Microwave Engineering
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Lecture 12
Impedance Matching using Shunt Stub, Double Stub and Quarter Wave Transformer

We have seen the impedance matching using a series stub. Let us now consider another type of stub matching, the impedance matching with a shunt stub. We first derive the analytical expressions, and then we will solve an example problem of shunt stub matching using Smith chart.

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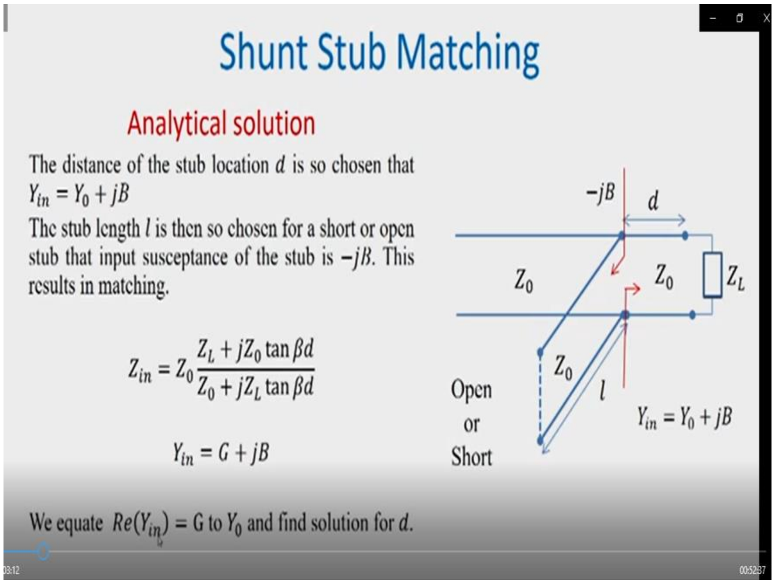
The distance of the stub location d is so chosen that $Y_{in} = Y_0 + jB$

The stub length l is then so chosen for a short or open stub that input susceptance of the stub is $-jB$. This results in matching.

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan \beta d}{Z_0 + jZ_L \tan \beta d}$$

$$Y_{in} = G + jB$$

We equate $Re(Y_{in}) = G$ to Y_0 and find solution for d.



So, we start with an impedance Z_L. Now, this Z_L is to be matched to a transmission line having characteristic impedance Z_0 using a shunt stub. So, first of all, what we do, we find out a distance d from the load impedance Z_L where if we look we find out Y_{in} at this point becomes equal to Y_0 that means 1 by Z_0 and it will have a susceptance component jB.

Now, so the distance of the stub location d is so chosen that Y_{in} becomes equal Y_0 plus jB. Now, once we do that next, we put a stub of length l, and it may be an open or a short stub and the length l is selected in such a way that if we look from this point into the stub we get minus jB. So the stub length l is chosen for a short or open stub that input susceptance of the stub is minus jB, and this plus jB minus jB will get canceled, and that will result in matching.

So, we know that Z_{in} is given by Z_0 Z_L plus jZ naught tan Beta d divided by Z_0 plus jZ_l tan Beta d and with this, we develop the analytical solution. So we can write Y_{in} equal to 1 by Zin and which will be G plus jB. We have already stated that we choose this d in such a way that this G becomes equal to Y_0. That means we equate the real part of Y_{in} equal to G to Y_0 and find the solution for d.

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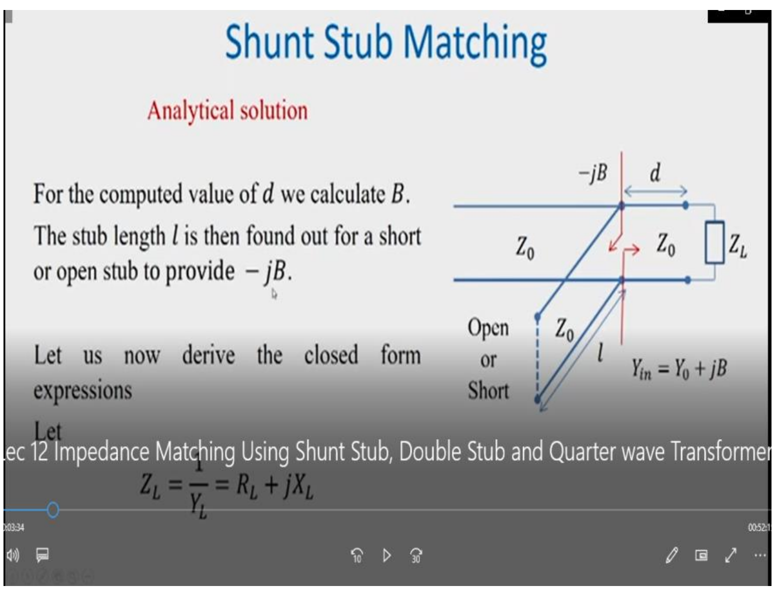
Let us now derive the closed form expressions

Let

$$Z_L = \frac{1}{Y_L} = R_L + jX_L$$



Then, once we have the computed value of d with this we find out B and then the stub length l is found for a short or open stub to provide minus jB.

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Similar to series stub matching, let $t = \tan\beta d$

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan\beta d}{Z_0 + jZ_L \tan\beta d}$$

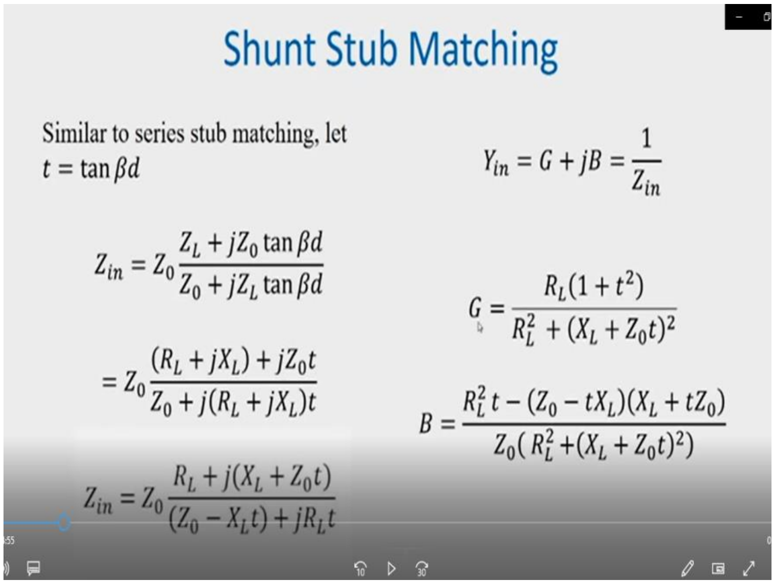
$$= Z_0 \frac{(R_L + jX_L) + jZ_0 t}{Z_0 + j(R_L + jX_L)t}$$

$$Z_{in} = Z_0 \frac{R_L + j(X_L + Z_0 t)}{(Z_0 - X_L t) + jR_L t}$$

$$Y_{in} = G + jB = \frac{1}{Z_{in}}$$

$$G = \frac{R_L(1 + t^2)}{R_L^2 + (X_L + Z_0 t)^2}$$

$$B = \frac{R_L^2 t - (Z_0 - tX_L)(X_L + tZ_0)}{Z_0(R_L^2 + (X_L + Z_0 t)^2)}$$



So, as in the case of the series stub, matching let us replace this tan Beta d by t. Then this equation Z_{in} can be put in this form tan Beta d replaced with t, and then we can group the real and imaginary terms, and we get this resulting expression. Z_{in} is equal to Z_0 RL plus J X_L plus Z_0 t divided by Z_0 minus X_Lt plus jR_Lt. Now Y_{in} is equal to G plus jB is equal to 1 by Z_{in}. So, from this expression we can calculate the expressions for G and B. So that we can do by multiplying the numerator and denominator by the conjugate of this term, then this will become actually, once we multiply this conjugate it will become real, and then we group again the real and imaginary part and in that way we can find the expression for G and B.

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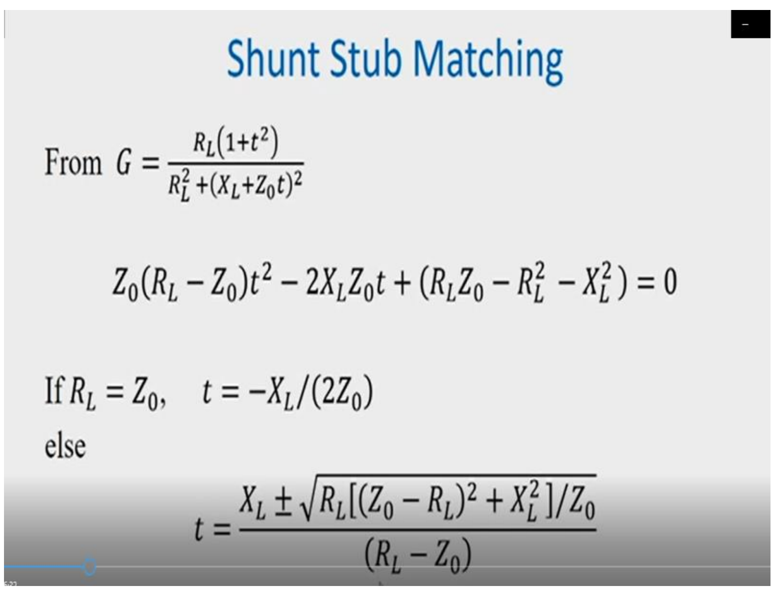
From $G = \frac{R_L(1+t^2)}{R_L^2 + (X_L+Z_0 t)^2}$

$$Z_0(R_L - Z_0)t^2 - 2X_L Z_0 t + (R_L Z_0 - R_L^2 - X_L^2) = 0$$

If $R_L = Z_0$, $\quad t = -X_L/(2Z_0)$

else

$$t = \frac{X_L \pm \sqrt{R_L[(Z_0 - R_L)^2 + X_L^2]/Z_0}}{(R_L - Z_0)}$$



Now, we start with the expression for G, and once we equate it to Y_0 then we can solve for t by forming this quadratic equation. So, here you put G is equal to Y_0 equal to 1 by Z_0 and then rearrange the terms, then we will get this quadratic equation. Now, here you note that if R_L is equal to Z_0 then this term becomes zero that means the real part of the load impedance if it is same as the characteristic impedance of the line it is to be matched then we get a very simple solution for t which is – X_L by 2 Z_0 because this R_L Z_0 will get canceled with RL square so we will be essentially having minus $X_L$square divided by 2X_L Z_0 and this will give t equal to minus X_L by 2 Z_0 but if R_L

is not equal to Z_0 then t is equal to X_L plus minus root R_L Z_0 minus R_L square plus X_L square whole thing divided by Z_0 and in the denominator we have R_L minus Z_0.

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We get two solutions for d which are given by

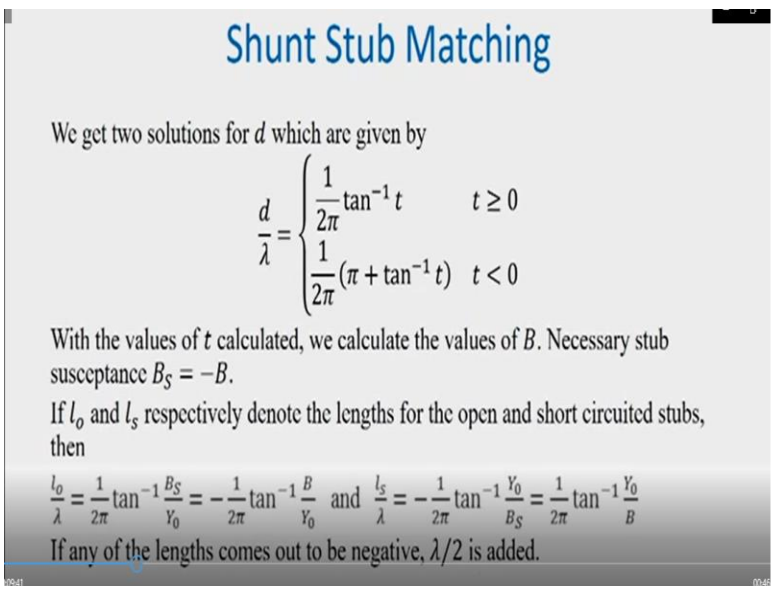
$$\frac{d}{\lambda} = \begin{cases} \frac{1}{2\pi}\tan^{-1} t & t \geq 0 \\ \frac{1}{2\pi}(\pi + \tan^{-1} t) & t < 0 \end{cases}$$

With the values of t calculated, we calculate the values of B. Necessary stub reactance $B_S = -B$.

If l_o and l_s respectively denote the lengths for the open and short circuited stubs, then

$\frac{l_o}{\lambda} = \frac{1}{2\pi}\tan^{-1}\frac{B_S}{Y_0} = -\frac{1}{2\pi}\tan^{-1}\frac{B}{Y_0}$ and $\frac{l_s}{\lambda} = -\frac{1}{2\pi}\tan^{-1}\frac{Y_0}{B_S} = \frac{1}{2\pi}\tan^{-1}\frac{Y_0}{B}$

If any of the lengths comes out to be negative, $\lambda/2$ is added.



So, since the equation in t is a quadratic equation, we will get two values of t, two solutions for t and correspondingly we will get two solutions for d. We write d by Lambda is equal to 1 by 2pi tan inverse t, when t is greater than zero, this comes directly from t is equal to tan Beta d and Beta is 2 pi by Lambda so when t is less than zero to get a positive length d we solve 1 by 2 pi, pi plus tan inverse t to be equal to d by Lambda and once the values of t are calculated we can calculate the value of B by substituting the value of t in the expression for B. Since we are having two solutions for t, we will also get two solutions for B and the necessary stub susceptance Bs is equal to minus B.

Let us denote the length of the open and short-circuited stubs by L_o and L_s then we can find out Lo by Lambda is equal to 1 by 2 pi tan inverse Bs by Y_0 and this can be written equal to minus 1 by 2pi tan inverse B by Y_0 and similarly for the open-circuited stub L_s by Lambda is equal to minus 1 by 2 pi tan inverse Y_0 by Bs which can be written as 1 by 2pi tan inverse Y_0 by B. So, we have found out the solution for d, the location of the stub from the load impedance and also we have found out the expression for the length of the stub and depending upon whether an open circuit at stub or a short-circuited stub is being used, we can calculate their lengths.It may be noted that if this length while computing comes out to be negative then we add Lambda by 2.

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Let us consider an example where $Z_L = 100 + j60\ \Omega$ is to be matched to a $50\ \Omega$ line. By applying the analytical solutions we get:

$t = 3.4091 = t_1$ and $t = -1.0091 = t_2$. We get two solutions for d

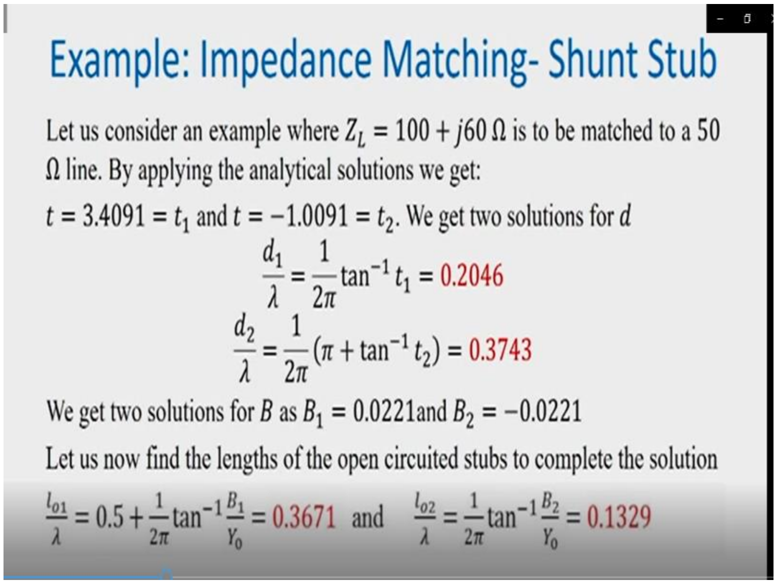
$$\frac{d_1}{\lambda} = \frac{1}{2\pi}\tan^{-1} t_1 = 0.2046$$

$$\frac{d_2}{\lambda} = \frac{1}{2\pi}(\pi + \tan^{-1} t_2) = 0.3743$$

We get two solutions for B as $B_1 = 0.0221$and $B_2 = -0.0221$

Let us now find the lengths of the open circuited stubs to complete the solution

$\frac{l_{o1}}{\lambda} = \frac{1}{2\pi}\tan^{-1}\frac{B_1}{Y_0}$ =0.3671 and $\frac{l_{o2}}{\lambda} = 0.5 + \frac{1}{2\pi}\tan^{-1}\frac{B_2}{Y_0}$ =0.1329



Now we take the example of an impedance matching using shunt stub, So let the load impedance to be matched is given by Z_L is equal to 100 plus j60 Ohm, and this is to be matched to a 50 Ohm line. So by applying the analytical solutions, we get a value of t to be equal to 3.4091 which we call as t1 and another value of t is Minus 1.0091which we call t2 and we get two solutions for d, in fact, d1 by Lambda comes out to be 0.2046 and d2 by Lambda this is 0.3743.

Similarly, we get two solutions for B, B_1 is 0.0221 and B_2 is Minus 0.0221. Now with these values, we can now calculate the length of the open-circuited stub so two solutions are again possible. So Lo1 by Lambda is 0.3671 and this one if you consider other value of B_2 this tan inverse will come out to be negative so we have to add Lambda by 2, here you note that Lo2 by Lambda we are calculating so this becomes 0.5 plus 1 by 2pi tan inverse B2 by Y naught which comes out to be 0.1329. Now, these are the two locations of the stubs, and the corresponding open-circuited stub length can be used to complete the design.

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From Smith Chart

$$\frac{d_1}{\lambda} = (0.5 - 0.46) + 0.164 = 0.204$$

$$\frac{d_2}{\lambda} = (0.5 - 0.46) + 0.336 = 0.376$$

$$l_{o1}/\lambda = 0.132$$

$$l_{o2}/\lambda = 0.368$$

$$Z_L = 100 + j60$$

Analytical

$$t1 = 3.4091$$

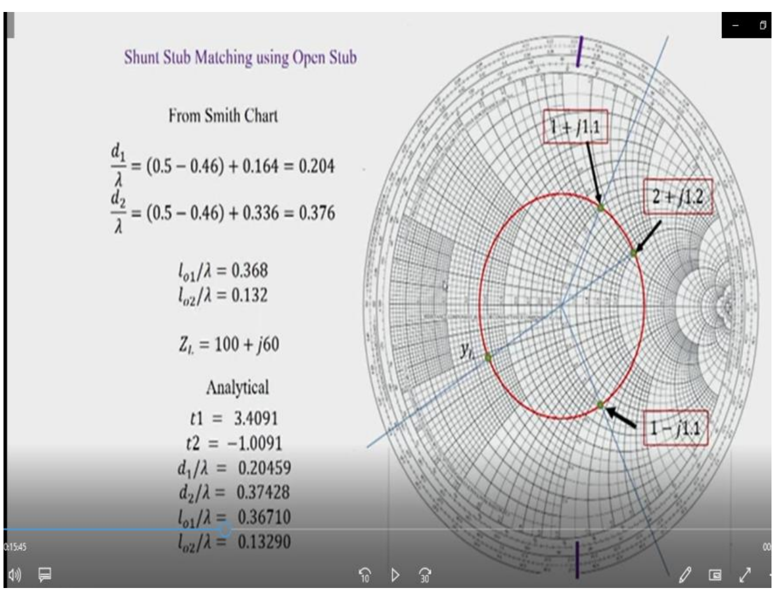
$$t2 = -1.0091$$

$$d_1/\lambda = 0.20459$$

$$d_2/\lambda = 0.37428$$

$$l_{o1}/\lambda = 0.36710$$

$$l_{o2}/\lambda = 0.13290$$



Next what we do, we verify our analytical design by comparing with Smith chart solution. So, our load impedance Z_L is 100 plus j60 which can be shown by the normalized impedance of 2 plus j12 in the Smith chart. Next, we draw the constant VSWR circle. Pass this line through the center of the Smith chart, and then we find out Y_L, the admittance, normalized admittance. Next what we do we use this Smith chart as an admittance chart, and as we move from this towards the 1 plus jB circle we intersect first at this point, which is 1 plus j1.1, and we draw a line from the center of the Smith chart through this point. Similarly, if we continue our VSWR circle we intersect at 1 plus jB circle for the second time at this point, and we find that here the value is 1 minus j1.1.

We drew another line through this point from the center of the Smith chart. Now, we calculate the distances d1 we can see that. So we find from the Smith chart that d1 by Lambda is equal to 0.204, and d_2 by Lambda is equal to 0.376, and the corresponding lengths of the open-circuited stub can be found out Lo1 by Lambda is equal to 0.368 and Lo2 by Lambda is equal to 0.132.

From the analytical solution we got t1 is equal to 3.4091, t2 is equal to Minus 1.0091 d1 by Lambda 0.20459, d_2 by Lambda 0.37428 and L1 by Lambda is equal to 0.36710, L2 by Lambda is 0.13290. So, we can see that the solution for d1, d2, and Lo1, Lo2 obtained graphically from the Smith chart is quite close to the solution that is obtained analytically.

So, we have seen how we can do impedance matching using single stubs either series or shunt stub. Now, these stubs can be fabricated along with the transmission line particularly when we use the planar transmission line. The stubs can be fabricated along with the lines. Generally, shunt stubs are preferred for microstrip or steep line type of transmission line, whereas series stub is preferred for short line or co-planar web guide type of transmission line.

Again as far as fabrication is concerned, when it is a planar transmission line like microstrip line, open-circuited stubs are advantageous because we need not use any wire connecting the steep conductor to the ground plane whereas short-circuited stubs are preferred when we consider coaxial type line.Let us now move on to another type of stub matching where we use more than one stub. So let us see how we can do impedance matching with double stub. Now, these two stubs may be spaced at a fixed distance and also located at a fixed distance from the load.

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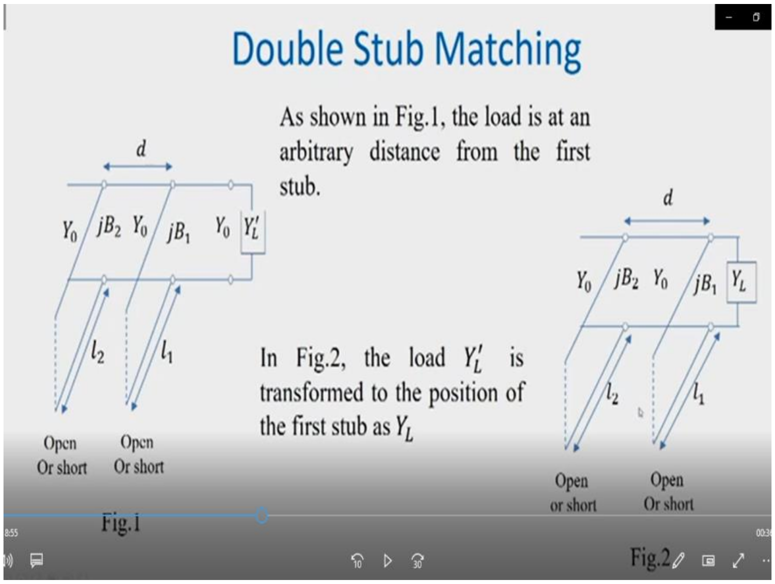

So, here we consider the load represented by Y_L dash, and these are the two stubs which are located from a fixed distance from the load they are separated by a distance d and the stubs of length l_1 and l_2, the stubs may be open or short. Now, what we do first of all this Y_L dash is transformed by this length of the transmission line, and now it is Y_L is the load that is seen at the location of the first stub. So, the load Y_L dash is transformed to the position of the first stub as Y_L.

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Analytical solution

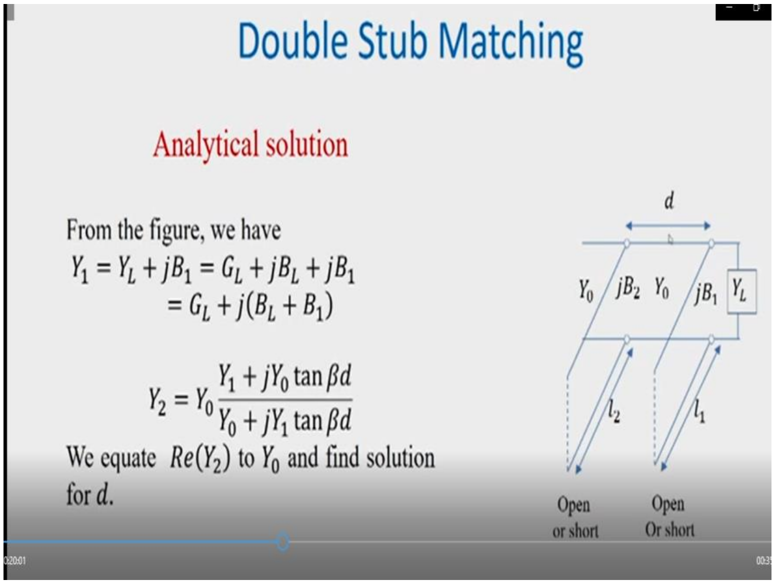
From the figure, we have

$$Y_1 = Y_L + jB_1 = G_L + jB_L + jB_1$$

$$= G_L + j(B_L + B_1)$$

$$Y_2 = Y_0 \frac{Y_1 + jY_0 \tan \beta d}{Y_0 + jY_1 \tan \beta d}$$

We equate $Re(Y_2)$ to Y_0 and find solution for d.



Now what we need to determine again the values of B, l_1, and l_2 depending upon the type of the stub that is being used. So, we can write Y_1 that is the admittance at the location of the first stub. Y_L plus jB_1 which can be written in this form GL plus jB_L plus $B_1$and then once we transform this Y_1 to the location of the second stub Y_2 then we get Y2 is equal to Y_0 Y_1 plus jY naught tan Beta d divided by Y_0 plus jY_1 tan Beta d and then for matching we need real part of Y_2 to be equal to Y_0 and once we do that we also get the solution for this distance d.

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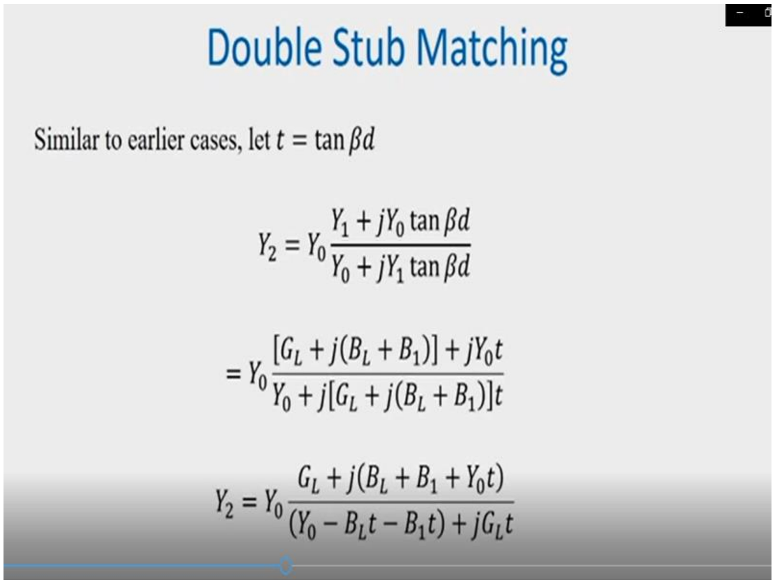
Similar to earlier assumptions, let $t = \tan \beta d$

$$Y_2 = Y_0 \frac{Y_1 + jY_0 \tan \beta d}{Y_0 + jY_1 \tan \beta d}$$

$$= Y_0 \frac{[G_L + j(B_L + B_1)] + jY_0 t}{Y_0 + j[G_L + j(B_L + B_1)]t}$$

$$Y_2 = Y_0 \frac{G_L + j(B_L + B_1 + Y_0 t)}{(Y_0 - B_L t - B_1 t) + jG_L t}$$



So, similar to our earlier cases, let us put t equal to tan Beta d then substituting tan Beta d by t here and substituting Y_1 we can write Y_2 in this form and after the rearrangement of terms Y_2 can be put in this form G_L plus jB_L plus B_1 plus Y naught t divided by Y_0 minus B_L minus t B_1t plus jG_Lt and this entire term multiplied by Y naught.

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On equating $Re(Y_2)$ to Y_0

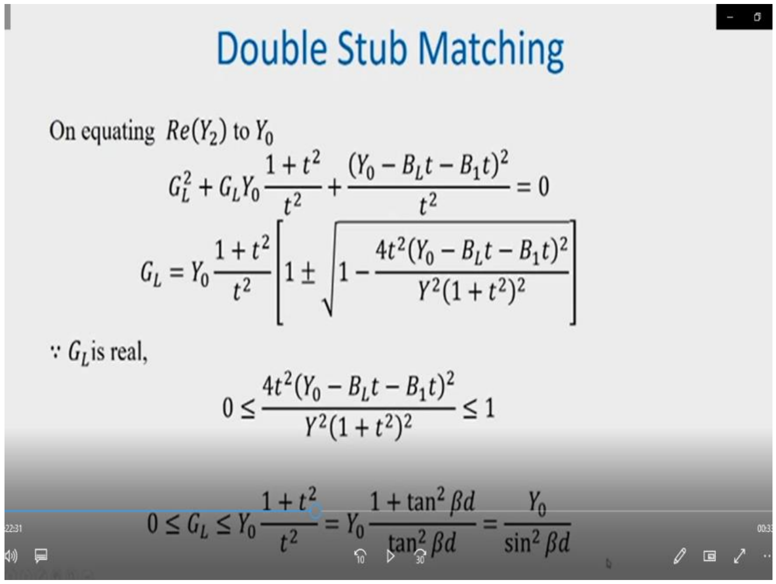
$$G_L^2 + G_L Y_0 \frac{1+t^2}{t^2} + \frac{(Y_0 - B_L t - B_1 t)^2}{t^2} = 0$$

$$G_L = Y_0 \frac{1+t^2}{t^2}\left[1 \pm \sqrt{1 - \frac{4t^2(Y_0 - B_L t - B_1 t)^2}{Y^2(1+t^2)^2}}\right]$$

$\because$ G_L is real,

$$0 \leq \frac{4t^2(Y_0 - B_L t - B_1 t)^2}{Y^2(1+t^2)^2} \leq 1$$

$$0 \leq G_L \leq Y_0 \frac{1+t^2}{t^2} = Y_0 \frac{1+\tan^2\beta d}{\tan^2\beta d} = \frac{Y_0}{\sin^2\beta d}$$



Now, we equate the real part of Y_2 with Y_0 and then we get an equation of this form. This is a quadratic equation. We can find out G_L, and what we find that G_L is real, therefore this term please noticed both numerator and denominator they are positive values because of the squares and therefore if we want to get this G_L to be real this entire term has to be less than 1, so this becomes greater than zero and less than 1, this particular term 4t square Y_0 minus B_Lt minus B1t whole square divided by Y square into 1 plus t square whole square.

So this term is greater than zero and less than 1, and this will us the real solution for G_L, and from this equation now we can write G_L is between 0 to Y naught by sin square Beta d.

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$$B_1 = -B_L + \frac{Y_0 \pm \sqrt{(1+t^2)G_L Y_0 - G_L^2 t^2}}{t}$$

$$B_2 = \pm \frac{Y_0 \sqrt{(1+t^2)G_L Y_0 - G_L^2 t^2} + G_L Y_0}{G_L t}$$

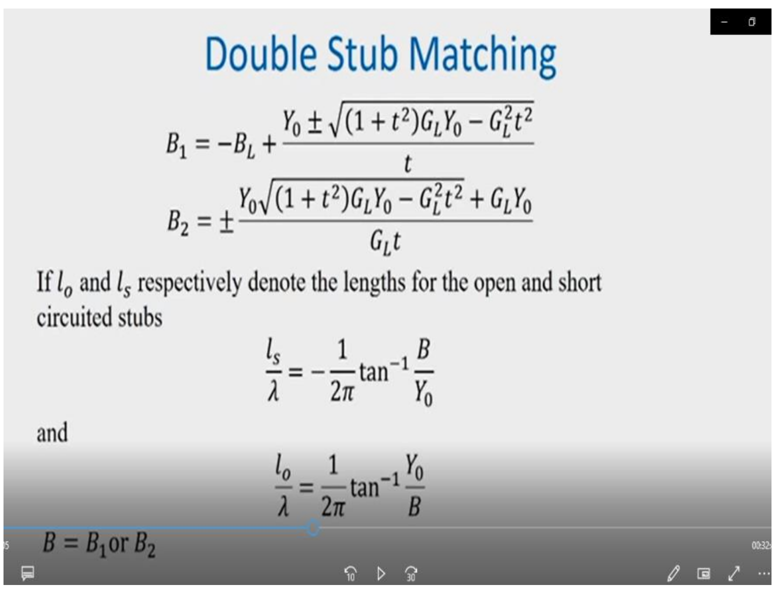
If l_o and l_s respectively denote the lengths for the open and short circuited stubs

$$\frac{l_o}{\lambda} = -\frac{1}{2\pi}\tan^{-1}\frac{B}{Y_0}$$

and

$$\frac{l_s}{\lambda} = \frac{1}{2\pi}\tan^{-1}\frac{Y_0}{B}$$

$B = B_1 \text{or } B_2$



Similarly, we can find out the expression for B_1 and B_2 and once we calculate B_1 and B_2 we can calculate the lengths of the open and short stub by this relationship where this B is either B_1 or B_2 and therefore we get the location which is fixed already d, the separation for that particular location now we get the solution for the short and the open-circuited stubs their lengths in terms of Lambda. We have seen the double stub matching analytically, and we have derived the closed-form expression for computation of stub lengths.

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$$Y_i = Y_B + Y_{sB} = Y_0$$

In normalized form, $1 = y_B + y_{sB}$

Since y_{sB} is purely imaginary we must have,

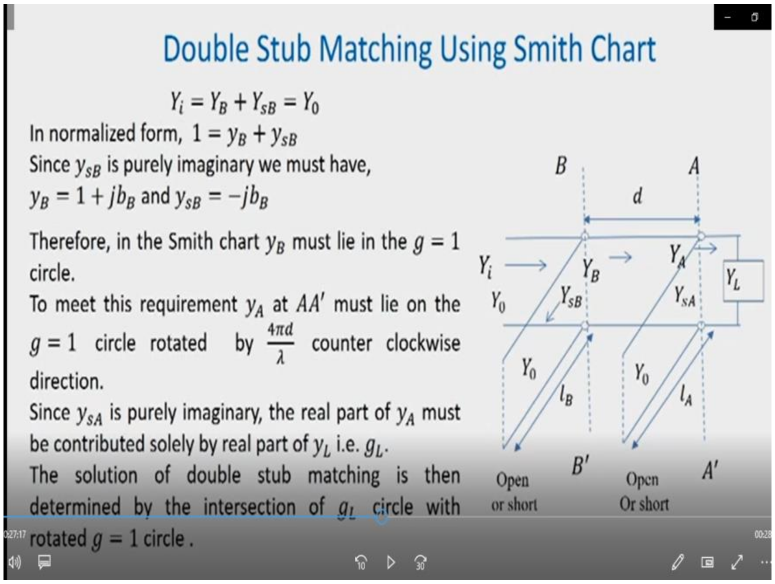
$y_B = 1 + jb_B$ and $y_{sB} = -jb_B$

Therefore, in the Smith chart y_B must lie in the $g = 1$ circle.

To meet this requirement y_A at AA' must lie on the $g = 1$ circle rotated by $\frac{4\pi d}{\lambda}$ counter clockwise direction.

Since y_{sA} is purely imaginary, the real part of y_A must be contributed solely by real part of y_L i.e. g_L.

The solution of double stub matching is then determined by the intersection of g_L circle with rotated $g = 1$ circle .



Let us now consider double stub matching using Smith chart. As shown in the figure, Y_L is the load impedance which has been transformed to the location of the first stub, and the two stubs are connected, separated by a distance d. We have the reference locations AA dash and BB dash at the location of the two stubs. Now if we look at this double stub matching circuit, we find that Yi, the input admittance is equal to YB, the admittance looking from here plus YsB the admittance of the stub. The stubs may be open or short, as shown.

Here we need to find out the lengths of the stub lA and lB. we can write this equation in the normalized form 1 is equal to normalized YB plus normalized YsB. Since YsB is purely imaginary, this YsB is purely imaginary and we must have YsB equal to minus jbB, the susceptance of the stub so that we finally get Yi equal to 1 and therefore we find that in Smith chart YB must lie in the g is equal to 1 circle. We are using the Smith chart as an admittance Smith chart, and since the real part of YB is 1 it must lie on the g equal to 1 circle.

To meet this requirement YA at AA dash must lie on the g is equal to 1 circle rotated by 4 pi d by Lambda counterclockwise that means towards the load because then only once we come back to g is equal to 1 circle we will get the required value of YB. Since YsA, the stub admittance is purely imaginary the real part of YA must be contributed solely by the real part of Y_L and the real part of yL is gL, and therefore the solution of double stub matching is then determined by the intersection of gL circle with rotated g is equal to 1 circle. So let us explain this on a Smith chart.

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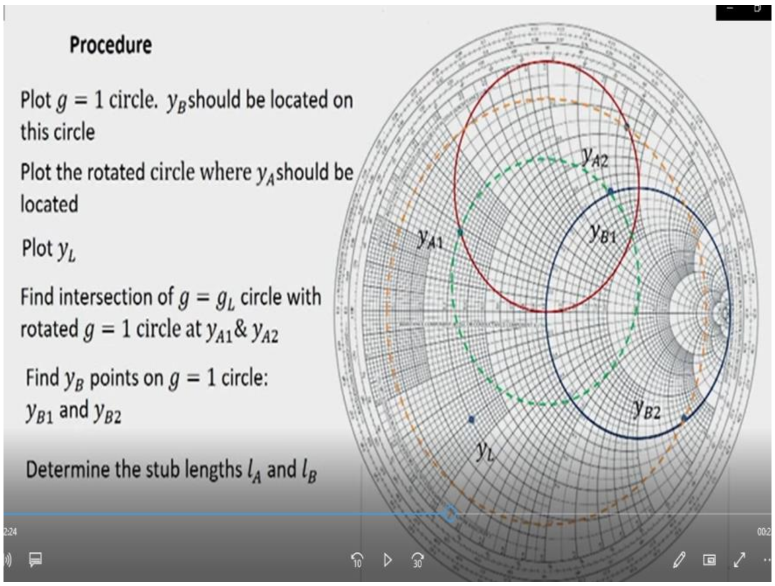

So, we follow the procedure as shown. We plot g is equal to 1 circle and YB should be located on this circle because after matching we require Yi equal to 1. Plot the rotated circle where YA should be located. Now, this rotated circle, we have seen that it is to be rotated counter clockwise by 4 pi d by Lambda.

Here we are considering an example where d is equal to Lambda by 8, therefore, this circle has been rotated by pi by 2. Next, what we do, we plot Y_L. So this is the Y_L point shown. Now as we have said that the g is same as gL on this particular circle when we find YA the real part of YA should be equal to g_L so what we do we move across this constant g circle which is equal to g_L and find the intersection point, first intersection point here and this is YA1. Please note that the real part of this YA1 is same as g_L, and then as we continue moving, we get over this circle, we get another intersection point YA2. Again the real part of these is same as g_L.

Now, these are the location of YA on the rotated g is equal to 1 circle. Now this has to be transformed to g equal to 1 circle to get the corresponding YB points and in order to do that we now move by the distance Lambda by 8 or an angular movement of 4 pi Lambda by d that means here in this case pi by 2 and follow a VSWR circle so that, we find out the intersection YA1 on the rotated g equal to 1 circle actually corresponds to this point YB1 on the g equal to 1 circle.

Similarly, we find that this point will be transformed to this point, which is shown as YB. Now we have all four points located. So we can see that at this point YB1 is having a value 1 plus jB in that form and we can find out the corresponding stub length that will compensate for the imaginary part. Similarly, here at YA1 point we have the real part to be g_L and the imaginary part has a value so this starting from this point this additional susceptance when added will transform YL to YA1 values, and we can calculate the lengths of the stubs that would be necessary for this additional susceptance.

So we can calculate for the other two points YA2 and YB2 as well. So, we can determine the stub lengths lA and lB that would be necessary, and the lengths will depend on whether we are choosing open-circuited at stub or short-circuited at stub. So, this is how we can use double stub for impedance matching. Please note that the separation between the stubs, d is fixed, and we find out the lengths of the stubs. Also, Y_L has been considered at the location of the first stub. In practice, Y_L will be the transformed value that means a load impedance transformed by a section of the transmission line to that location.

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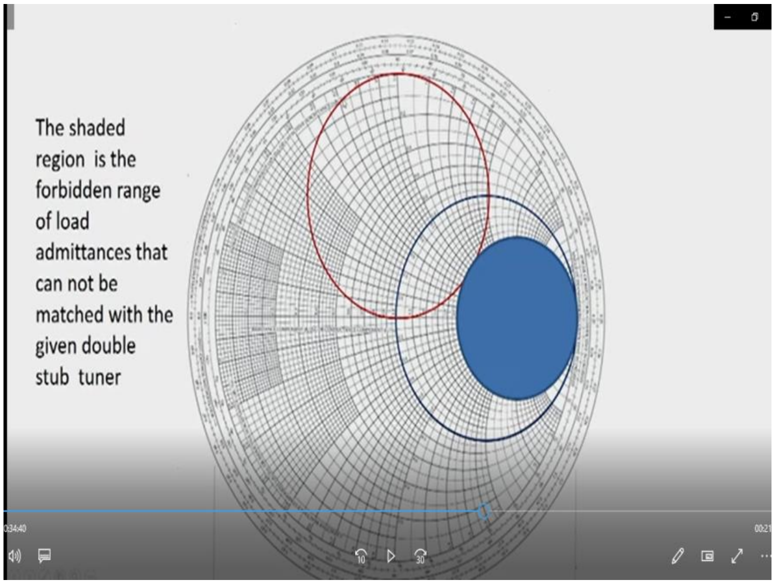

Another point you should remember, we have shown a shaded region and what we noticed that if Y_L lies within this shaded region and if we follow a constant g circle we cannot have an intersection with rotated g equal to 1 circle that means for the given arrangement of the stubs we cannot find the solution for this range of Y_L. Please note that this region for which solution is not possible the forbidden region, it changes depending upon the orientation of this rotated circle. For example, if d is less than Lambda by 8 as shown here then this circle will shrink whereas if it is more than Lambda by 8 this circle will actually increase in size.

So, this actually shows the forbiddence range of the load admittances that cannot be matched with the given double stub tuner for which we have drawn the rotated g is equal to 1 circle.

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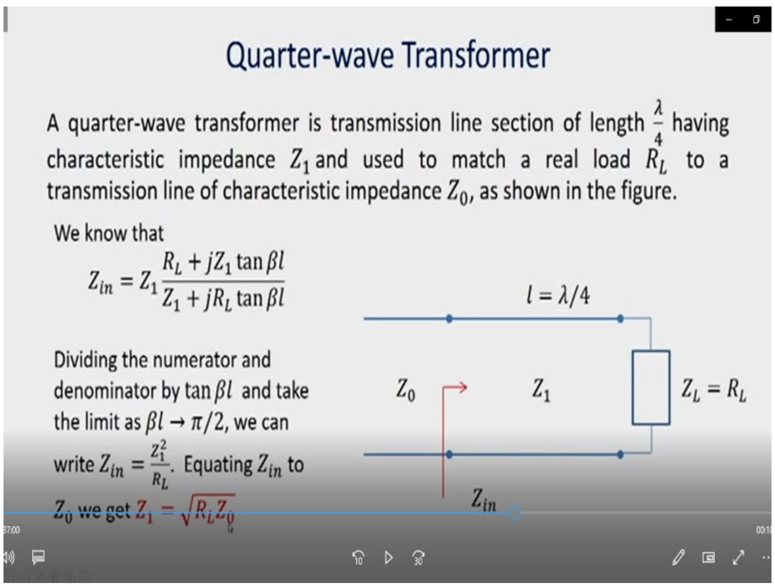
We know that

$$Z_{in} = Z_1 \frac{R_L + jZ_1 \tan \beta l}{Z_1 + jR_L \tan \beta l}$$

Dividing the numerator and denominator by $\tan \beta l$ and take the limit as $\beta l \to \pi/2$, we can write $Z_{in} = \frac{Z_1^2}{R_L}$. Equating Z_{in} to Z_0 we get $Z_1 = \sqrt{R_L Z_0}$



Let us now move on to another type of impedance matching network which is called a quarter-wave transformer. So, a quarter-wave transformer it is essentially a transmission line of section of length Lambda by 4, and it is having a characteristic impedance Z_L and it is used to match a real load RL to a transmission line of characteristic impedance of Z_0 and we show this arrangement in the figure. So here we have the characteristic impedance Z_0 for the transmission line, and our Z_L is equal to RL and here in between we have a quarter-wave transformer having a length l equal to Lambda by 4 and its characteristic impedance is Z_1.

So what we can do we can find out the input impedance looking at this point, and this can be written as Z_{in} is equal to Z_1 R_L plus jZ_1 tan Beta l divided by Z_1 plus jR_L tan Beta l. Now, what we can do we can divide numerator and denominator by tan Beta L, and we see that when l become

Lambda by 4 Beta l tends to pi by 2 and therefore we are left with Z_1 into jZ_1 divided by jR_L that means Z_1 square by R_L and equating this Z_{in} to Z_0 we get Z1 to be equal to under root R_L into Z_0. So what we find that a quarter-wave transformer can match a real load RL to Z_0 provided we select Z_1 to be equal to under root R_L Z_0.

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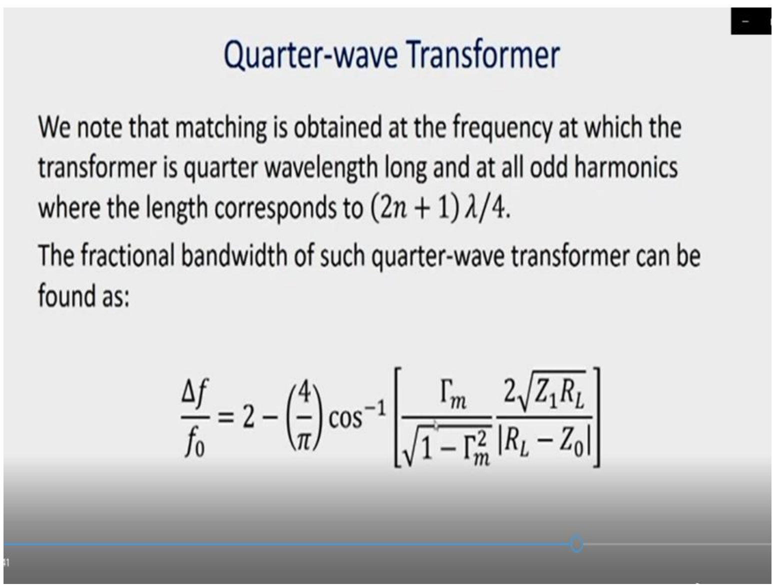
We note that matching is obtained at the frequency at which the transformer is quarter wavelength long and at all odd harmonics where the length corresponds to $(2n + 1)\,\lambda/4$.

The fractional bandwidth of such quarter-wave transformer can be found as:

$$\frac{\Delta f}{f_0} = 2 - \left(\frac{4}{\pi}\right)\cos^{-1}\left[\frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}}\frac{2\sqrt{Z_1 R_L}}{|R_L - Z_0|}\right]$$

Γ_m is the magnitude of the acceptable value of reflection coefficient



So we note that matching is obtained at the frequency at which the transformer is quarter wave long. So, this is very important to note because once we have a physical section of a transmission line this will be a quarter wave long only at a particular frequency. So, exact matching is obtained only at a particular frequency and also at all odd harmonics where the length corresponds to 2n plus 1 Lambda by 4 that means 3 Lambda by 4, 5 Lambda by 4. For all these lengths also we will get exact matching.

Now, as I have told that the length of the quarter wave transformer is exactly Lambda by 4 only at the operating frequency f naught. At other frequencies f greater than f naught or less than f naught this length will not be exactly Lambda by 4 and there will be some amount of mismatch and we can calculate the fractional bandwidth which is defined we by delta f by f naught, f naught is the operating center frequency and delta f gives the frequency deviation from f naught.

This can be found out and this is given by this expression 2 minus 4 by pi cos inverse, Gamma m, here this Gamma m is the maximum value of the reflection coefficient that we are ready to accept and depending upon this Gamma m the fractional bandwidth is given by delta f divided by f naught, f naught is the center frequency or the designed frequency, delta f is the frequency deviation and this is given by 2 minus 4 by pi cos inverse of 2 Gamma m, Gamma m is the magnitude of the reflection coefficient that we are ready to accept. You can see that with different value of acceptable level of reflection coefficient fractional bandwidth will be different. So this is cos inverse to Gamma m divided by under root 1 minus Gamma m square into 2 underroot Z_1 into R_L divided by mode of RL Minus Z_0.

So this expression will give the fractional bandwidth for the quarter wave transformer and we can actually have different fractional bandwidth depending upon the maximum value of the reflection coefficient at the pass bent edges that we are ready to accept.

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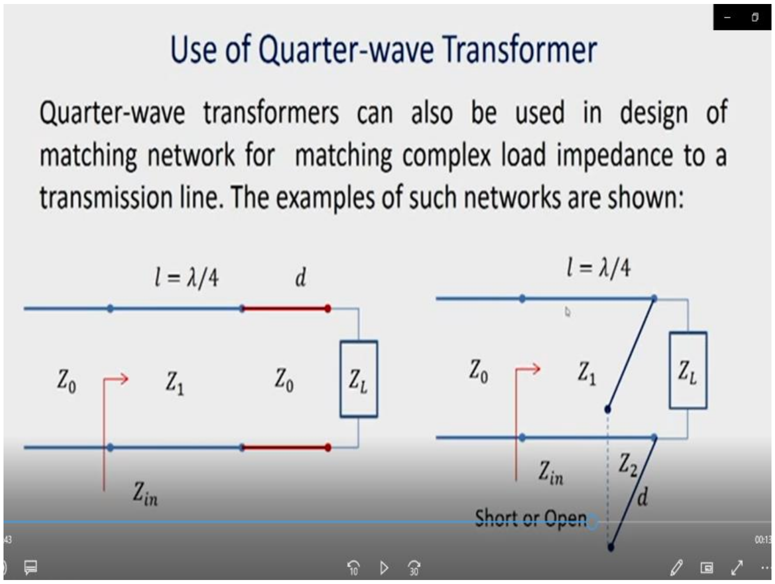

Now, quarter wave transformer can also be used with other circuitery in the design of a matching network for matching a complex load impedance to a transmission line. So, we show some examples of such networks. For example, here Z_L is a complex load to be matched to Z naught. What we can do? We can take a transmission line section of length d and transform this Z_L in such a way that the imaginary part becomes zero. So, once we can do that, here we will be left with a real transformed value of the load and that real value can be matched to the transmission line using a quarter wave transformer.

Another example is shown here where suppose we put a stub at the location of the load Z_L to cancel out the reactive part now it is shown with a shunt stub that means this shunt stub and Z_L combined will give a resistive load impedance here which can be transformed to Z_0 by using these quarter waver transformer.Let us now move on to another topic which is the theory of small reflections.

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A single section transformer is shown in figure.

The partial reflection and transmission coefficient of the single section transformer are:

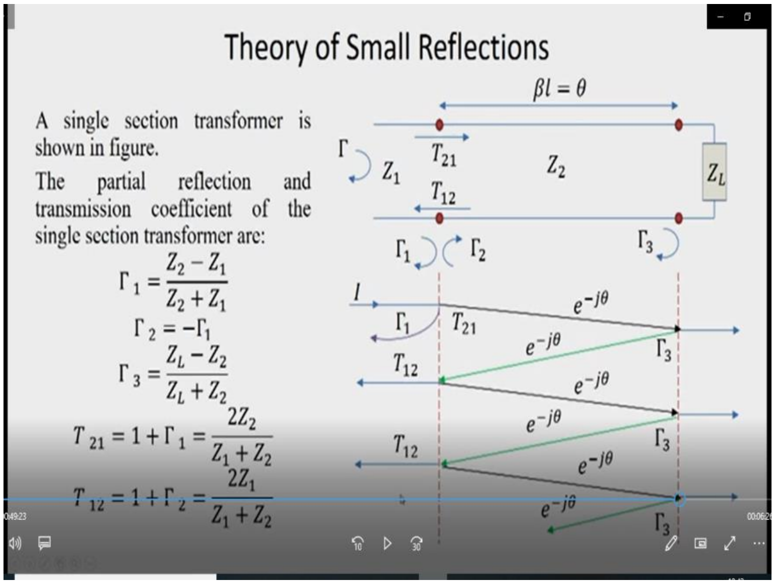
$$\Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

$$\Gamma_2 = -\Gamma_1$$

$$\Gamma_3 = \frac{Z_L - Z_2}{Z_L + Z_2}$$

$$T_{21} = 1 + \Gamma_1 = \frac{2Z_2}{Z_L + Z_2}$$

$$T_{12} = 1 + \Gamma_2 = \frac{2Z_1}{Z_L + Z_2}$$



So here what we are considering, we are considering a load impedance Z_L, a transmission line of characteristic load impedance Z_1 and in between we have another section of transmission line with characteristic impedance Z_2 and the electrical length of this section is Beta l is equal to Theta. So this is essentially a single section transformer and we define the partial reflection and transmission coefficient.

So here Gamma is the overall reflection coefficient. Now at this junction when the wave travels it sees a mismatch of impedance and the partial reflection coefficient Gamma 1 is dependent upon this mismatch $Z_2 - Z_1$ divided by Z_1 plus Z_2 as you will see. So the wave traveling in this transmission line seeing characteristic impedance Z_1 finds a mismatch here and Gamma 1 is the partial reflection coefficient.

Similarly, a part of the wave will also travel to the second section of the transmission line and we have T21 is the transmission coefficient from transmission line section 1 to 2. Now, this wave traveling in this second section finds another mismatch when it reaches the load impedance Z_L. So we define gamma 3 as the partial reflection coefficient and Gamma 3 will depend only on Z_2 and Z_L.

Similarly the reflected wave from the load it will also get reflected by this interface and this is denoted by Gamma 2 which will again depend upon Z_2 and Z_1. So we can write these partial reflection coefficients. Gamma 1 is Z_2 minus Z_1 divided by Z_2 plus Z_1. Gamma 2 will be Z_1 minus Z_2 divided by Z_1 plus Z_2 which is same as minus of Gamma 1 and Gamma 3 will be Z_L minus Z_2 divided by Z_L plus Z_2. So having defined Gamma 1, Gamma2 and Gamma 3, now we define T_{21} which is equal to 1 plus Gamma 1 and it becomes 2 Z_2 by Z_1 plus Z_2.

Similarly T_{12} which is 1 plus Gamma 2, it becomes 2 Z_1 divided by Z_1 plus Z_2. Now let us consider here suppose a wave of unity magnitude it is incident here then as it first times reaches this interface we will have Gamma 1 as the reflected wave and T_{21} is the transmitted one then it will travel down this transmission line so the phase changes represented by e to the power minus j Theta where Theta is actually Betal and then it gets reflected from this point where there is a mismatch between the transmission line and the load impedance.

So, it will essentially get multiplied by Gamma 3 here. So if we start with the wave incident here we can see that here we will get T_{21}, at this stage it will be T_{21} e to the power minus j Theta and then multiplied by Gamma 3 then, it will travel down so it will become T_{21} Gamma 3 e to the power minus j2 Theta and then at this interface it will get reflected again and we have Gamma 2 as the partial reflection coefficient and T_{12} will be the transmission coefficient here.

So, finally the wave that will immerse from here will be T_{21} e to the power minus j 2 Theta into Gamma 3 into T_{12}. So these are all waves, reflected waves from this interface and these are all partial reflected waves. So we can write the total reflection coefficient Gamma in terms of these partial reflected signals.

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Thus, we can express the total reflection coefficient as a sum of partial reflection and transmission coefficients:

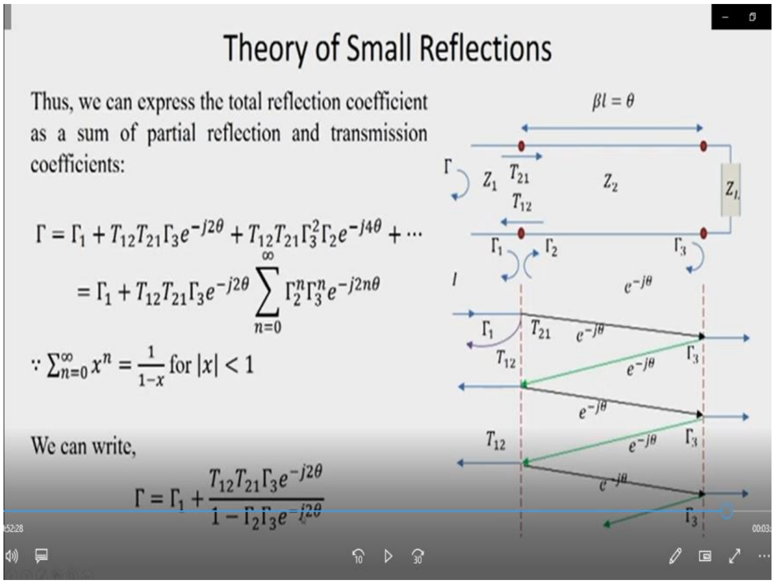
$$\Gamma = \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-j2\theta} + T_{12}T_{21}\Gamma_3^2\Gamma_2 e^{-j4\theta} + \cdots$$

$$= \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-j2\theta}\sum_{n=0}^{\infty}\Gamma_2^n\Gamma_3^n e^{-j2n\theta}$$

$\because \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$

We can write,

$$\Gamma = \Gamma_1 + \frac{T_{12}T_{21}\Gamma_3 e^{-j2\theta}}{1 - \Gamma_2\Gamma_3 e^{-j2\theta}}$$



So we can write Gamma is equal to Gamma 1, first one, then T_{21} e to the power minus j Theta, Gamma 3 e to the power minus j Theta T_{12} that means this term becomes T_{12} T_{21} Gamma 3 e to the power minus j 2 Theta. Similarly, from here the partial reflected term is T_{12} T_{21} Gamma 3 square because now it will undergo another reflection here and another Gamma 2 will come here because this signal is reflected from here so we will have T_{12} T_{21} Gamma 3 square Gamma 2 and total phase shift will be 1, 2, 3, 4 so e to the power minus j4 Theta.

Now, it will be, this term will be a common term in all the terms and this can be put in a sumession form as shown. It may be noted that since Gamma2 Gamma3 they are reflection coefficient, their magnitude is less than 1 and we know that when we have a sum of x rest to the power n where magnitude of x is less than 1 it can be expressed as sum of x rest to the power n, n extending to be zero to infinity is equal to 1 by 1 Minus x when mode of x is less than 1 and if we apply this relation we can write this submission replaced by 1 minus Gamma 2 Gamma 3 e to the power minus j2 Theta and hence we get Gamma to be equal to Gamma 1 plus T_{12} T_{21} Gamma 3 e to the power minus j2 Theta divided by 1 minus Gamma 2 Gamma 3 e to the power minus j2 Theta.

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Using, $\Gamma_2 = -\Gamma_1, T_{21} = 1 + \Gamma_1,\ T_{12} = 1 - \Gamma_1$, we can write

$$\Gamma = \Gamma_1 + \frac{T_{12}T_{21}\Gamma_3 e^{-j2\theta}}{1 - \Gamma_2\Gamma_3 e^{-j2\theta}}$$

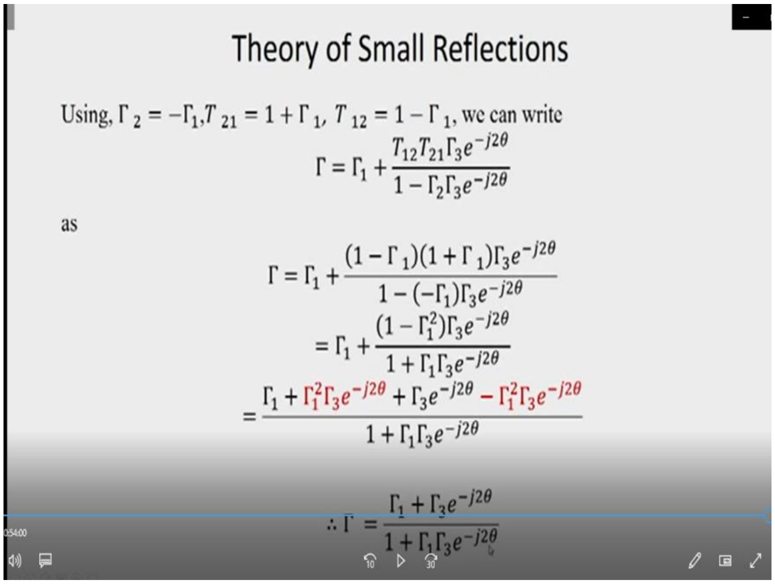
as

$$\Gamma = \Gamma_1 + \frac{(1-\Gamma_1)(1+\Gamma_1)\Gamma_3 e^{-j2\theta}}{1-(-\Gamma_1)\Gamma_3 e^{-j2\theta}}$$

$$= \Gamma_1 + \frac{(1-\Gamma_1^2)\Gamma_3 e^{-j2\theta}}{1+\Gamma_1\Gamma_3 e^{-j2\theta}}$$

$$= \frac{\Gamma_1 + \Gamma_1^2\Gamma_3 e^{-j2\theta} + \Gamma_3 e^{-j2\theta} - \Gamma_1^2\Gamma_3 e^{-j2\theta}}{1+\Gamma_1\Gamma_3 e^{-j2\theta}}$$

$$\therefore \Gamma = \frac{\Gamma_1 + \Gamma_3 e^{-j2\theta}}{1+\Gamma_1\Gamma_3 e^{-j2\theta}}$$



Now we use the relation Gamma 2 is equal to minus Gamma1, T_{21} is equal to 1 plus Gamma1 and T_{12} is equal to 1 plus Gamma 2 which is equal to 1 minus Gamma1 and then we can write this expression as by replacing T_{12} and T_{21} and also replacing Gamma 2 by minus Gamma 1 you can put it in this form. Now, finally it becomes Gamma1 plus 1 minus Gamma1 square Gamma 3 e to the power minus j2 Theta divided by 1 plus Gamma 1 Gamma 3 e to the power minus j2 Theta. So, this we can write once we multiply this denominator term by Gamma 1 and then the terms shown in red they will cancel out and you will be left with Gamma is equal to Gamma1 plus Gamma3 e to the power minus j2 Theta divided by 1 plus Gamma1 Gamma3 e to the power minus j2 Theta.

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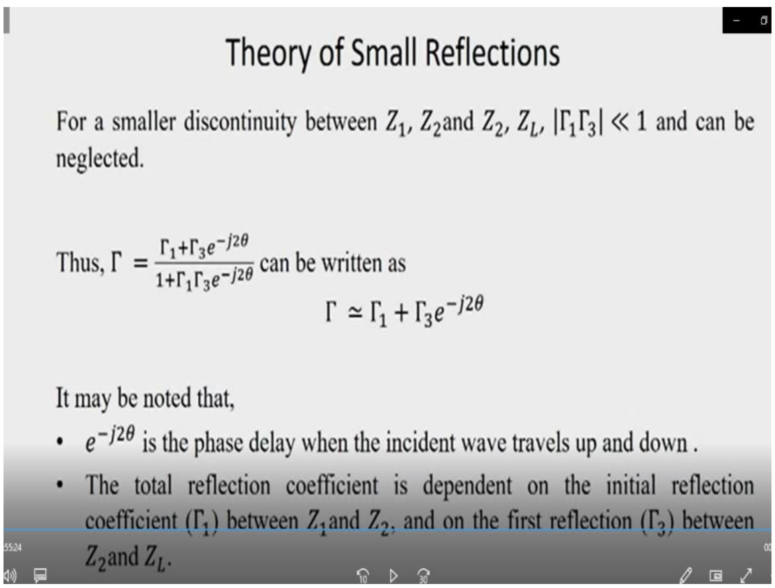
For a smaller discontinuity between Z_1, $Z_2$and Z_2, Z_L, $|\Gamma_1\Gamma_3| \ll 1$ and can be neglected.

Thus, $\Gamma \ = \frac{\Gamma_1+\Gamma_3e^{-j2\theta}}{1+\Gamma_1\Gamma_3e^{-j2\theta}}$ can be written as

$$\Gamma \ \simeq \Gamma_1 + \Gamma_3e^{-j2\theta}$$

It may be noted that,

- $e^{-j2\theta}$ is the phase delay when the incident wave travels up and down .
- The total reflection coefficient is dependent on the initial reflection coefficient (Γ_1) between $Z_1$and Z_2, and on the first reflection (Γ_3) between $Z_2$and Z_L.



Now suppose we have the condition that discontinuity between Z_1 and Z_2 and Z_2 and Z_L these are small so that both Gamma1 and Gamma3 they are very small and their product, magnitude of the product of Gamma1 and Gamma 3 will be very, very small compared to 1 and neglecting this term we can write, we neglect these with respect to 1 and we write Gamma is equal to Gamma1 plus Gamma3 e to the power minus j2 Theta. So, it may be noted that e to the power minus j2 Theta is the phase delay when the incident wave travels up and down and the total reflection coefficient is dependent on the initial reflection coefficient between Z_1 and Z_2 and on the first reflection between Z_2 and Z_L.