Digital Signal Processing & Its Applications

Professor Vikram M. Gadre
Department of Electrical Engineering,
Indian Institute of Technology, Bombay

Lecture No. 35 a

Introduction to the DFT

So a warm welcome to the 35th lecture on the subject of Digital Signal Processing and its Applications. We have spent several lectures until the last one to look at certain basic principles in filter design. In fact now we are at a position where we can realize filters as well, with different structures. We have talked about filter design and realization. We have not quite exhausted all the possibilities either of design or realization.

The subject of design and realization of filters is more than 25 years old. So there have been several developments in the field and we cannot possibly capture all of them in a short span of time. Even so we have given some of the best known structures and approaches. And therefore as you do your assignment on filter design and realization you at least get a complete knowledge of one approach to filter design and realization with the lattice structure. With that then we now need to carry out the next step, namely, if we happen to have designed a filter and realized it, we must also be able to evaluate it.

And other than the question of evaluating a filter, it is also important for us to be able to depict the frequency of the transform domain on a computer. All this while we have assumed that the transform domain only needs to be calculated. We have not quite thought about how it should be depicted on a computer. And if we do depict it on a computer we have no choice but to discretize that domain. Nothing continuous can really be represented on a computer. So what we are going to do today is to discretize the frequency axis.

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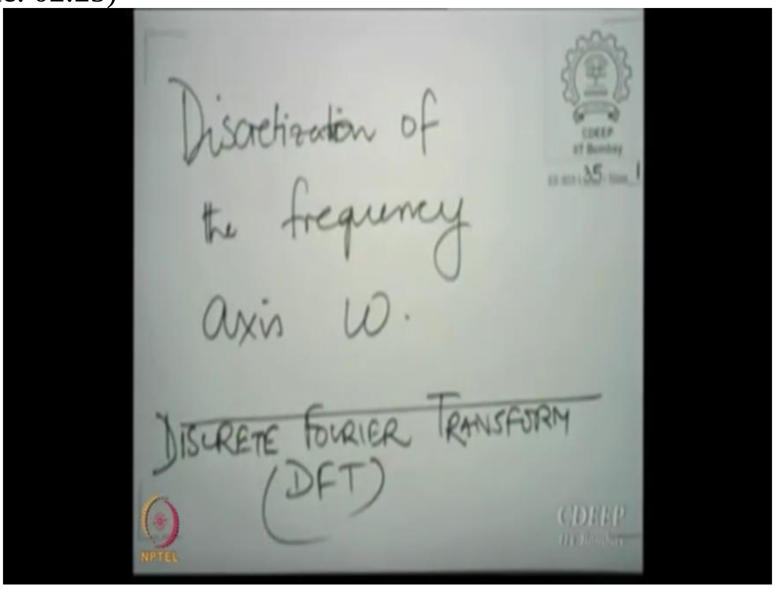

And that would lead us to what is called the discrete Fourier transform, as opposed to the discrete time Fourier transform. You see the discrete Fourier transform, or the DFT, as we would refer to it in the future, is discrete both in time and frequency.

The discrete time Fourier transform uses a signal or a sequence that is discrete in time but it is continuous in frequency. In contrast, the discrete Fourier transform, about which we are going to talk today, is discrete in both domains.

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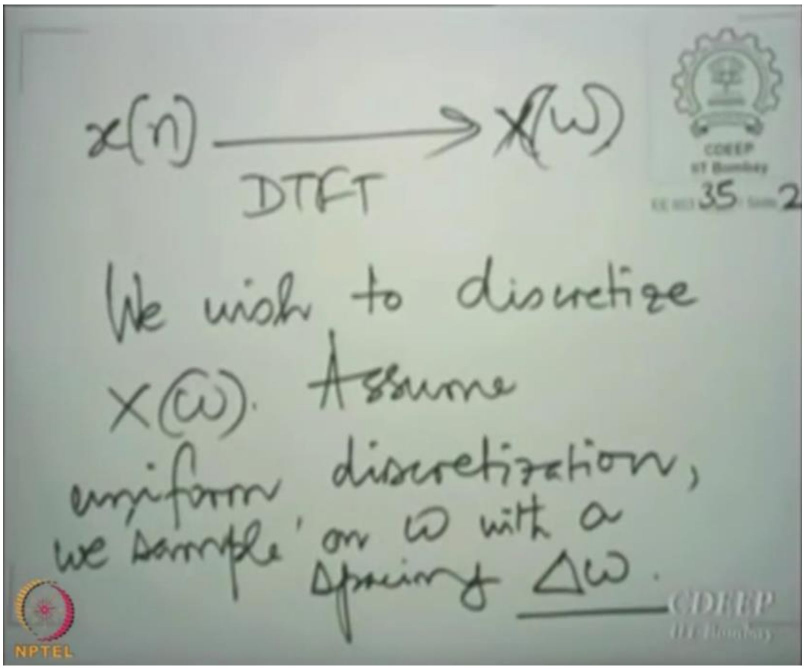

Let us begin with the following question. We have a sequence $x[n]$ with DTFT equal to $X(\omega)$. We wish to discretize $X(\omega)$, and let us for the moment assume uniform discretization. What I mean by that is we sample at a uniform rate, just like uniform sampling of a time domain signal. Now please remember that here sampling means sampling in the ω domain, not in the time domain. We sample on ω with a uniform spacing of, say,$\Delta\omega$.

When sampling in time, we would take one sample every T units of time. Now we are saying we will take one sample on the frequency axis at every $\Delta\omega$ interval on the frequency axis. You must remember that when you are taking the discrete-time Fourier transform of a sequence, the DTFT immediately has periodicity implicit in it. So there is really no point in taking samples all over the ω axis, you need to take samples only in one period of 2π. And this time instead of taking the standard interval $-\pi$ to π, let us take the standard interval 0 to 2π.

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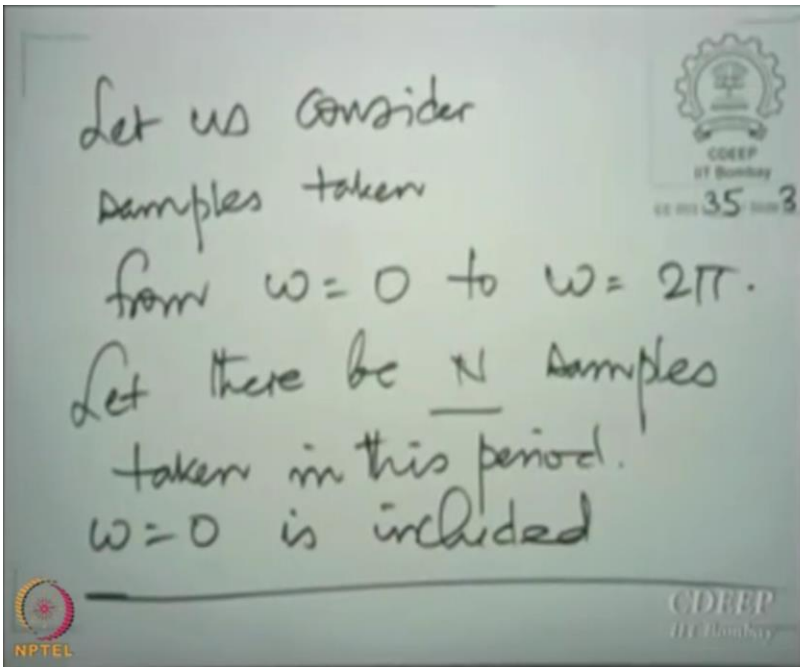

Let us consider the samples taken from $\omega = 0$ to $\omega = 2\pi$, and let there be N samples taken in this period. For good reasons, $\omega = 0$ is included as we would intuitively do normally. Now if you have N samples and if you have included $\omega = 0$, what you are saying effectively is that the spacing is $\frac{2\pi}{N}$. Thus,

$$\Delta\omega = \frac{2\pi}{N}$$

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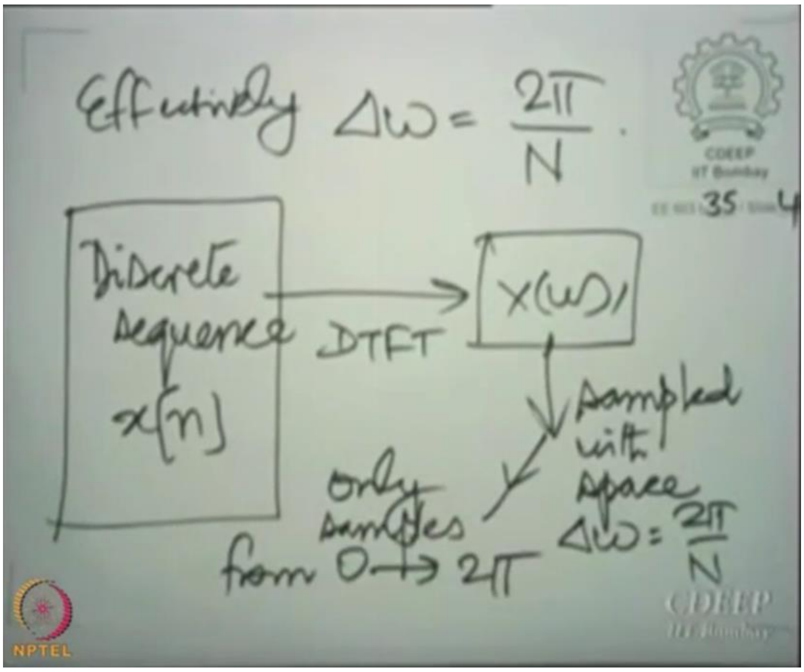

Let us analyze the situation. We have a discrete sequence $x[n]$. We have taken its DTFT that gives $X(\omega)$. We have sampled with a spacing $\Delta\omega = \frac{2\pi}{N}$ and we take only samples between 0 and 2π, for obvious reasons. All the other samples are simply repetitions of these.

So as we expect we have N samples there. Now we try to answer or analyze the adequacy or the inadequacy of these samples for understanding the sequence, and therefore the discrete-time Fourier transform. You see what was the question that we began with when we began this course? We said that if we want to deal with continuous time signals on a computer, we need to sample them.

Now when you sample them what is going to happen is a loss of information in some sense. Now how does that loss of information manifest? We saw it manifests not by not allowing a possible signal, but allowing too many possible signals which could have had those samples. So the loss of information is by creation of ambiguity.

Something similar is expected here. You see here we are performing sampling twice. We are sampling in time and then we are also sampling in frequency. For a moment, we could forget that we have sampled in time and we only look at the consequence of sampling in frequency. And we shall invoke the principle of duality.

Duality is a very basic principle in the Fourier transform. I shall not formally state it, but informally what duality says is that if you make a statement involving the time and the frequency domain for continuous time signals, then an equally correct statement can be obtained by reversing the roles of frequency and time. Now let us state the informal understanding of duality, an informal statement of duality.

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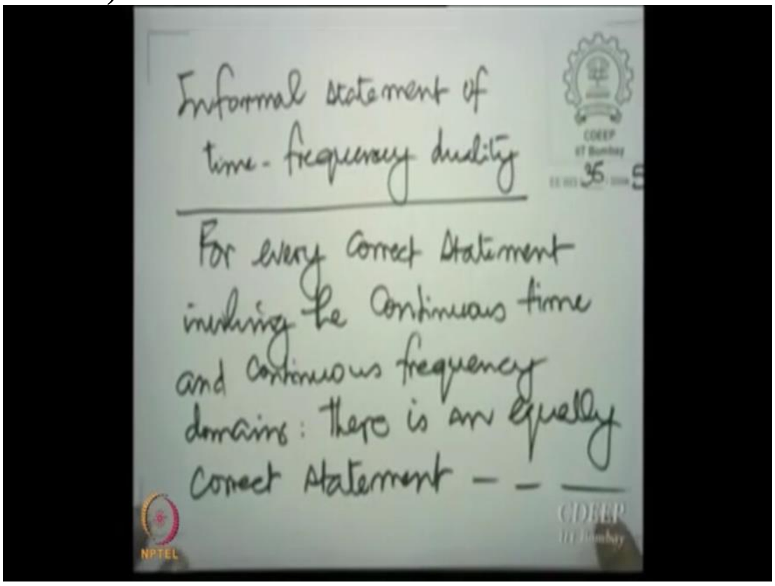

For every correct statement involving the continuous time and continuous frequency domains, there is an equally correct statement where the role of time and frequency has been reversed.

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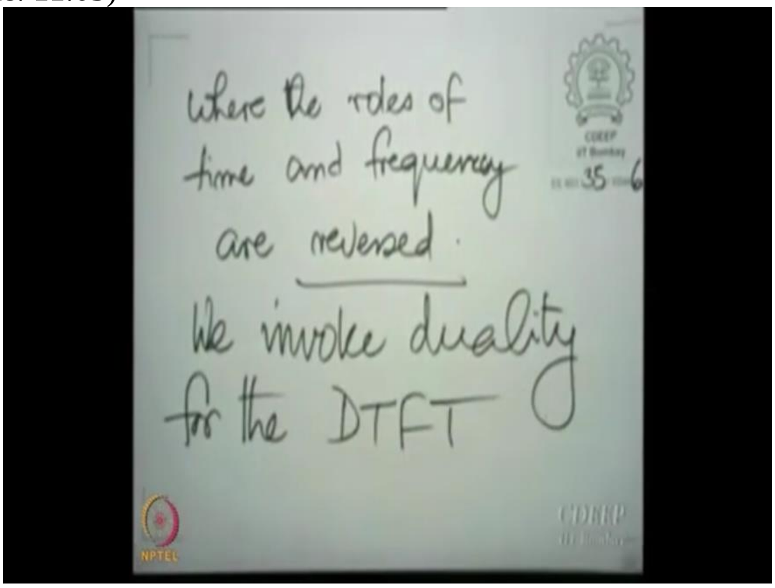

Now strictly this is for continuous time and continuous frequency, but with some skill we can also adapt it to discrete time and continuous frequency as we have here. For the moment, we can treat discrete time as a train of impulses, or a train of pulses approaching a train of impulses.

We have always focused on the portion of the frequency axis between 0 and π or between $-\pi$ and π, but we need not do so necessarily. We could consider the entire frequency axis, keep the periodicity and then work with that for the requirement of duality. So what I am saying is we invoke duality for the DTFT. And how do we invoke it? We analyze the effect of sampling.

When we sampled in time with a spacing of T, the consequence was to take the original spectrum, translate it by every multiple of $\frac{1}{T}$, and then add up each of these translations. So we took the original spectrum, the original Fourier transform, moved it by every multiple of the sampling frequency, that is $\frac{1}{T}$. If you were talking about the angular frequency axis, you would move it by every multiple of $\frac{2\pi}{T}$.

So we move it by every multiple of $\frac{2\pi}{T}$ or $\frac{1}{T}$ as suitable, and add up these carbon copies. If there is an overlap between these added spectra then you have aliasing. If there is no overlap, there is no aliasing. There is no distortion of the original spectrum if these carbon copies do not overlap with the original. Now exactly similar is the consequence of sampling in ω.

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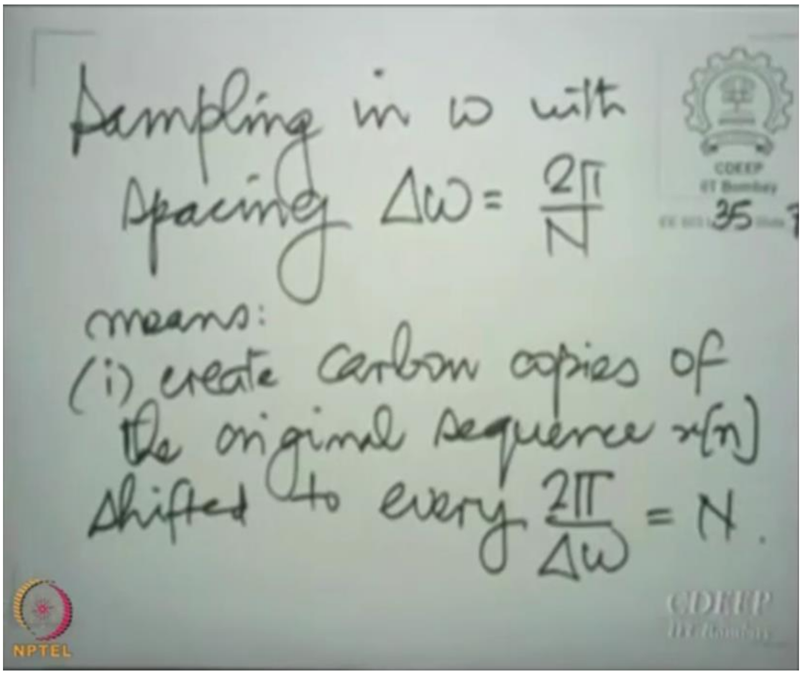

You see sampling in ω, with spacing $\Delta\omega = \frac{2\pi}{N}$ means:

(i) create carbon copies of the original sequence $x[n]$ shifted to every $\frac{2\pi}{\Delta\omega} = N$. Here we are talking about angular frequency, so the shift is by $\frac{2\pi}{\Delta\omega} = N$. Since you are talking about angular frequency you need to take the 2π divided by the spacing. If you were talking about the Hz frequency or the normalized Hz frequency then you would just take the reciprocal.

So that means you need to create carbon copies of the original sequence shifted to every integer multiple of $\frac{2\pi}{\Delta\omega}$, that is $x[n + rN] \; \forall \; r \in \mathbb{Z}$. Those are the carbon copies.

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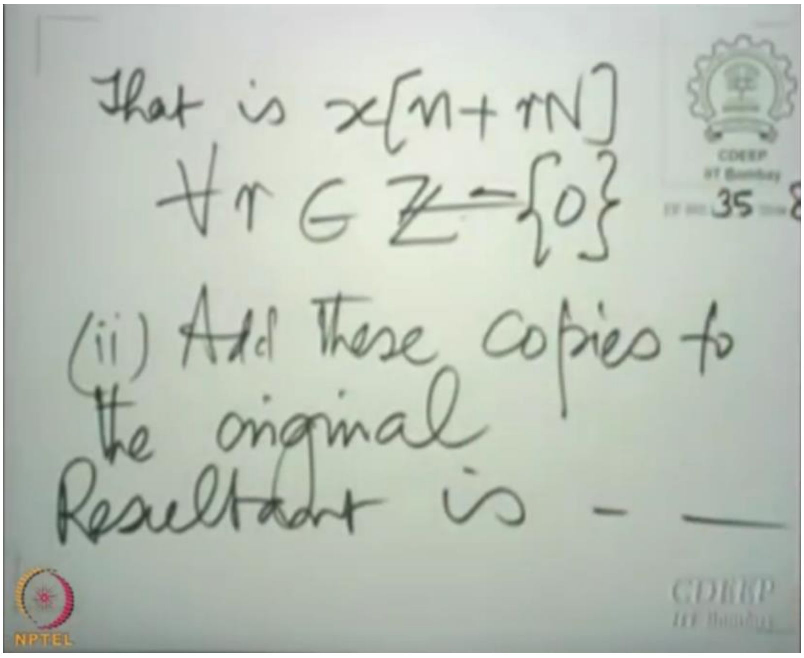

(ii) add these carbon copies, that means the resultant is essentially a sum of these carbon copies, including the one at $r = 0$. If you want to be very careful in delineating the copies from the original, you could say $\forall\, r \in \mathbb{Z} - \{0\}$ when expressing the sum of the copies. If you want to be very, very fussy you could write it like that.

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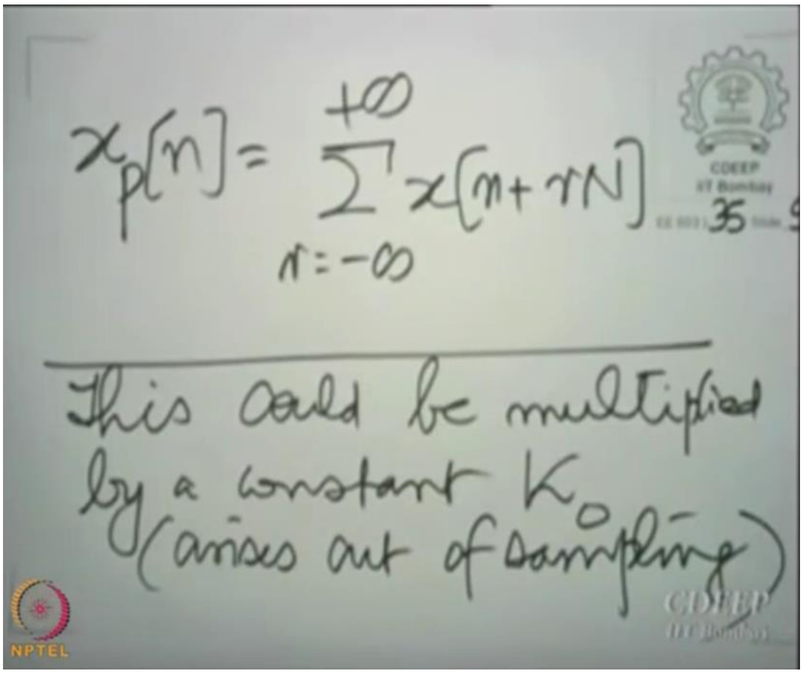

Let us call the resultant as $x_p[n]$, which is a periodic version of $x[n]$, given by:

$$x_p[n] \ = \sum_{r=-\infty}^{+\infty} x[n + rN]$$

So we have the original sequence translated by every multiple of N and added. Of course, this sum could be multiplied by a constant. It is called κ_0. This constant arises out of sampling.

During the process of sampling, each sample could be implicitly multiplied by a constant.But that is a minor issue. We shall either neglect that constant or assume that we have taken care in the process of sampling that it is made 1. If required, we could simply include the constant. It just multiplies the whole signal by a constant. It is not a very serious issue.

Now, just like we had aliasing in frequency, we could have aliasing in time. If these carbon copies of the original sequence overlap with the original sequence there is time domain aliasing.

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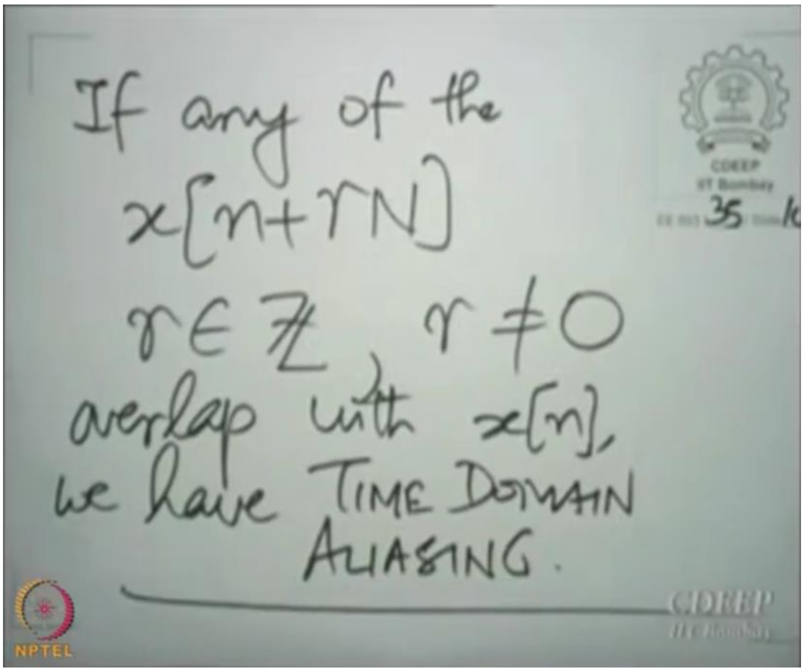

If any of the $x[n + rN], r \in \mathbb{Z}, r \neq 0$, overlap with $x[n]$, we have time-domain aliasing. And if there is no overlap, we don't have any aliasing. It is very clear from here that if a sequence is of infinite length and if you sample the ω axis you cannot avoid time domain aliasing. And that is not surprising. It is perfectly dual to what happens in frequency. If a signal is not bandlimited, and if you sample it in time, then its spectrum will have to be aliased without exception.

So time-domain aliasing is inevitable when you have an infinite length sequence, and frequency domain aliasing is inevitable when you have a signal which is not bandlimited. On the other hand, just like the case of bandlimited signals, if a sequence is of finite length, then it is possible to avoid time-domain aliasing. And the rule is very simple.

If you have sampled with a sampling rate of $\frac{2\pi}{N}$ on the frequency axis, you can avoid time-domain aliasing if the range over which the sequence is non-zero is $\leq N$.

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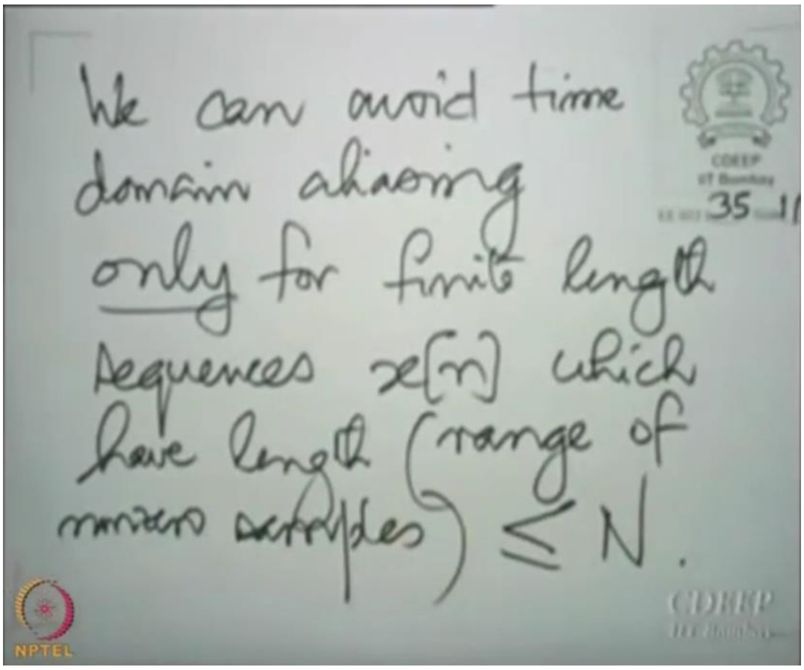

We can avoid time domain aliasing only for finite length sequences $x[n]$, which have lengths $\leq N$. Now when I say length, I mean the range of non-zero sample values. What it means is that if the sequence is shifted by N units, there is no overlap, either backward or forward.