Nonlinear and Adaptive Control

Dr. Shubhendu Bhasin
Department of Electrical Engineering
Indian Institute of Technology - Delhi

Lecture – 08
Robust Model Reference Adaptive Control part-2

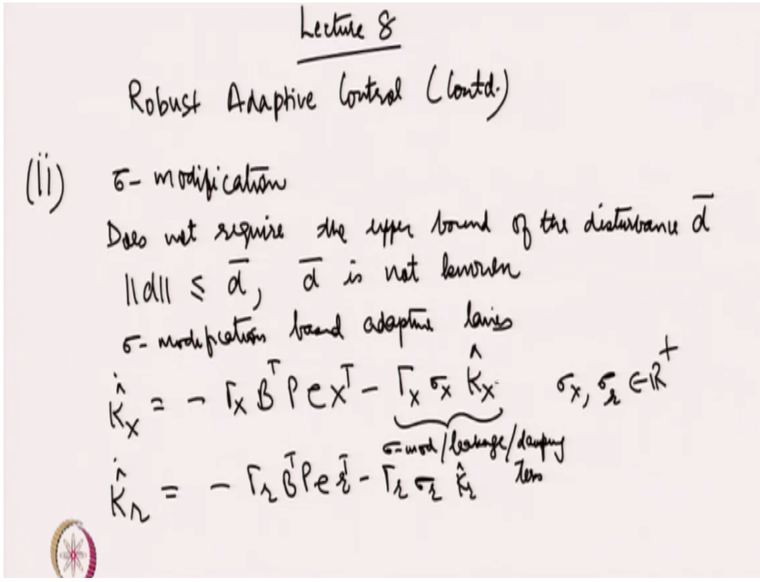
Welcome everyone to lecture 8 of this course on nonlinear and adaptive control. In the last class, we had studied how to design adaptive controllers for plants with external disturbances where the structure of these disturbances may not be known or they could be unmodeled dynamical effects. We have discussed that we could design a robust controller to account for such effects, but since the focus of this class is adaptive control, we are in fact going to robustify the adaptive update laws.

So in the last class we had discussed one of the ways in which we can add robust elements to these classical adaptive control laws. We had studied that, that zone approach were within a dead zone there is no adaptation. The adaptation of the parameter estimate happens only if the error is greater than a certain threshold and this threshold is dependent on the upper bound of the disturbance so if the error is larger, then the classical adaptive control laws will be used.

Once the error becomes smaller then the threshold which is called the dead zone there will be no adaptation. So what this ensures is that the parameter estimates stay bounded for all time. So using this approach we are able to guarantee boundedness of all the signals and also the control input.

One of the features of this technique is that the upper bound of the disturbances required to be known however in many cases we do not know what the upper bound of this disturbance is going to be. So this is a very practical situation because we may not be able to figure out what that upper bound d bar would be. So in that case if it is possible to still guarantee that the parameter estimates always remain bounded.

(Refer Slide Time: 02:43)



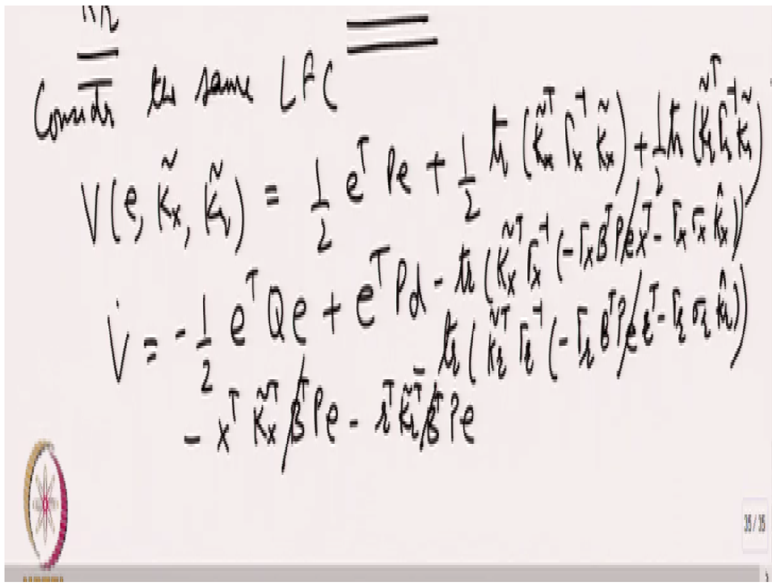
So in this class we will talk about 2 of these techniques. So the second technique for a robust adaptive control is sigma modification. So it does not require the upper bound of the disturbance. So although we know that the disturbance is upper bounded by d bar, but d bar is not known. So we of course cannot use the dead zone in this case because we will not be able to indicate what that dead zone is without knowing the value for d bar.

So what can be done in this case? So we can use the following sigma modification adaptive laws so kx hat dot. We have designed previously was given by - gamma x B transpose P * e x transpose. So what we do here is we add an additional term which we call as a sigma modification term like this so this is the adaptation term let us not put it in this way we just say that this term is the sigma mod term or the leakage term or the damping term.

So we do the similar modification in the update law for kr hat. So what do you think this term can do? So if you look closely this term consists of this sigma x and sigma r which are positive constants and it also includes kx hat. So this is kx hat dot and then we have a - kx hat. So this term damping effect on kx hat this can also be thought of a stabilizing effect on the kx hat dynamics. It is very similar to when we have x dot = - kx.

So in that scenario x would decay and go to 0. So similar effect is provided by these 2 terms where the first term is the adaptation term where the kx hat and kr hat are trying to learn the ideal values of kx and kr and just for the purpose of stabilization we have these damping terms which try and slow down the adaption of kx hat and kr hat and make sure that the estimates remain bounded for all time. So of course we need to prove how Lyapunov stability or boundedness can be proved in such a scenario so let us go ahead and jump into Lyapunov analysis and see if these modifications can indeed lead to some notion of stability or boundedness.

(Refer Slide Time: 07:44)

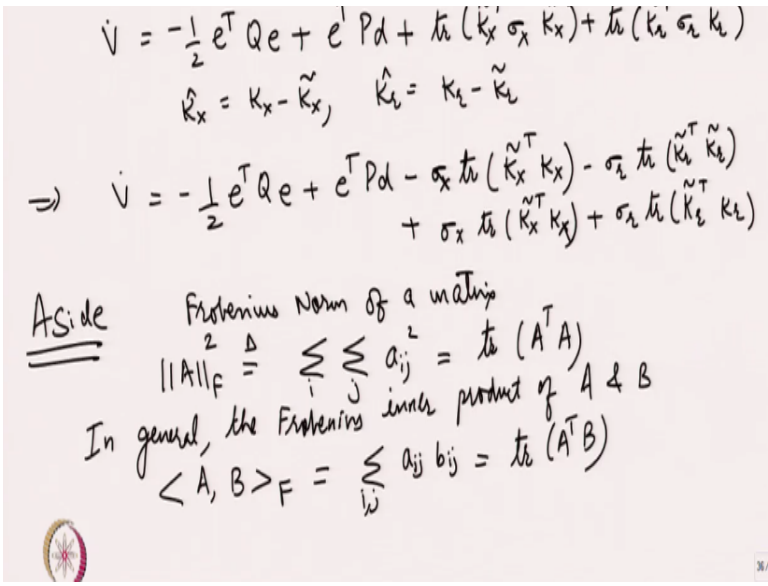


So let us consider the same Lyapunov function candidate as before so v is given by e as a function of ke, kx tilde, and kr tilde is given by 1/2 e transpose Pe + 1/2 trace of kx tilde transpose gamma x inverse kx tilde + 1/2 trace of kr tilde transpose gamma r inverse kr tilde. So this is our Lyapunov function candidate. So now we go head and take the time derivative. So we can skip a few steps there so it is a repetition of what we have done many time in this course so far.

So take the time derivative and V dot is given by - 1/2 e transpose Q e + e transpose pd. This term is coming from the disturbance which is acting on the plant and then we have a - trace of kx tilde transpose gamma x inverse and then we use the adaptive laws that we have mentioned here so - gamma x b transpose Pe x transpose - gamma x sigma x kx hat similarly for kr hat we substitute for the adaptive law.

Further we of course I missed the terms from the tracking error dynamics which cancel by these adaptive laws so we have this - x transpose kx tilde transpose * B transpose * Pe - r transpose * Kr tilde transpose * B transpose * Pe. So we have seen before that these terms in fact get cancel by these adaptive laws, the classical adaptation term in the adaptive law and what we are left with is this damping term and let us see how this damping term helps in accounting for this e transpose Pd term which is a disturbance term. So let us look at V dot now.

(Refer Slide Time: 11:10)

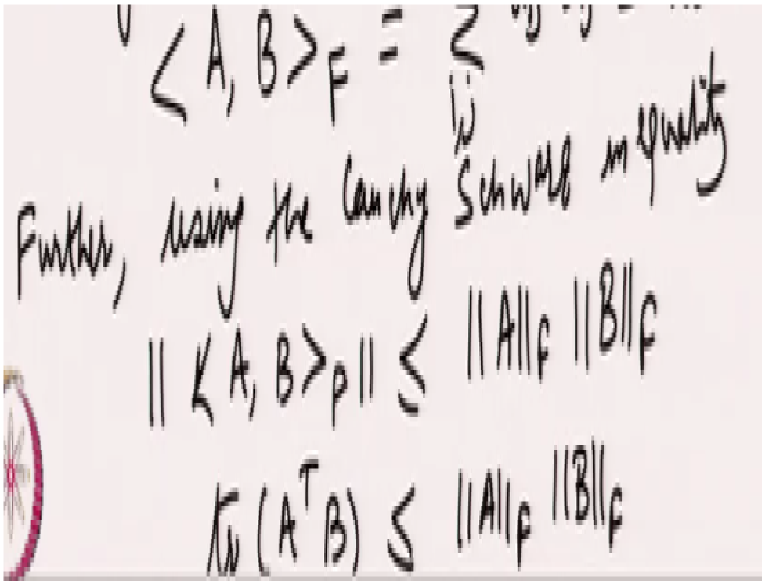


So V dot is - 1/2 e transpose * Qe + e transpose * Pd + trace of kx tilde transpose * sigma of course a scalar and can be moved out let us have it here right now + trace of kr tilde transpose * sigma r * kr hat. So now we can substitute for kx hat and kr hat and we know that kx hat is kx - kx tilde and kr hat is kr - kr tilde. So what we get is v dot - sigma x trace of kx tilde transpose kx - sigma r trace of kr tilde + sigma x trace of kx tilde transpose kx + sigma r trace * kr tilde transpose * kr.

Now I am going to consider aside and I will come back to this expression a little later, but aside I want to talk about the Frobenius norm and the Frobenius product of a matrix. So you may be aware that the Frobenius norm of a matrix is given as is defined as the square of the Frobenius norm is defined as the sum over all the elements of the matrix of the square is the square root of the sum of the squares of the elements.

So aij square which is also same as the trace of A transpose A. So in general the Frobenius inner product of matrices A and B is denoted by this expression and is given by the elementwise product of the elements of A and B so what we have is overall elements aij * bij and this is also the trace of A transpose B.

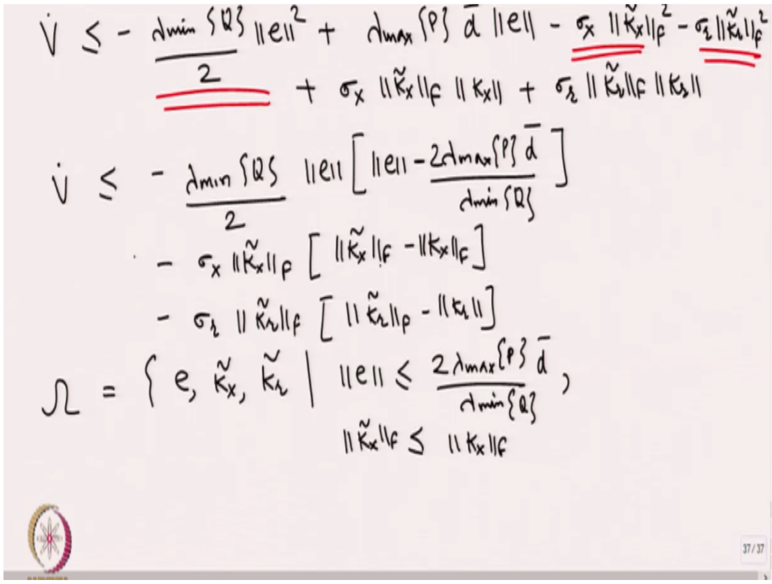
(Refer Slide Time: 16:55)



Further using the Cauchy Schwarz inequality for inner products we can say that the norm of this inner product is <= to the product of the Frobenius norms of the matrices A and B. So this is same as writing a trace of A transpose B <= the Frobenius norm of A * the Frobenius norm of B. This is using the Cauchy Schwarz inequality. So now using this aside we can now go back to the Lyapunov derivative.

So here what we can see also there is a slight error. So what we can see here is that the trace of Kx tilde transpose Kx tilde is same as saying that this is the square of the Frobenius norm of Kx tilde similarly this is the square of the Frobenius norm of Kr tilde and this can be replaced as by this can be in fact upper bounded the norm of this can be upper bounded by the product of the Frobenius form of Kx tilde and the Frobenius norm of Kx similarly for this term.

(Refer Slide Time: 19:08)



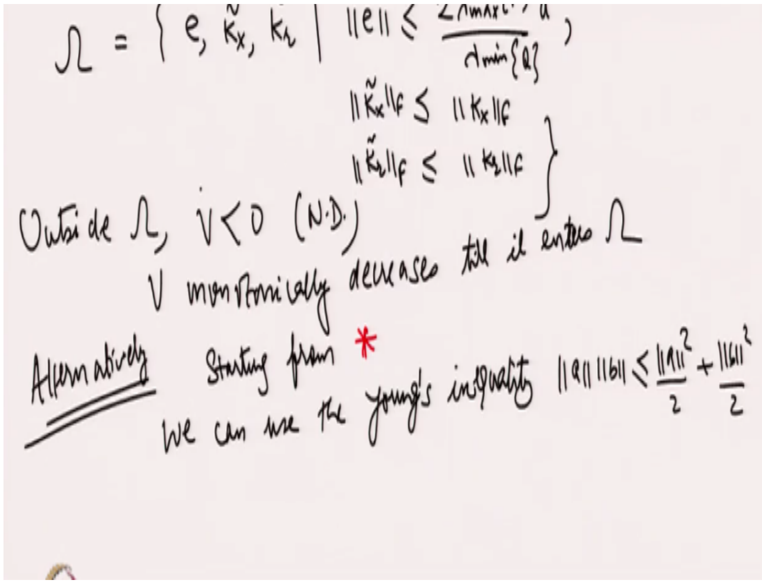
So now we can upper bound this expression of V dot as so the first term we can upper bound like we did before and we can also take the upper bound of the second term which is the disturbance term by using a maximum Eigen value of the matrix P * the upper bound of the disturbance d bar * the norm of e. So although d bar is not known it is okay for this to be to appear in Lyapunov analysis. As long as it does not appear in our control or adaptive laws it is fine.

Then - sigma x of kx square - sigma r + sigma x of + norm of kr tilde * Frobenius norm of kr. So what do we see from this expression for V dot. So what we see is that this term is good because it is a negative term. This term is also good because it is a negative term. This term is good because it is a negative term whereas all the other terms are not so good because so those are positive terms. So we need to find a way to dominate these positive terms using these negative terms.

So one way that we can go ahead with this analysis by combining these terms so that is say V dot is <= - lambda min of Q I take it outside and then I have e - 2 lambda max d bar/ lambda min of Q. Similarly, I combine the other terms - sigma x kx - sigma r. Now what we can say from here is that if you look at these terms in these square brackets.

And if these terms are positive terms then we can say that v dot is <= 0 and since v is positive and v dot is <= 0 what we can say is that v is bounded and we can further say that the error states e * kx tilde * kr tilde are also bounded. So we can construct a set. So let us construct set given by this and such that norm of e is <= 2 lambda max of P/lambda min of Q * d bar, kx tilde * F <= Kx.

(Refer Slide Time: 24:39)



And notice here that the norm of kx is simply a number so although we do not know what that number is but it is a constant. So kr tilde <= kr * F. So this is the set definition. It is basically all the values of e, kx tilde and kr tilde such that these errors are <= some constant. That constant may not be known but as long as they are within these values. We say that they are elements of the set. So this is a compact set.

This is a compact set where we say that all the errors are in fact bounded e, kx tilde, and kr tilde are bounded because the right hand side here are all these constant expressions. So if you are stage are outside this set so outside this set we say that we are outside this set then these terms in the square bracket are in fact positive and which means that V dot is in fact < 0. So it is negative definite because this includes all the 3 states so norm of e, norm of kx tilde, and norm of kr tilde.

So none of the states is missing here and that is why we say that v dot is in fact negative definite not negative semi definite. So if you are outside the set then V dot is < 0 which means that V will monotonically decrease. So V monotonically decreases from its initial value V of 0 and infinite time till it enters. So V monotonically decreased still the states e, kx tilde, and kr tilde in fact enter this set and inside this set of course we say that the error states are bounded.

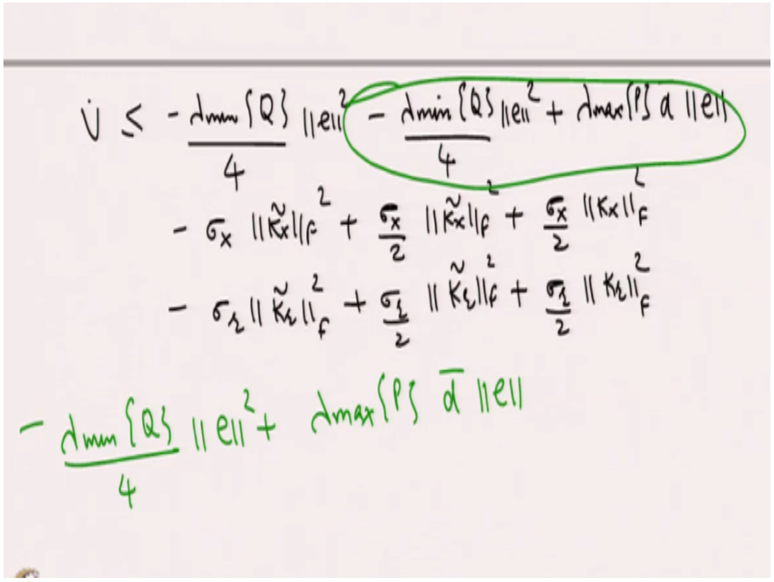
And because outside of this set since v is positive V dot is < 0 then we can say that v is bounded and e, kx tilde, and kr tilde are also bounded. So again what we have shown here is that all the error states the tracking error states and the parameters and the estimation error states are bounded for all time. So outside the set v dot is < 0 which means that V monotonically decreases from its initial value which means that the tracking error, the parameter estimation error would stay bounded.

Inside the set we cannot say about the sign for V dot. We cannot say whether V dot is <= 0 or > 0 since this is the sign of V dot is unknown but what is known is that this is the compact set where all the states, the error states are bounded. So in both the scenarios the parameter estimation errors are bounded and hence we can say that this sigma modification leads to boundedness of the parameter estimation error.

Alternatively, we can do this analysis in a more regress way so we can either prove by using the method that I have just shown or we can proceed with the Lyapunov analysis starting from say this equation. So let us call it equation star. So let us proceed from star and see how alternatively we can prove that the parameter estimation errors are bounded in fact.

We can probably say more than that so starting from star so we can use the Young's inequality that says that norm of a * norm of b <= norm of a square/2 + norm of b square/2. So we use that on certain terms in this expression. So we use that on this term and we use that on this term and we use that on this term. So on these 3 terms in fact we let us use it from this term.

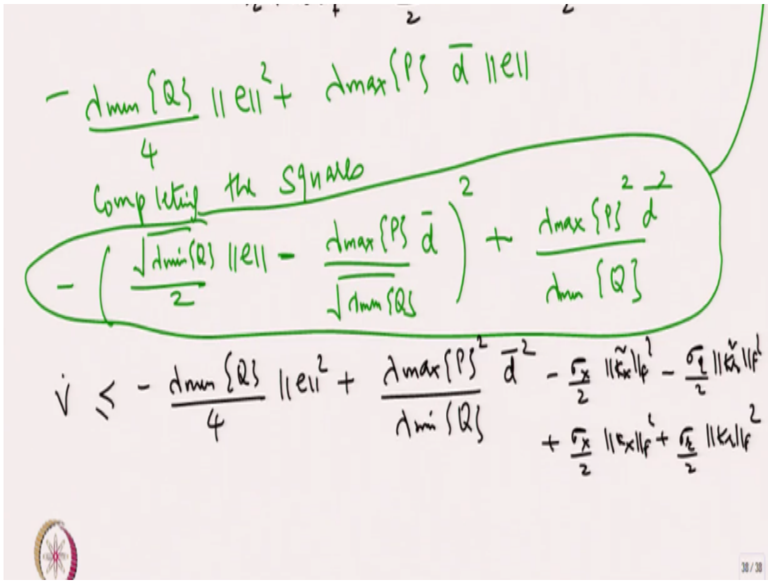
(Refer Slide Time: 31:33)



So we can say that V dot is <= so let us split the first term into 2 equal parts and we will see why we are doing this. So this kind of manipulation you should get used to doing when doing this Lyapunov analysis because this is used to upper bound certain terms and also damp out certain positive terms so we are dividing this into 2 parts + what we have is lambda max * d bar * e. Then what we have is the negative term involving kx tilde square we also have then we use Young's inequality and what we get is sigma x/2 * kx tilde square + sigma x/2 * kx square.

Similarly, for kr tilde we get sigma r/2 * kr * F + sigma r/2 * kr square. So sometime I forget the Frobenius norm. I hope that you understand that wherever I am missing I mean that these norm matrix that are in fact Frobenius norms. So now let us consider this term here. Let us consider this term here and try and manipulate this so what this term is - lambda min/4 * e square * lambda max * d bar * norm of e. So let us complete the squares.

(Refer Slide Time: 34:24)

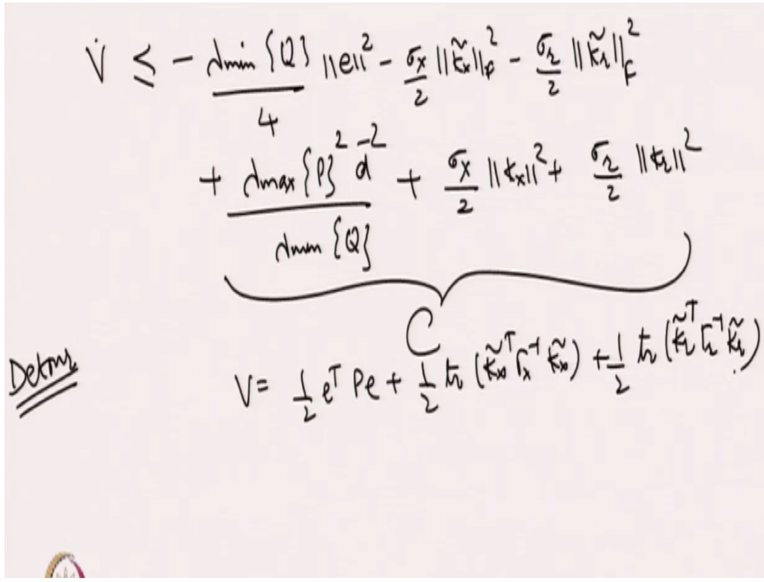


So completing the squares, so how do we complete the squares on this expression? So we add so we have a - a square term for example and this is say our 2 * AB and what we need to get is a - b square term. So from these 2 expression we extract what b is and then we add and subtract that term. So let us for this case let us complete the squares and what we get is our A in fact is square root of lambda min of Q/2 * e so the b is lambda max/square root of lambda min of Q * d bar square.

And then because we add and subtract the term that we subtract it is used to form this square term and the term which we added is left as a residue so that is lambda max square d bar square over lambda min of Q. So you can work this out. So if you substitute for this in the expression above what so we just replace this whole expression instead of this. So then what we can do is we can ignore or we can throw out this negative term and what that will do is that will further upper bound V dot.

So we can further upper bound V dot by throwing away this negative term. So what we are left with is V dot is <= - lambda min of Q/4 * e square + lambda max of P/lambda min * d bar square and then we have the other terms so - sigma x/2 kx square - sigma r/2 * kr square. Then we have these positive terms + sigma x/2 kx + sigma r/2 Kr * norm square.

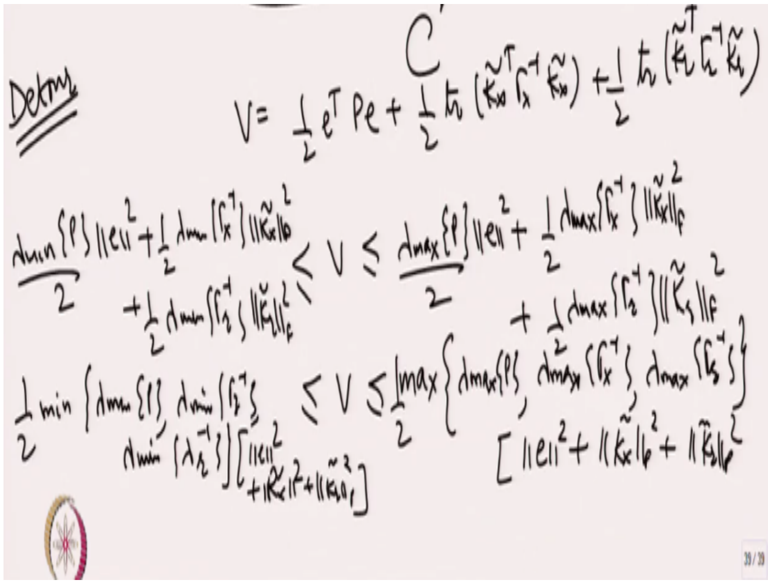
(Refer Slide Time: 38:12)



So now we collect the negative terms together and then and the positive terms together. So what we get is V dot is <= - lambda. So we just rearrange here. - lambda x/2 * kx tilde square - lambda r/2 * kr square and then we have the positive terms. So what we see from here is that all these positive terms are in fact constants. So let us call them as C and what about these negative terms. Can we somehow collect these negative terms and comment on how V of t evolves as t goes on.

So what we have to do for that is it is first to go back to the Lyapunov function. So now the motivation here is that we want to collect all these terms into something which is a function of V so we will see how that really helps us, but just have a detour what we look at is the original v. So V should remember was 1/2 e transpose Pe + 1/2 trace of kx tilde transpose gamma x inverse kx tilde + 1/2 trace of kr tilde transpose gamma r inverse and kr tilde.

(Refer Slide Time: 40:39)



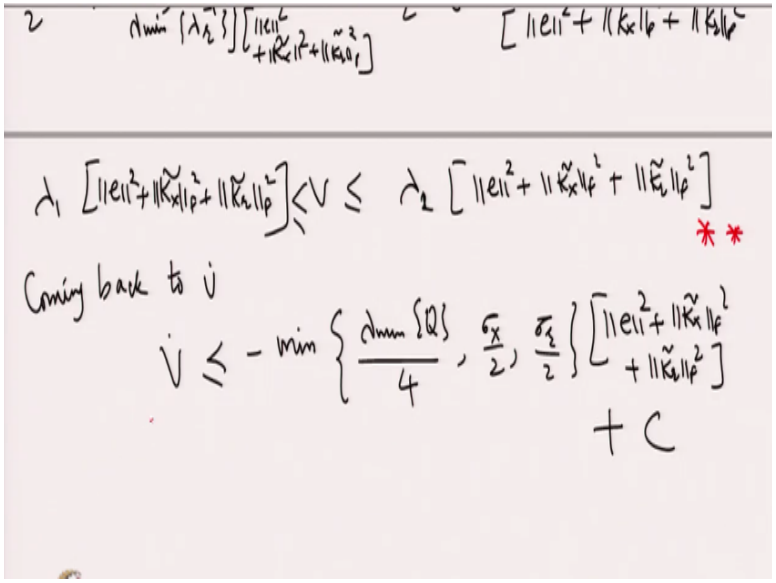
Can we upper and lower bound V in terms of norms of these errors? So which we can in fact do that so V can be upper bounded by lambda max of P * norm of e square + again using properties of trace and Frobenius norm we can say that this is 1/2 * lambda max * gamma x inverse * Frobenius norm of Kx tilde square + 1/2 * lambda max of gamma r inverse * norm of kr tilde square.

The lower bound can similarly be calculated as lambda min of p * e square + 1/2 * lambda min of gamma x inverse * kx tilde square + 1/2 * lambda min * gamma r inverse kr * tilde square. So now can we collect these error terms together? So again we can further upper and lower bound the expression for V as so remember this is the detour. We were on V dot and we just started developing the upper and lower bound for Lyapunov function itself.

So if we consider the maximum coefficients of all these then we can further upper bound V using the maximum of lambda max so 1/2 the maximum of lambda max * lambda max of gamma x inverse and lambda max of gamma r inverse. So we can * what we have is norm of e square + norm of kx tilde square + norm of kr tilde square. Similarly, here we can have 1/2 the minimum of lambda min of P * lambda min of gamma x inverse * lambda min of lambda r inverse * norm of e square + kx tilde square + kr tilde square.

So it came untidy here, but I hope you get the idea.

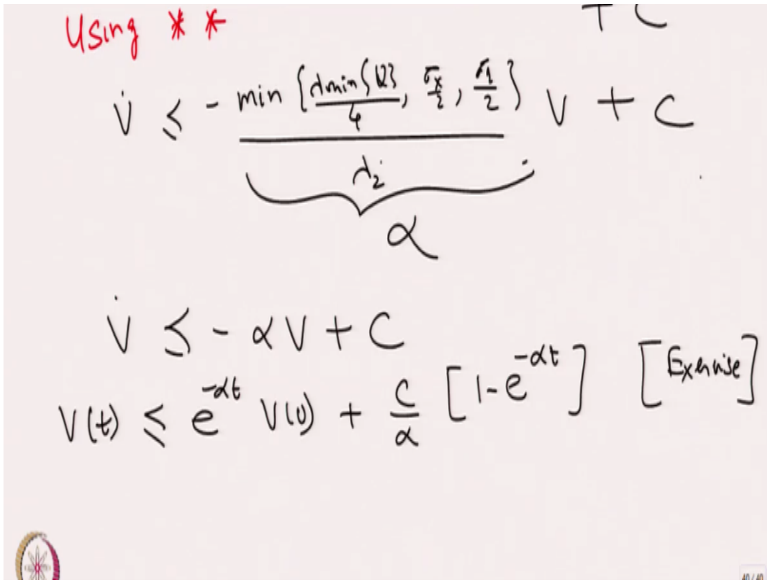
(Refer Slide Time: 44:36)



So what we can in fact say in a tidy way is that V can be upper and lower bounded by some lambda 1 * e square + kx + norm of kr so let us make this as lambda 2 and lambda 1 of e square + kx. So now can we use that in our expression for V dot. So we can again rewrite our V dot by collecting these negative terms together so if you look at V dot we can collect these 3 terms together and we can further upper bound V dot as so coming back to V dot so the d2 ends here.

So V dot is further upper bounded as - minimum of lambda min of Q, lambda x/2, lambda r/2. Then we have so this is how we collect these terms together and we will see how this helps us + c. So now we can say that we can use this equation number double star and we can replace this term in the square brackets by something which is smaller than that so which is v/lambda 2. So we see here that v/lambda 2 is smaller than the sum of squares of these error terms. So if you replace v/lambda2 in place of these square brackets we end up further upper bounding V dot.

(Refer Slide Time: 47:55)

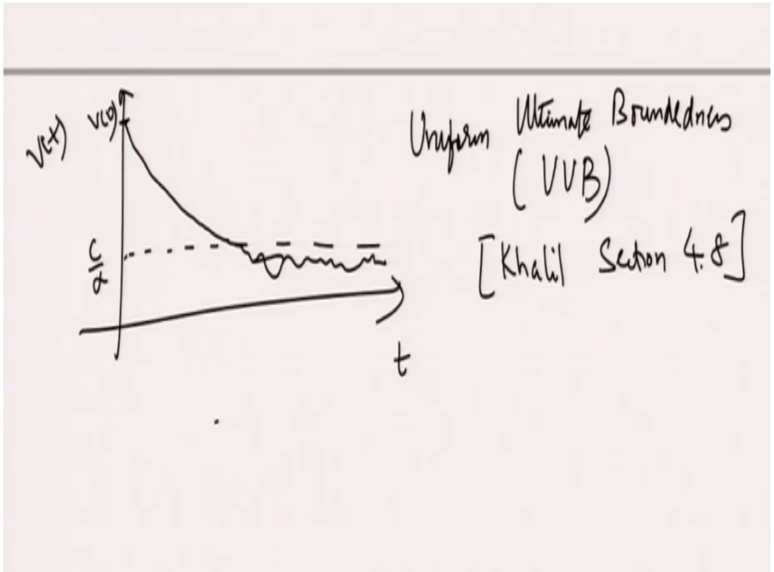


So what we do is we say that so using V dot is <= - minimum of lambda min/lambda 2 * v + C. Let us call this a coefficient of V/alpha. So then we can write V dot as - alpha v + c. So what this means that alpha is a constant and C is also a constant and this is in fact a linear differential inequality which you can solve so the solution of this linear differential so if this term suppose C is not there then this is simply V exponentially decaying to 0 from its initial condition V of 0.

(())(49:24) of this term C what we see is that V will not decay down to 0 in fact it will reach a region and stay within hat region for all future time. So it will exponentially decay to a region and then for all future time it will stay within that region. So this is how this v of t evolves with time. So we can say that V of t when solving this linear differential inequality what we get is V of t to be < e to the power - alpha t * V of 0.

So assuming time starts at t = 0 + C/alpha 1 - e to the power - alpha t. So this is you know for you as an exercise to solve this and prove that from this we get this. So it is not very hard to solve it you have to use your linear system concepts because it suggests the linear system which is driven by an input which is constant in this case.

(Refer Slide Time: 50:19)



So what this means is so from this expression we can say that we can combine the exponential terms together and we see that starting from v of 0, v of t exponentially decays until it reaches a region C/alpha and within that region it scales for all future time. It is not able to come out of that region. So what I am trying to say is that if we construct what our V(t) involves the time so starting so let us say we have this region given by C/alpha and let us say that v(0) initially starts from something which is > C/alpha.

So starting from this it exponentially decays until it reaches this region and then it goes inside this region and stays there for all future time. So it is similar to what the kind of argument that we had made earlier that the states are outside the set then V dot is < 0 which means that v monotonically decreases until it reaches this compact set and it stays there for all future time. So since V does that and it is V is a sum of squares of these error states so we say that the error states e of t, kx tilde, and kr tilde also behave in a similar way.

You can prove that mathematically. So this kind of a result is called as uniform ultimate boundedness. In short it is UUB. So it is not a Lyapunov stability. We do not use any of the Lyapunov theorems here but we were able to show is that all the error states they remain bounded in fact we say more than that we say that the error states starting from a region which is > C/alpha they exponentially converge and in finite time they enter this compact set and for all future time they stay within this compact set.

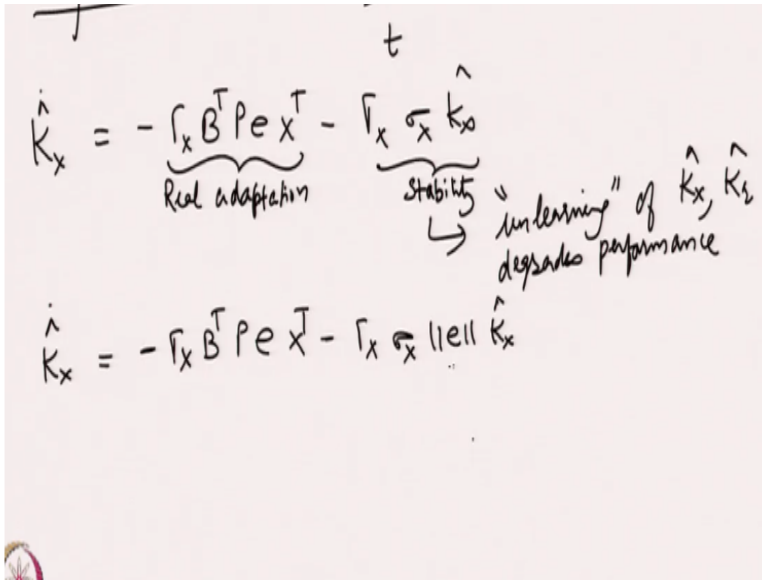
So somehow if we can reduce the size of this compact set we can in fact say that these sets become very close to 0 which is what is desired. So if we want to know more about this UUB result you can refer the book by Khalil and you can look at the section 4.8. So what sigma mod dos that it does not require the upper bound on the disturbance and it is still able to guarantee Lyapunov some kind of boundedness result.

In fact, we have a uniform ultimate boundedness result which is fairly good for practical systems which are subjected to external disturbances, but even in this case suppose we use this modification and the real plant does not have any disturbance so it is actually a disturbance free case, but if we use this sigma modified update law then the cause of the damping term because of this leakage term we will always have we will always have this region within which the state finalize.

So we will not be able to prove that the tracking error will asymptotically go to 0 for the disturbance free case. All we can say is that the tracking error will be uniformly ultimately bounded we may be able to reduce the size of this UUB region, but we cannot say that the tracking error asymptotically goes to 0 for the disturbance free case, but for the case where there might be a disturbance this sigma modification is robust to such disturbances.

So they provide this damping term which slows down the convergence of kx hat and kr hat and thus provide the stability in this case.

(Refer Slide Time: 55:53)



So if we look at the adaptive law for kx hat and kr hat we find that there is a small issue with these laws and there is (()) (56:00) further improvement. So as I said this is the real adaptation term. This term provides the adaptation for kx hat and this term which is the damping term is added for stability and it prevents the phenomena of parameter drift. So in the case that there is a disturbance this makes sure that the estimates are always bounded.

And we do not need to make any assumptions about the knowledge of the upper bound of the disturbance or the knowledge of the unknown parameters so in this place sigma mod is a fairly effective method, but what happens when the error becomes very small so as the error becomes very small we see that the effect of this adaptation term becomes very small and this term the sigma mod term may become dominant and in the case where e becomes very small.

So what happen in that scenario is that kx hat dot is mainly driven by the second term and what does the second term do it tries to drive kx hat to 0 which may not be the true in fact which is not the true value for kx hat. kx hat has an ideal value of kx. So this adaptation term is trying to make sure that kx hat somewhere become close to kx whereas this stabilizing term is trying to drive kx hat to 0.

So although it guarantees that we have boundedness, but this also leads to unlearning. So this term leads to unlearning. Whatever kx hat has learnt so far because e becomes very small this adaptation term has less effect and kx hat slowly starts to decay to 0 so this is the unlearning phenomenon of kx hat and kr hat. So that degrades performance so as I said if you bring in robustness your performance will degrade.

There is always a trade of between robustness and performance and this clearly shows that the performance might degrade although we have been able to show that this is robust to external disturbances. So as much as possible we would like this adaptation term to dominate because that is the real learning term and this damping term should only be able to keep the parameters from drifting away in the presence of disturbances.

So this is what we want. So if the sigma x which is the gain of the sigma mod term is very high then that could lead to the unlearning phenomenon or in the case where the tracking error becomes very small so we have to guard again such phenomena. So how do we overcome such a case that we will cover in the next lecture, but I would like to give you just a little bit of a hint as to what we might be able to do. Suppose we modify the update law as - gamma x * B transpose * Pe * x transpose - gamma x * sigma x * norm of e * kx hat.

Suppose I introduce this norm of e in the damping term so what does it do? So one thing is for sure is that when the error becomes small this terms, the damping term also becomes small. So the unlearning effect can be mitigated in some way by including the norm of e term in the second term as well of the update law.

So not only will this term becomes small, but this second term also will become small if the tracking error becomes very small. So the only thing we need to prove is that with this update law can we make sure that the signals are bounded. Can we make sure that the parameter estimates remain bounded? So that is what we will prove in the next class. Thank you.