basis the synthesis begins, knowing the K_N and now we have to obtain K_{N-1}, K_{N-2} right down to K_1. So, we need to go one step backward. So, we need a backward recursion.