Digital Signal Processing & Its Applications
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Lecture 33b
Backward recursion and rational, casual system function realization

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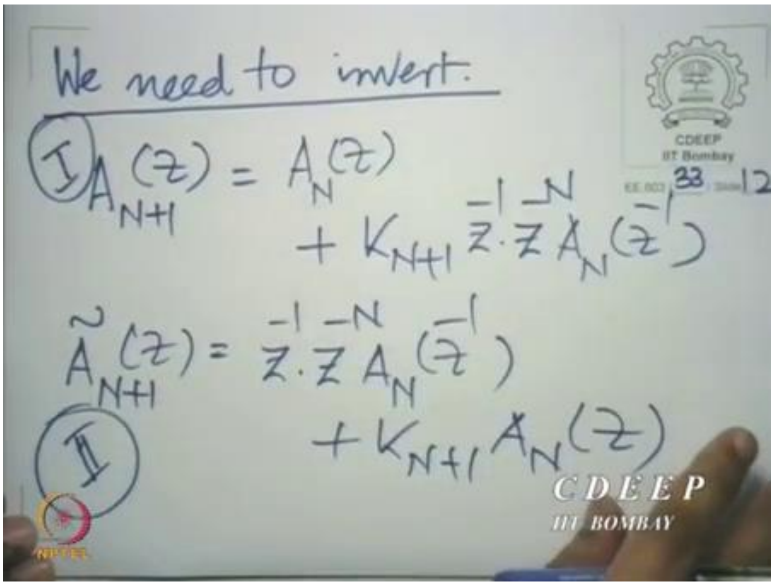

We need to invert the following. You say A_{N+1} let us write down the recursion, but this time let us note that we already have a relation between A_N and $\tilde{A}_N$. So, $A_{N+1}(Z) = A_N(Z) + K_{N+1} Z^{-1} Z^{-N} A_N(Z^{-1})$, let us write that down straightaway.

And similarly $\tilde{A}_{N+1}(Z) = Z^{-1} Z^{-N} A_N(Z^{-1}) + K_{N+1} A_N(Z)$. So, what we are going to do is to get the A_N from the A_{N+1} now, you see, so we need to invert this. Now, that is very easy to do, I just need, let me call this relationship I and let me call this relationship II. All that I need to do is to cut off this term really, I can later on get rid of the $A_N(Z^{-1})$ term that is very easy to do, I can just subtract K_{N+1} times this from this.

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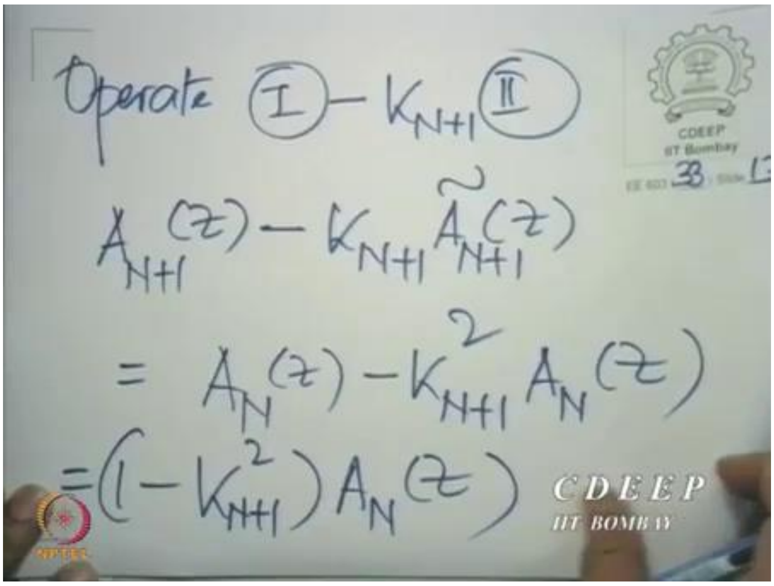

So, all that I need to do is to operate I - K_{N+1} II and that gives me a very simple elegant relationship it says $A_{N+1}(Z) - K_{N+1} \tilde{A}_{N+1}(Z) = A_N(Z) - K^2_{N+1} A_N(Z)$. As simple as that. Now you see this is of course very easy to write $(1 - K^2_{N+1}) A_N(Z)$. So, now we have $A_N(Z)$ in terms of A_{N+1} . In fact, we will have to construct $\tilde{A}_{N+1}(Z)$ from $A_{N+1}(Z)$ we know how to do that by the inductive relation.

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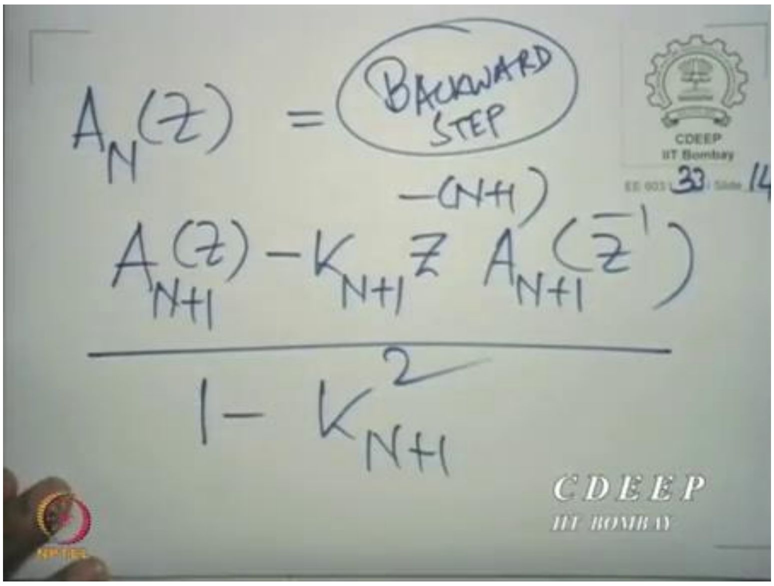

So, in fact what this says is that $A_N(Z)$ is then $[A_{N+1}(Z) - K_{N+1} Z^{-(N+1)} A_{N+1} (Z^{-1})] / [1 - K^2_{N+1}]$. So, we have the most important step completed; the backward step, from A_{N+1} you can go 1 step back. Now, we can think of this as peeling off one stage.

You were want to visualize the lattice structure like a piece of cabbage. Where you have peel after peel after peel, it stays like a peel. So, what we have done is to peel off one layer of that cabbage and you see the inner layer now, you can peel off. Now you see this is recursive. So you know how to go from A_{N+1} to A_N and the same approach can be used to go from A_N to A_{N-1} and you can keep doing this until you reach A_1 and once you reach A_1, of course, you are done.

Now, in this process, the properties of A_N would be maintained, because if you look at it, again, here, you see it is very clear that, see the whole idea was that we would need to maintain the property that the coefficient of Z^0 is 1 and so on. Now that would continue to be maintained in this recursion. I leave it to you to verify that, but one word of caution, there is only one situation in which we have trouble here and that relates to when this denominator becomes degenerate.

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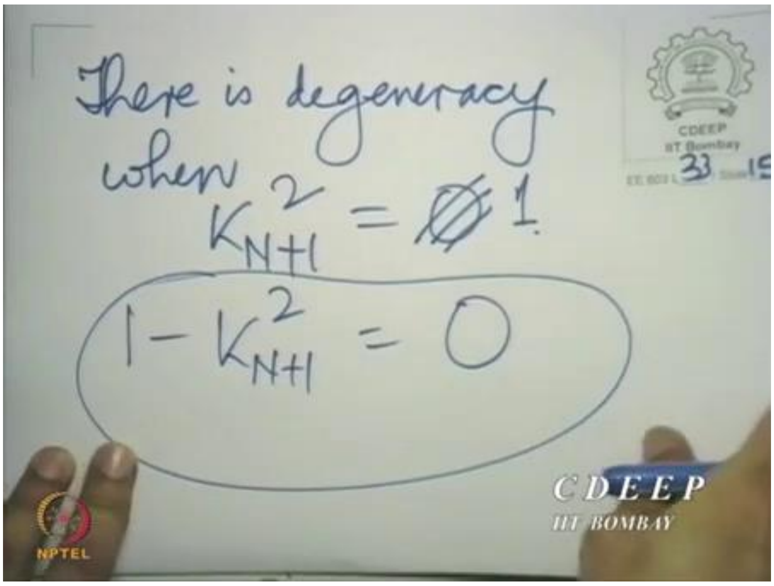

So, there is degeneracy. When $K_{N+1} = 0$, that means in this lattice recursion at any stage, if you find that this K turns out to be 1, then you have a degeneracy, you cannot realize it with a lattice structure.

Student: (())(06:48)

Professor: I am sorry, yes, I meant to say K_{N+1}, I meant to say $1 - K^2_{N+1} = 0$. Yeah, that is correct. K_{N+1} in fact, I would write, I would prefer to say $K^2_{N+1} = 1$ and of course, we are assuming that the coefficients are all real, Ks are real. So, it could be either plus or minus 1. So, plus or minus 1 lattice parameter, this is called, the K_N is called the lattice parameter.

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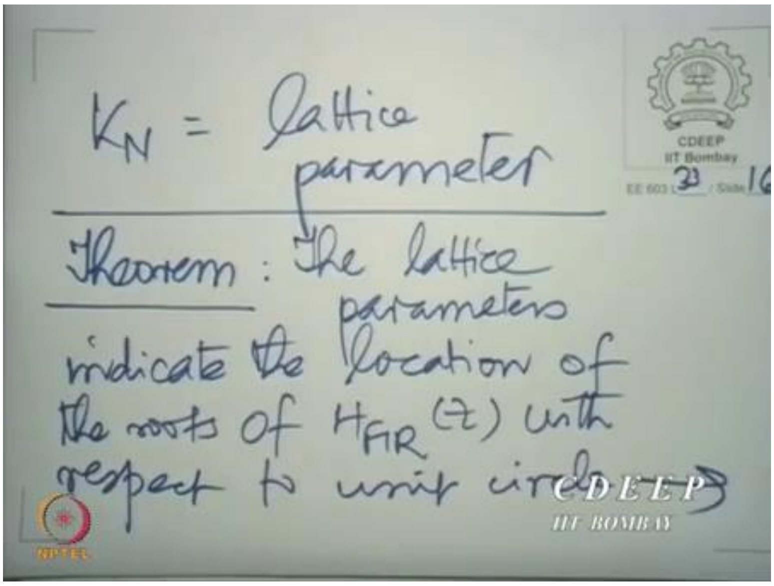

Let us take note of this term. K_N is called the lattice parameter. So, degeneracy results when the lattice parameter takes the value plus or minus 1 and degeneracy means that system function cannot be realized with the lattice. Now in fact, of course, an important question why a degeneracy takes place and why these lattice parameters are so interesting, I mean this just looks like any other structure, what is so special about this, we will now put down a very important theorem, which will explain why this is so important. I shall not prove this theorem, but I should state it. The lattice parameters indicate the location of the roots of $H_{FIR}(Z)$ with respect to the unit circle in the following specific way.

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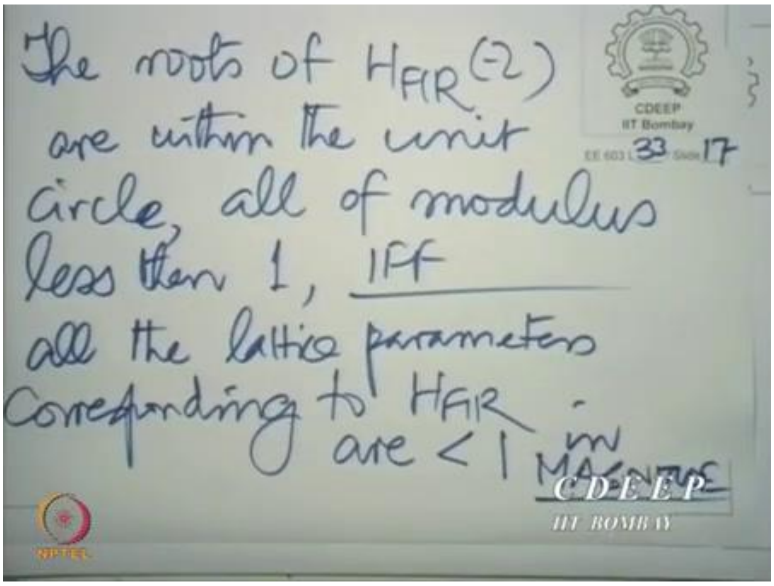

The roots of $H_{FIR}(Z)$ are within the unit circle that means, all of modulus less than unity, now note strictly less than unity, if and only if all the lattice parameters corresponding to H_{FIR} are less than 1 in magnitude. That means if the system function $H_{FIR}(Z)$ has all its lattice parameters less than 1 in magnitude, you are guaranteed that it has its roots inside the unit circle and vice versa. If you require that $H_{FIR}(Z)$ have all its roots inside the unit circle, then all the lattice parameters must be less than 1 in magnitude. Now why is this important, in fact not so much for an FIR function.

It is important for the denominator in a rational function, that means what we want to do is to exploit this property to study the denominator of a rational system and that is the next step that we would like to take and that is the reason why as somebody asked, why are we creating that auxiliary function $\tilde{A}_N$, the auxiliary function $\tilde{A}_N$ is being created to allow us to exploit this property to realize a rational system function in general, we are actually going to use this case to realize the denominator and the numerator would be realized with some other type of coefficients and then we will get insight into the stability of the rational system function by looking at the case.

Now, what does it mean about the K equal to 1 case or mod K equal to K square equal to 1 case, it means there, there is trouble. If you find the any of the case becomes plus minus 1, then you have a problem with the solutions. The if and only if relation, so you know, if you are trying to look at, so you know what you could, I mean, irrespective of whether you are utilizing a rational system function.

What this tells you is that if I am given the denominator polynomial in a rational system function and of course, I have told you that in a causal rational system function, you can always write the denominator with a leading 1. In the causal, I repeat in the causal rational system function, you are always in a position, to write the denominator with a leading 1, the coefficient of $Z^0 = 1$.

Once you do that, take the denominator as it is, think of it as a system function and realize it with the lattice structure and then if you want to check whether the system is stable or not, all the solutions must of course lie within the unit circle and that implies that all the lattice parameters must be strictly less than 1 in magnitude. So it is a very simple test for stability of rational system function. Let us make a note of that. So, this, yes, there is a question.

Student: We have specified what happens that it is inside and what happens it is all in K is equals to 1. What happens if the magnitude is greater than 1?

Professor: The question is, what happens if the magnitude is greater than 1 or in fact, for that matter, what happens to the magnitude is equal to 1? Well, all that we are saying at the moment, is that if all the magnitudes are not less than 1 strictly, you are sure that there is a root either on the unit circle or outside?

That is all we can say at the moment, we are not going further at the moment. So, there is a one to one, there is stability implies and is implied by or you know what I mean is, if you want to check that all the roots of a polynomial and their inverse are within the unit circle, then it is necessary and sufficient that all the corresponding lattice parameters be less than 1 in magnitude. If that condition is violated, you are sure that there is a violation also of the roots being inside the unit circle.

Now, in what way we do not know, whether the roots are on the unit circle or the roots are also outside, that we are not going into at the moment. In fact, we have not even proved this. This proof is a little involved, time permitting, we will see if we can give a small proof, I would also like to offer it to you as a challenge, but it is rather difficult challenge.

Well, actually, that the, I can tell you if some of you are really interested in taking up the challenge and trying to prove this, the central idea in the challenge is to look at what happens when you create an all pass like function as you have here. You know the relation between A_N and $\tilde{A}_N$ and the recursive relation between A_N A_{N+1} is at the heart of this proof.

You know, anyway, I do not want to spend too much of time on that safe to say that the proof is a little involved and we shall take it up if time permits. But we must know the consequences, the theorem should be understood, theorem is not too difficult to understand. It says that necessity and sufficiency, you essentially, it states the necessity and sufficiency of all the lattice parameters being less than 1 in magnitude for all the roots of that polynomial to be inside the unit circle. Now, what why is this useful, that is what we are going to do next.

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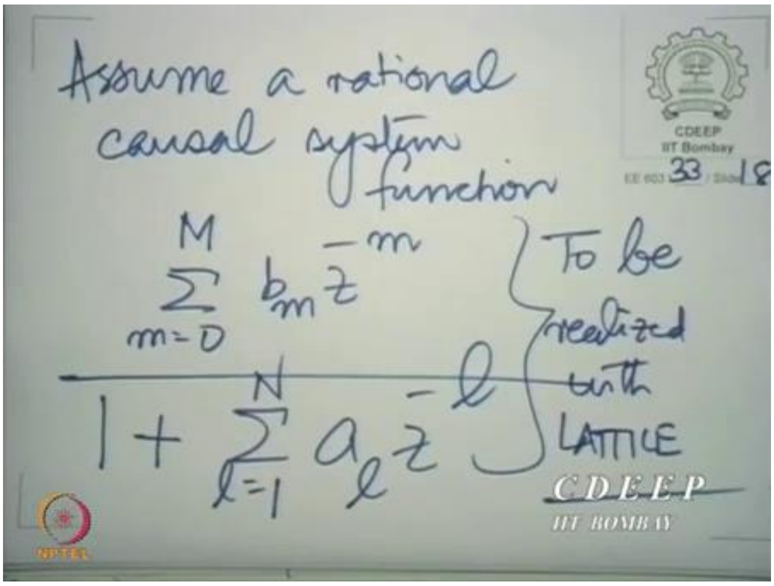

What we are going to do next is to assume then that we have a rational causal system function as follows. As usual it is of the form $\sum_{m=0}^{M}$ b_m z^{-m} / 1 + $\sum_{l=1}^{N}$ a_l z^{-l} We have to realize with the lattice, now obviously you cannot do as it is. So, far lattice only allows us to realize FIR system functions and the real use of the lattice is in studying this. So, what we need to do is to make a little change in the equations or the recursive equations of the lattice, we could go back, let us go back and look at those recursions once again. So, in fact, maybe we do not even need to do that.

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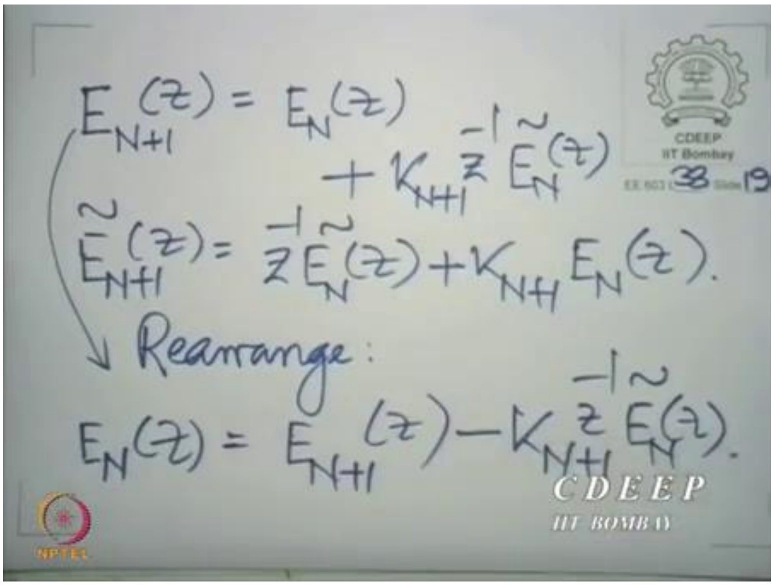

Let us write them right from fundamentals. We will not even write the system functions now, we will write the sequences right from the beginning, you go back to the very first pair of equations that we wrote in terms of sequences on the lattice. So, we had $E_{N+1}(z) = E_N(z) + K_{N+1}$ Z^{-1} $\tilde{E}_N(Z)$ and $\tilde{E}_{N+1}(Z) = Z^{-1}$ $E_N(Z)$ $\tilde{E}_N(Z)$, I am sorry plus K_{N+1} $E_N(Z)$. Now all that we do is to make a little rearrangement of this.

So, we just rearrange it and rewrite this as $E_N(Z) = E_{N+1}(z)$ - K_{N+1} Z^{-1} $\tilde{E}_N(Z)$. That is an interesting situation that we have here. We have $\tilde{E}_{N+1}$ written in terms of $\tilde{E}_N$ and E_N and we have E_N we can write it in terms of E_{N+1} and $\tilde{E}_N$. Let us write down the structure of this implies. Now please know that we have not changed the basic equations, the basic equations are the same. Therefore, the basic relationships remain the same, the basic relationships do not change, but the structure changes and the structure would look like this.

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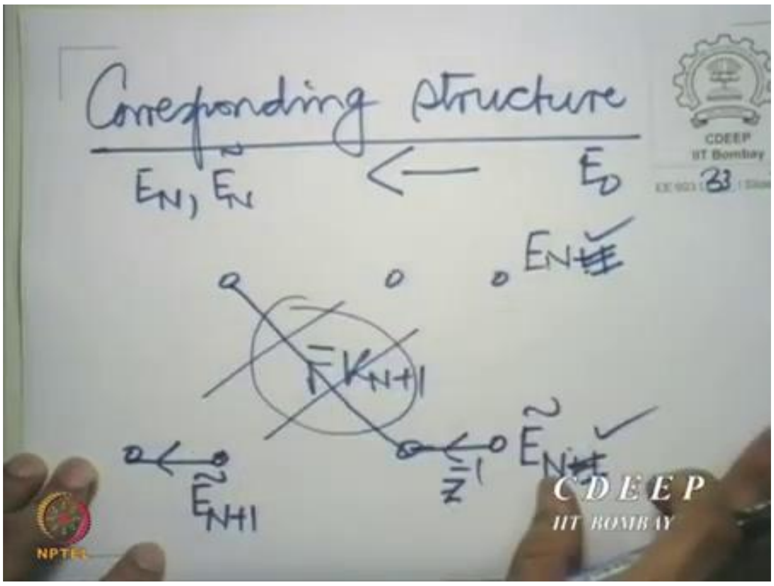

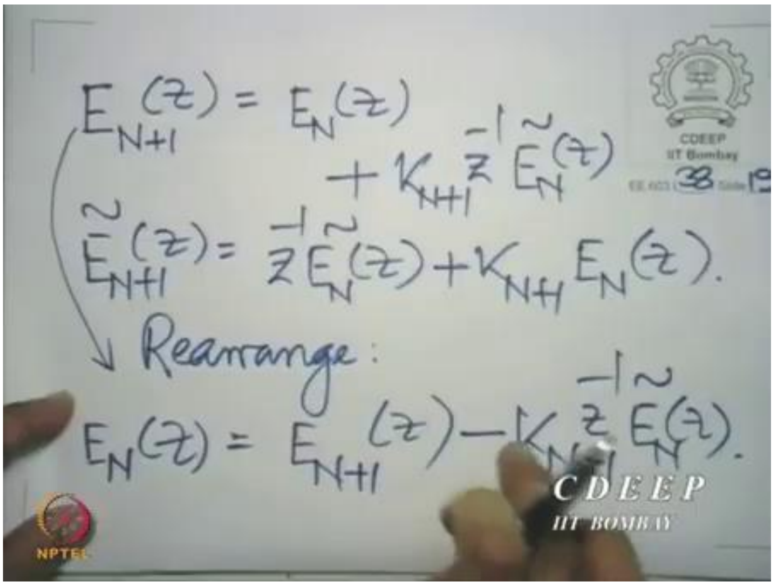

You see what we will do is we will now draw the structure the other way. So, we will move this way from E_0 to E_N or $\tilde{E}_N$. So, we assume that somewhere here you have E_0 and $\tilde{E}_0$ and you are moving this way. So, we write 1 stage. So, we have $\tilde{E}_N$ here please not E_N and of course, E_N there, and go back to the equation. You have an $\tilde{E}_N(Z)$ multiplied by Z^{-1}, so there we are.

Now notice this equation say, this equation says Z^{-1} times $\tilde{E}_N$ needs to be multiplied by - K_{N+1} and added to E_{N+1}. So in fact, let us make this E_{N+1} here for convenience. Oh, well, I am sorry. I think this is all right. So, we need to multiply this by K_{N+1} rather - K_{N+1}, Now what about this equation here. $\tilde{E}_{N+1}(Z)$, so you have not, let us keep forming the structures you know, it is a little complicated.

You have an $\tilde{E}_{N+1}$ being formed here. $\tilde{E}_{N+1}$ is formed with a Z^{-1} $\tilde{E}_N(Z)$ and K_{N+1} times $E_N(Z)$, so you need a K_N from here and this as it is, you see $\tilde{E}_{N+1}(Z)$ is Z^{-1} $\tilde{E}_N(Z)$, Z^{-1} $\tilde{E}_N(Z)$ is essentially this plus K_{N+1} $E_N(Z)$. So, you need a K_{N+1} to come from there. Let me write it like this. Maybe I need you to map let us remove this branch for the moment. Let us take this later. Let us draw only one pair of branches to make it look much neater.

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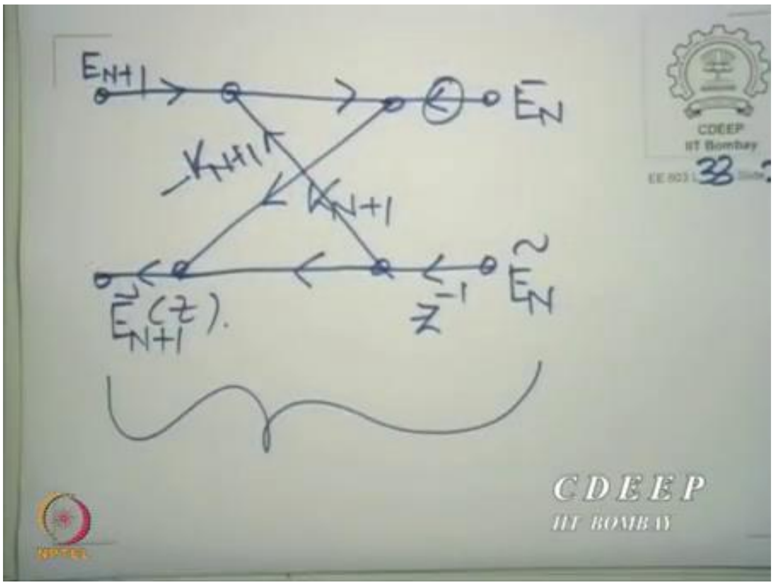

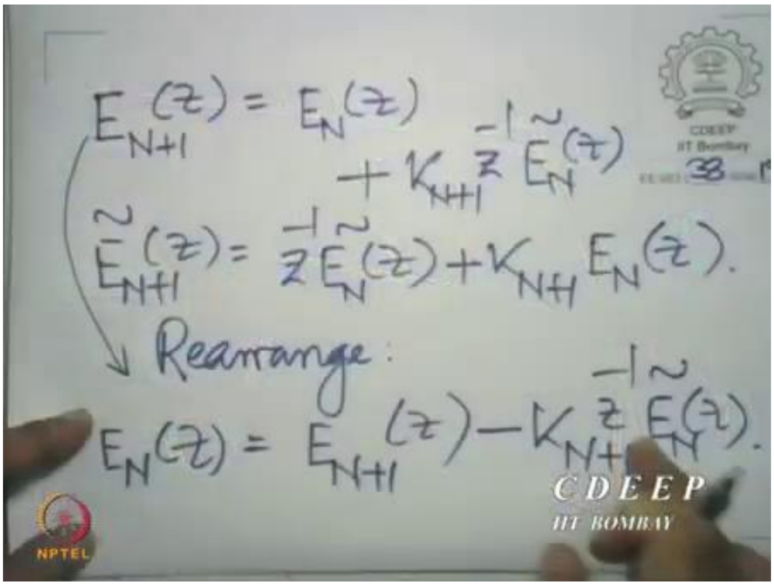

We have $\tilde{E}_N$, we have a Z^{-1}, we will realize this equation first. So we add this to this multiplied by K_{N+1}. So, what we have done is to realize this equation here. Now you see the second equation is E_{N+1} - K_{N+1} times Z^{-1} $\tilde{E}_N$. You see, so that is a funny situation to be in.

You have an E_{N+1} here. You need to take E_{N+1} and subtract from it, K_{N+1} times Z^{-1}, I am sorry, Z^{-1} times $\tilde{E}_N$, so you need to realize this equation in the second branch. Now how would you get that, you would need to take E_{N+1} as it is and subtract from it K_{N+1} times Z^{-1} $\tilde{E}_N$.

Now you are getting Z^{-1} $\tilde{E}_N$ here. Multiply that by - K_{N+1} and add it to E_{N+1} and this is what should give us E_N. In fact, now the, now we have a problem, you see now this is the kind of structure that we expect. We will need to see more of this. We will take this up again in the next lecture, we will sort of clean up the structure a little more.

Student: (())(24:42)

Professor: So we will have to do a little bit of shifting here, we will have to rearrange the structure to realize this set of equations. But the philosophy is the following, the equations are the same, the relationships are the same, but now we are reinterpreting the equations to give us an IR system function, rational system function, we need to spend more time on this and therefore, we continue this in the next lecture by repeating some steps. Thank you.