Digital Signal Processing & Its Applications
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Lecture No. 34a
Lattice Structure for an arbitrary rational system

So, warm welcome to the thirty fourth lecture on the subject of Digital Signal Processing and its applications. We continue in this lecture to look at the lattice structure and the construction of a lattice structure for rational system functions, last time we had seen how to construct it for FIR system functions or polynomials. But now, we need to see how to construct it for a rational system.

Now, we have taken a step in that direction last time, but we had not quite completed it, the step was to make a small transformation on the equation. Let me put that transformation back again.

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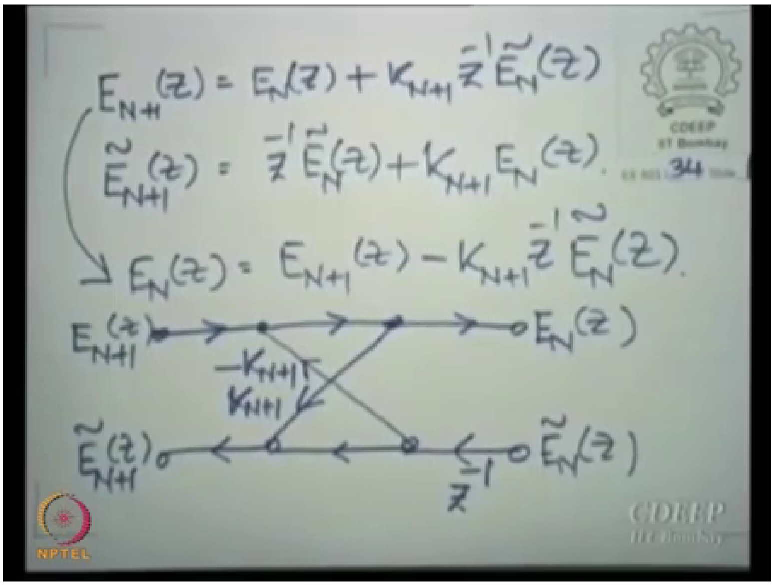

Let us begin right from the very equations relating the sequence. So, the sequences in successive stages are related to the following equations: $E_{N+1}(z) = E_N(z) + K_{N+1} z^{-1} \tilde{E}_N(z)$

$\tilde{E}_{N+1}(z) = z^{-1} \tilde{E}_N(z) + K_{N+1} E_N(z)$

and we said we would make a little transformation on this, we would rewrite this as $E_N(z) = E_{N+1}(z) - K_{N+1} z^{-1} \tilde{E}_N(z)$ and we noted that since the equations were the same the relationships would remain the same between quantities.

But which sequence we use as an input and which we use as an output could be our choice. So, the relationship, the mutual relationship would be the same, but we could choose which to use as input and which to use as an output. That was the central point here. Now first what we were trying to do is to write down one stage of this set of equations. So, we said we will reverse the direction we would put the initially, we had put the sequences in increasing order of index from left to right.

Now, we will put the sequences in increasing order of index from right to left. So, we would have an E_N here, we would have an $\widetilde{E}_N$here and you would have E_{N+1} here and $\widetilde{E}_{N+1}$ there and we would carry out a relationship between them. So, if you look at this relationship, what it says is that E_N(z) is a combination of $E_{N+1}(z)$ and $K_{N+1}z^{-1}\widetilde{E}_N(z)$.

In any case we require $z^{-1}\widetilde{E}_N(z)$ for both of these expressions and therefore, we can put a z^{-1} right there. So, we need to add up, $E_{N+1}(z)$ on $-\ K_{N+1}z^{-1}\widetilde{E}_N(z)$. So, what we generate here is essentially E_N, $E_{N+1}(z) - K_{N+1}z^{-1}\widetilde{E}_N(z)$. So, we have generated E_N here and this is of course transmitted forward.

Now, note this was a little change that we needed to make, we have just begun making that change at the end of the previous lecture, but we did not complete it. So, our arrow was going this way. But now we need to push the arrow this way. Now this is what we have generated here is E_N(z). So, you can keep transmitting it, it is E_N(z).

Now from the second equation, you need K_{N+1} times this sequence, K_{N+1} times this sequence, $E_N + z^{-1}\widetilde{E}_N(z)$. So, what we generate here is as desired $\widetilde{E}_{N+1}(z)$ and this can be transmitted to $\widetilde{E}_{N+1}(z)$. So, we have constructed one stage of the lattice now and what we note is that the relationships are the same, essentially, it is the same relationships. But we are now going to redefine the input and output in the set of relationships.

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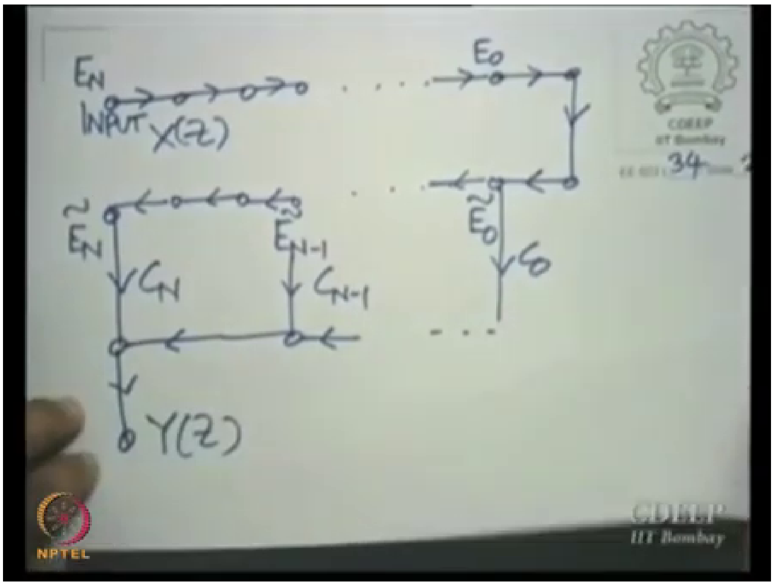

So, let us write down several stages of this lattice now. So, you have. Let us look at the Nth stage and let us go all the way up to 0th stage. You have E_0 and $\tilde{E}_0$ and what we do is, you see here, all our arrows now look at the upper branch here, all the arrows are forward here and all the arrows are reversed. So, as we go from E_{N+1} to E_N, then to E_{N+1} and then all the way down to E_0, we will be to join E_0 to $\tilde{E}_0$.

So, we have E_0 you see, the E_0 arrow goes forwards and the $\tilde{E}_0$ arrow goes backwards, so we need to join it here. So, E_0 needs to be transmitted to $\tilde{E}_0$. This is the entire structure. And if there are any stages, then this would be E_N and this would be $\tilde{E}_N$ and the input would be applied here.

So, here, we need to take an output from any of these points, you see the input is here and the output could be taken at really any of these points, but we need to be able to generate a most general form of a rational system function. So, what they will do is to take a linear combination here. So, if you have E_N here, $\tilde{E}_{N-1}$ here and so on, then we take a linear combination of these, that is the most general form that we can conceive of.

So, we could multiply this by c_0, this by c_{N-1} in general c_N and we could add it term by term and this is the output that we generate. So, we have $X(z)$ here at E_N and $Y(z)$ as a sum of or as a linear combination of the $\widetilde{E}_0$, $\widetilde{E}_{N-1}$ all this, $\widetilde{E}_0$ all the way up to $\widetilde{E}_N$. Yes, there is a question.

Student: (())(9:17)

Professor: Yes, so there is a question, there are so many nodes between $\widetilde{E}_N$ and $\widetilde{E}_{N-1}$, there are not so many. So, the question is, why are there so many nodes? There are not so many nodes, there are just enough nodes to create the lattice structure. So, you have one you see here, if you look at let us look at the structure again.

So, the question was, why do we have so many, we do not have so many nodes really, we have just enough nodes to create, this is the lattice part, this is the transmission part and so you know, here when you have a transmission, you have the lattice part. And you have, the only catch is that here you need 1 transmission here and 1 transmission here. Therefore exactly three parts are required; there is no extra part at all. So, therefore what we have here is a linear combination of the $\widetilde{E}$ outputs and therefore, we could write down $Y(z)$ by $E(z)$ as follows.

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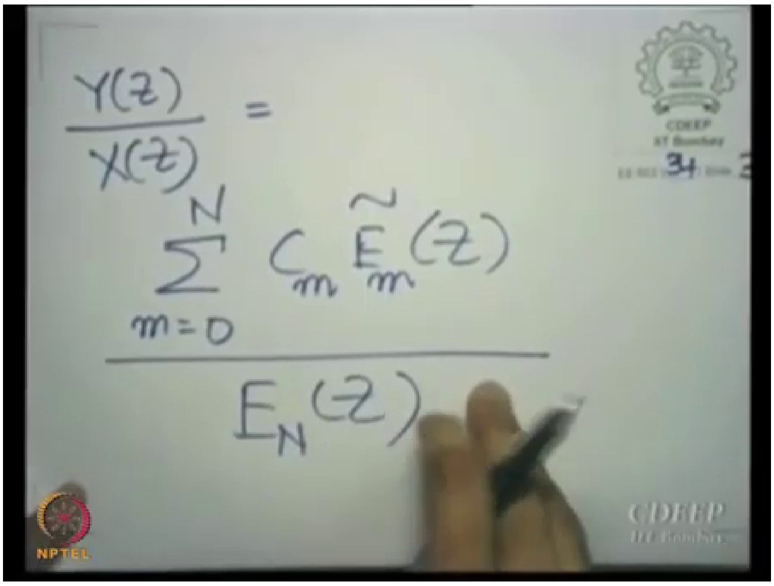

For $Y(z)$ by $X(z)$ as follows, it is essentially $\frac{\sum_{m=0}^{N} c_m \widetilde{E}_m(z)}{E_N(z)}$, this is the relationship and we can rewrite this relationship. Of course, needless to say, if we look at the way this has been constructed $\widetilde{E}_0$ is equal to E_0 here as well, as it was in our initial structure. This is a relationship. Now we multiply and divide by $E_0(z)$.

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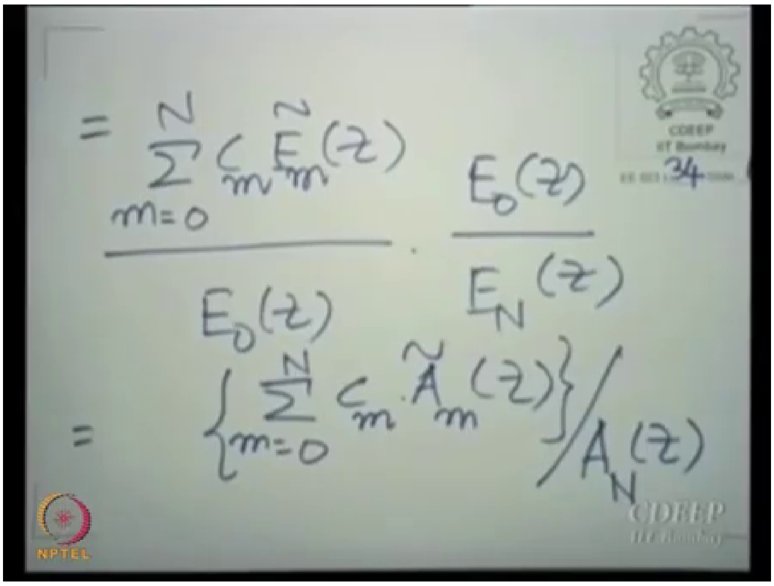

This can be rewritten as $\frac{\sum_{m=0}^{N} c_m \tilde{E}_m(z)}{E_0(z)} \bullet \frac{E_0(z)}{E_N(z)}$ and therefore, we can write the same relationship in terms of the $\tilde{A}_N$ and the A_N, this is the relationship and now this gives us a clue how we can realize an arbitrary rational system function, because $\frac{\tilde{E}_N(z)}{E_0(z)} = \tilde{A}_N(z)$ and $\frac{E_N(z)}{E_0(z)} = A_N(z)$. Now, $A_N(z)$ is already in the form that we wish it to be. So, you see when you have a general rational system function.

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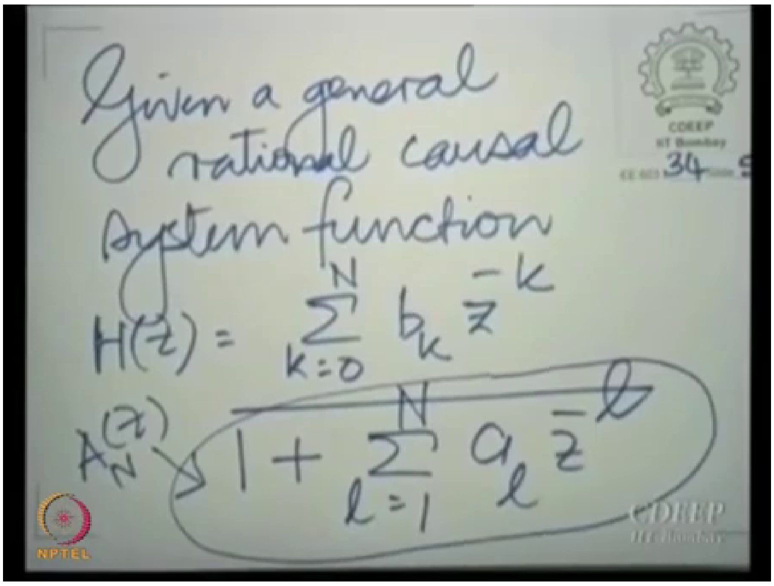

Rational causal system function, $H(z) = \frac{\sum_{k=0}^{N} b_k z^{-k}}{1+\sum_{l=0}^{N} a_l z^{-l}}$, then this essentially is equated to the denominator $A_N(z)$. So, this is a $A_N(z)$. So, what it really means is that you need to choose as many stages in the lattice as the degree of the denominator.

Now, there is a bit of a catch. Suppose the numerator degree is more than the denominator degree, then what do you do in that case, you need to first make a long division and remove a quotient. So, you see here when I write it like this, I have assumed that the degree of the demand of the numerator is not more than the degree of the denominator. If it is then first make a division, remove a term and leave the remainder on the numerator. Now, the term that is removed is essentially an FIR term.

So, now you realize the rational system function as a parallel combination of two parts, one with the lattice and one with an FIR system function. You have full freedom to realize the FIR system function either with a lattice or with a standard direct form 1 2 or cascade or parallel form that is up to you. The FIR part is anyways easy to realize, there is no problem of stability, there is not any serious problem of numerical instability and so on. IIR part where we need to be vigilant about the behaviour of the poles and so on. So, that part needs to be realized with the lattice.

So, we will assume that the numerator carries the remainder after division now and so you have already extracted, so when we put it in this form, I am assuming that you have already extracted that term which might be required by long division and the remainder has been put on the numerator here. If necessary, you may want to put one more branch in parallel with the lattice which corresponds to that term which you have removed by division, I am not showing it explicitly here, but we will just write it down.

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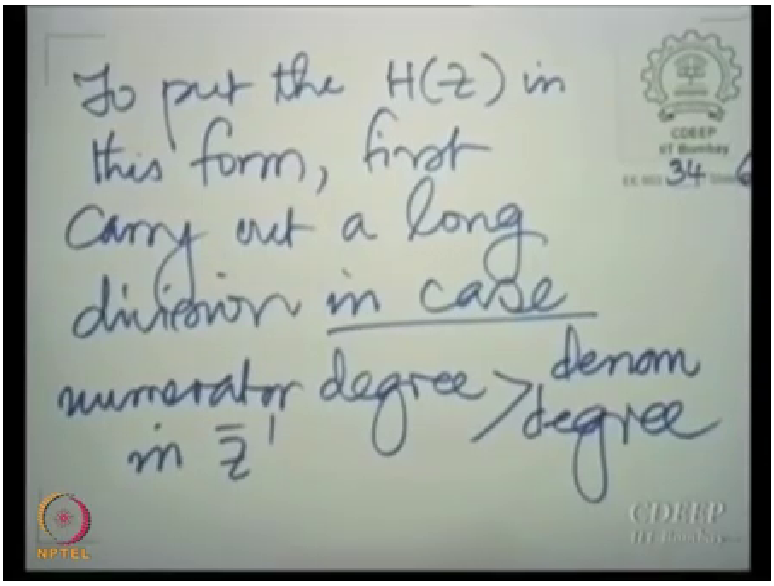

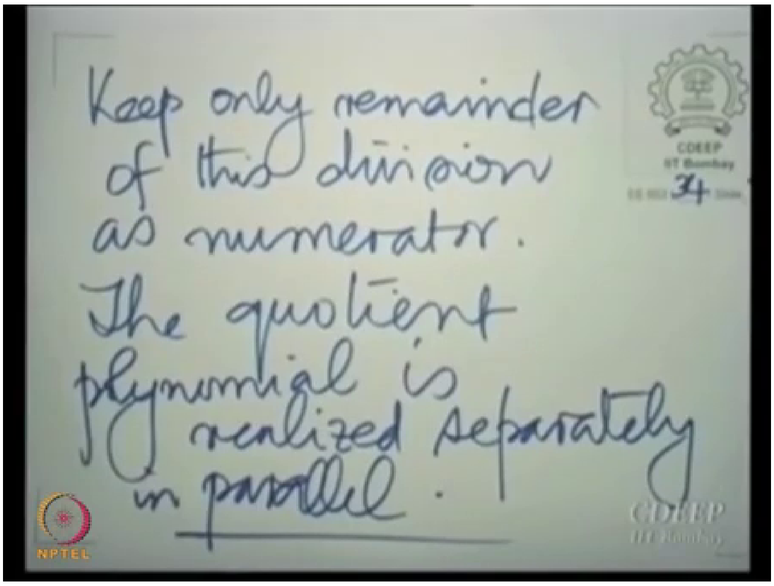

To keep rational system to put the $H(z)$ in this form, first carry out, a long division, in case the numerator degree in z^{-1} is greater than the denominator degree. Keep only the remainder of the division as the numerator, the quotient polynomial is realized separately in parallel.

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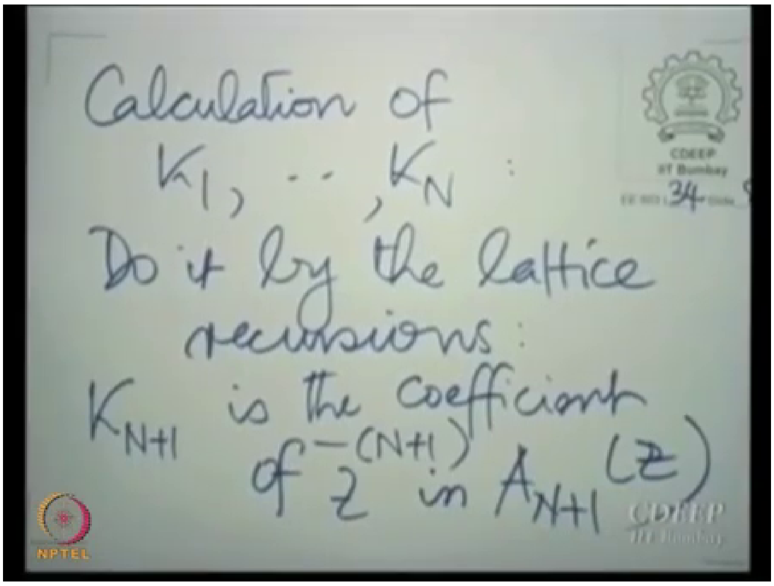

Now, you see it is very easy to calculate the lattice parameters, calculation of the lattice parameters here K_1 to K_N is easy, do it by the lattice recursion, we studied the lattice recursion in the previous lecture. We said essentially, K_{N+1} is the coefficient of the highest degree term. So, $z^{-(N+1)}$ in A_{N+1}, we had studied this last time.

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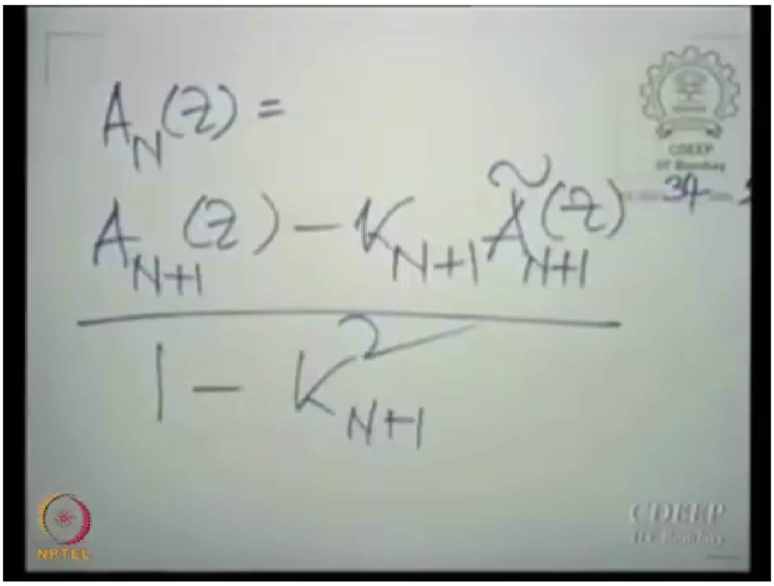

And $A_N(z) = \frac{A_{N+1}(z) - K_{N+1}\widetilde{A}_{N+1}(z)}{1 - K_{N+1}^2}$. We studied this recursion the last time. So, essentially we brought one stage backwards. So, I explained to you it was like peeling off one by one, the layers of a vegetable, the outer peel and the inner peel and so on.

So, you see you can now of course once you have reached A_N, then the coefficient of the highest power of the z^{-1} in A_N gives you K_N, and using K_N you can go peel of one more stage and you can keep peeling off stages until you reach A_0 which is 1. So, you can complete the largest recursion to obtain the K_1 to K_N. Now having completed the lattice recursions, we need to determine the c_0. So, let us go back to the structure we have drawn here, we need to, you see to complete the realization, we need to determine these c_0, c_0, c_1, c_{N-1} and c_N to realize the numerator now.

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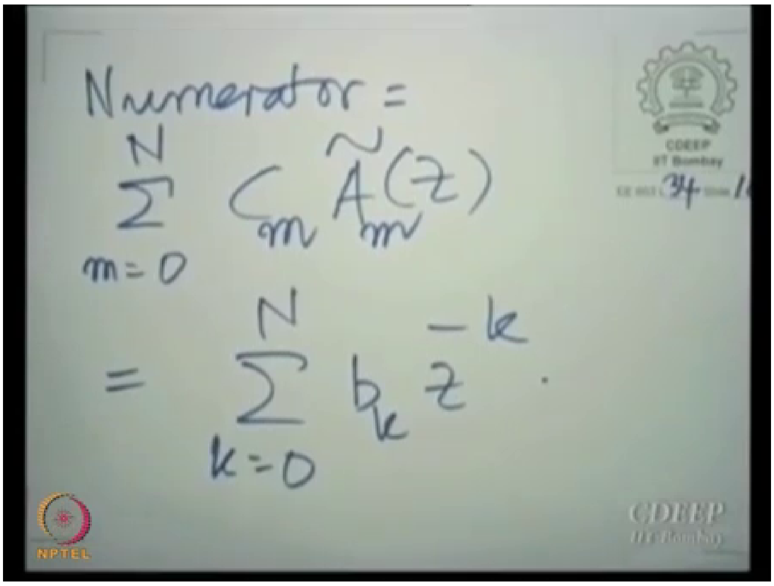

So, numerator is equal to $\sum_{m=0}^{N} c_m \tilde{A}_m(z)$. We have already done the long division and all that, so $\sum_{m=0}^{N} c_m \tilde{A}_m(z) = \sum_{k=0}^{N} b_k z^{-k}$. Now once again, employ the relationship between A_m and $\tilde{A}_m$. So, we have noted that the leading term that means, the coefficient of z^0 in all the A_Ns A_1, A_2, A_3 and so on, the coefficient of z^0 is 1 and further the coefficients of $\tilde{A}_N$ are the coefficients of A_N in reverse order and therefore, the coefficient of the highest power of z^{-1} in all the $\tilde{A}_N$. is 1 as well. Let us make this observation.

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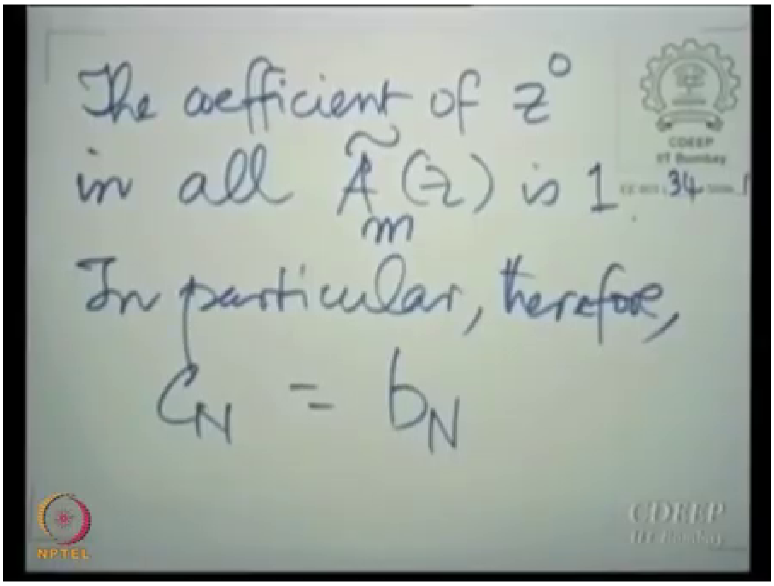

The coefficient of z^{0} in $\tilde{A}_N$ is 1, in particular, you see if you compare... where would the term z^{-m} arise? Only in $\tilde{A}_m$, none of the other $\tilde{A}_m$s could give the highest power of z^{-1} and what will the coefficient be of z^{-N} coming from here 1.

So, therefore, that would be multiplied by c_N. Therefore, the coefficient of comparing the coefficients of z^{-m} on both side $b_N = c_N$, $c_N = b_N$. Now once you have c_N, you can subtract that term from the left hand side and the right hand side and therefore, we can get.

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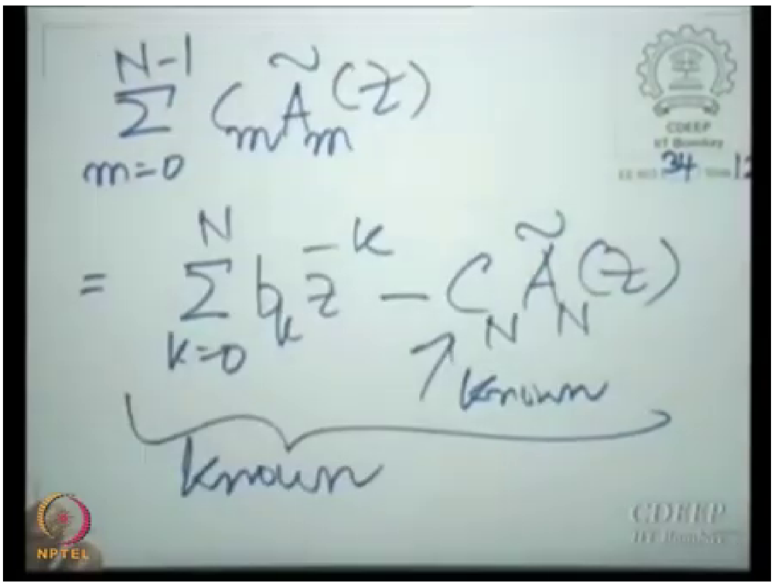

Summation $\sum_{m=0}^{N-1} c_m \widetilde{A}_m(z) = \sum_{k=0}^{N} b_k - c_N \widetilde{A}_N(z)$ and this is now known. So, this whole thing is known. Now since this whole thing is known, the same principle can be used to find c_{N-1}, the coefficient of c_{N-1} on the left hand side would only arise from $\widetilde{A}_{N-1}(z)$ and the coefficient of $z^{-(N-1)}$ $\widetilde{A}_{N-1}(z)$ is 1, that is going to get multiplied by c_{N-1} and whatever be the coefficient of $z^{-(N-1)}$ here is also going to be c_{N-1} therefore,

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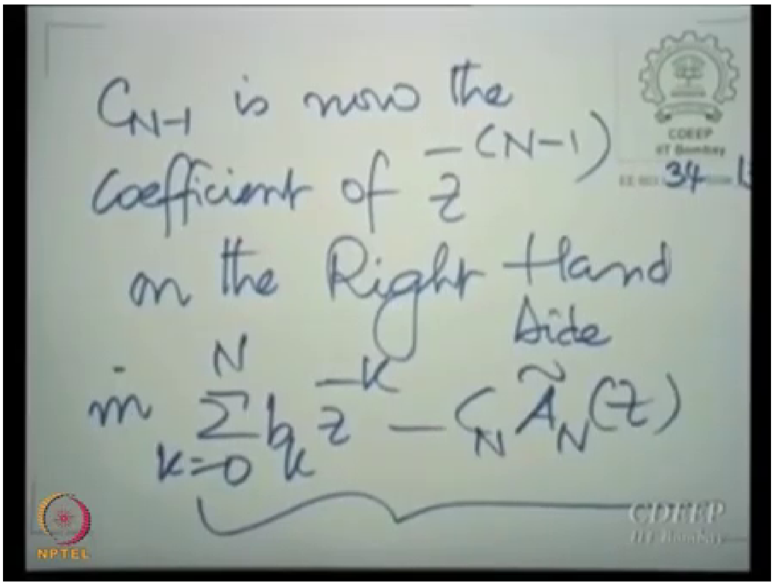

So, essentially what we are saying is c_{N-1}, is now the coefficient of $z^{-(N-1)}$ on the right hand side, in $\sum_{k=0}^{N} b_k - c_N \widetilde{A}_N(z)$, which is known. Now once you find c_{N-1}, you can apply exactly the same procedure to find c_{N-2}. So, this process can be continued recursively downwards until you reach c_0. So, let us make a remark.

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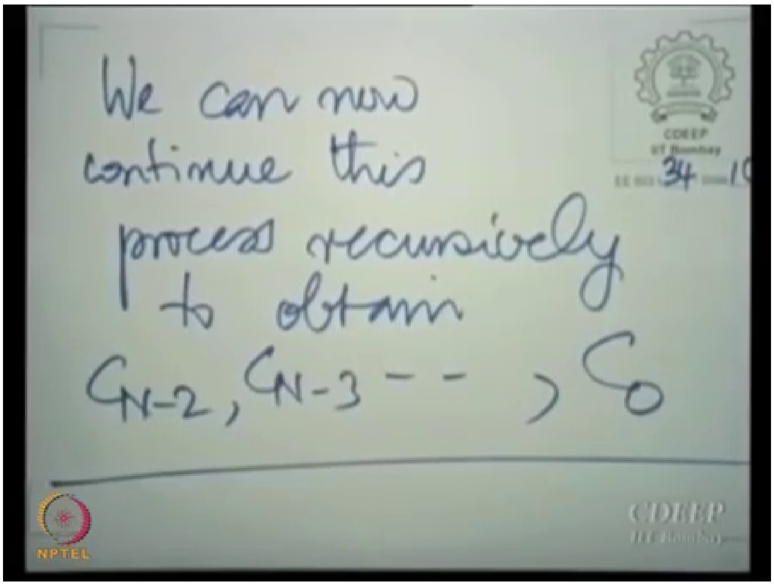

We can now continue this process recursively to obtain c_{N-2}, c_{N-3} all the way down to c_0. Subtract the corresponding term at the right hand side and look at the highest power of z^{-1}. Now having done that we have completed the lattice structure. So, we know in principle how to build the lattice structure. But now let us take an example. So, let us take a very simple example of a degree 2 rational system function.