Microwave Engineering
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Lecture 13:
Multi-section Matching Networks and Tapered Lines

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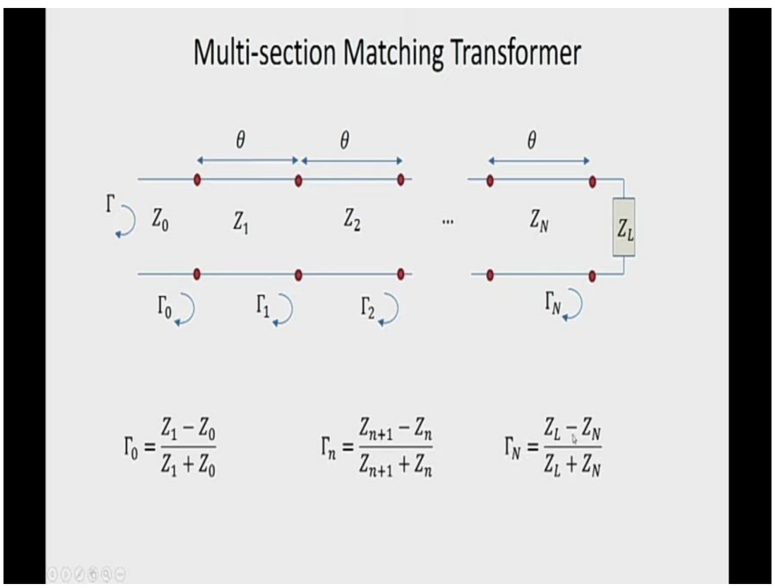
$$\Gamma_0 = \frac{Z_1 - Z_0}{Z_1 + Z_0}$$

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n}$$

$$\Gamma_N = \frac{Z_L - Z_N}{Z_L + Z_N}$$



So we have seen the theory of small reflections, let us now consider a multi-section matching transformer. The figure shows a multi-section matching transformer where we have capital N number of sections, each section is having an electrical length theta and the transmission line sections have impedances Z_1 and Z_2 up to Z_n. And this transformer is designed to match this load Z_l to this transmission line having characteristic impedance Z_0 Now we introduce the partial reflection coefficients gamma zero, gamma 1, gamma 2, gamma n at this interfaces. So whenever there is a change in impedance and gamma is the overall reflection coefficient.

We can write gamma 0 equal to Z minus Z_0 divided by Z_1 plus Z_0 and continuing in this manner. Gamma n is equal to Z_n plus 1 minus Z_n divided by Zn plus 1 plus Z_n. And finally gamma capital N is Z_l minus Z capital N divided by Z_l plus Z capital N.

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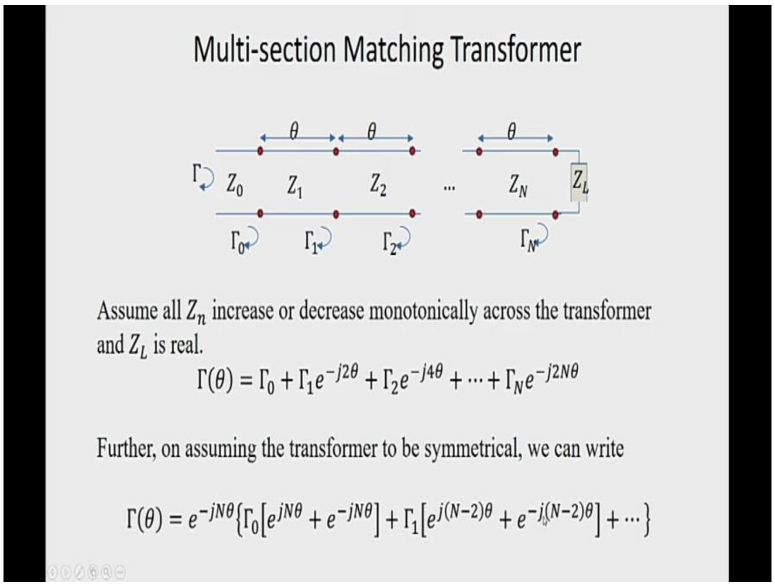
Assume all Z_n increase or decrease monotonically across the transformer and Z_L is real.

$$\Gamma(\theta) = \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \cdots + \Gamma_N e^{-j2N\theta}$$

Further, on assuming the transformer to be symmetrical, we can write

$$\Gamma(\theta) = e^{-jN\theta}\left\{\Gamma_0\left[e^{jN\theta} + e^{-jN\theta}\right] + \Gamma_1\left[e^{j(N-2)\theta} + e^{-j(N-2)\theta}\right] + \cdots\right\}$$



Now we assume that all Z_n they increase or decrease monotonically and Z_1 is real so that means this Z_1, Z_2, Z_n will increase either monotonically or decrease monotonically depending upon the values of Z_0 and Z_l. If we consider the theory of small reflections what we have discussed, we can write gamma the overall reflection coefficient as a function of theta as gamma nought plus gamma 1. This is the partial reflection coefficient minus j, e to the power minus j2 theta plus gamma 2 e to the power minus j4 theta and gamma capital N e to the power minus j2 capital N theta. Further we assume that the transformer be symmetrical that means what we mean is gamma nought is equal to gamma N. Please note that, impedance increase or decrease monotonically.

But this impedance level for example Z_1 and Z_0 is selected in such a way that gamma nought becomes equal to gamma L which is decided by Z_L and Z_n. So if we assume this type of symmetry then we can write gamma theta e to the power minus jn theta gamma nought e to the power jn theta plus e to the power minus jn theta plus gamma 1 e to the power jn minus 2 theta plus e to the power minus jn minus 2 theta plus and so on.

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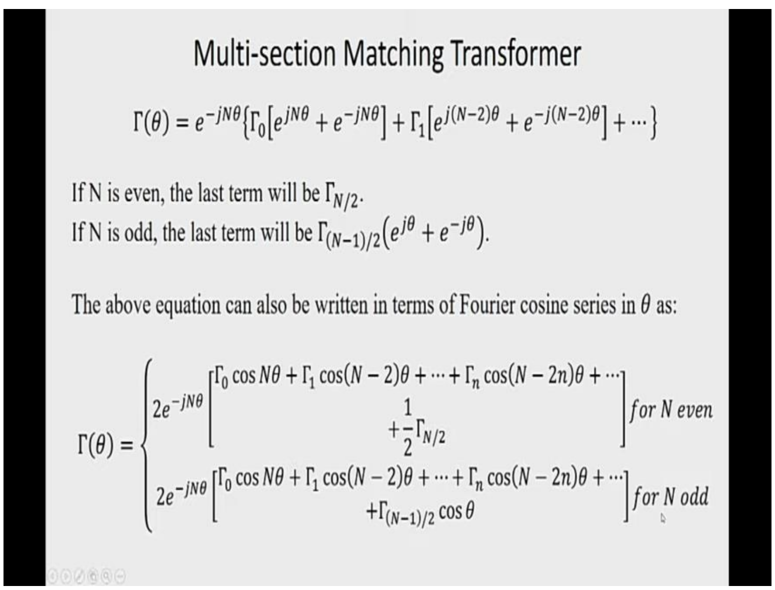
$$\Gamma(\theta) = e^{-jN\theta}\{\Gamma_0[e^{jN\theta} + e^{-jN\theta}] + \Gamma_1[e^{j(N-2)\theta} + e^{-j(N-2)\theta}] + \cdots\}$$

If N is even, the last term will be $\Gamma_{N/2}$.

If N is odd, the last term will be $\Gamma_{(N-1)/2}(e^{j\theta} + e^{-j\theta})$.

The above equation can also be written in terms of Fourier cosine series in θ as:

$$\Gamma(\theta) = \begin{cases} 2e^{-jN\theta}\begin{bmatrix}\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n)\theta + \cdots \\ +\frac{1}{2}\Gamma_{N/2}\end{bmatrix} & for\ N\ even \\ 2e^{-jN\theta}\begin{bmatrix}\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n)\theta + \cdots \\ +\Gamma_{(N-1)/2}\cos\theta\end{bmatrix} & for\ N\ odd \end{cases}$$



Now capital N it can be even, in that case, the last term will be gamma capital N by 2. If capital N is odd the last term will be gamma capital n minus 1 by 2 e to the power j theta plus e to the power minus j theta. And these two equations can be written in terms of Fourier Cosine series. In theta s, gamma is a function of theta is equal to 2 e to the power minus jn theta, gamma nought cos n theta gamma 1 cos n minus 2 theta and finally half gamma n by 2 for an even and gamma as a function of theta equal to 2 e to the power minus j capital N theta gamma nought cos n theta plus gamma 1 cos n minus 2 theta and finally gamma n minus 1 by 2 cos theta when capital N is odd.

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The passband response of a binomial matching transformer for a given number of sections is as flat as possible near the design frequency.

Reflection coefficient of such a response is given by

$$\Gamma(\theta) = A\left(1 + e^{-j2\theta}\right)^N$$

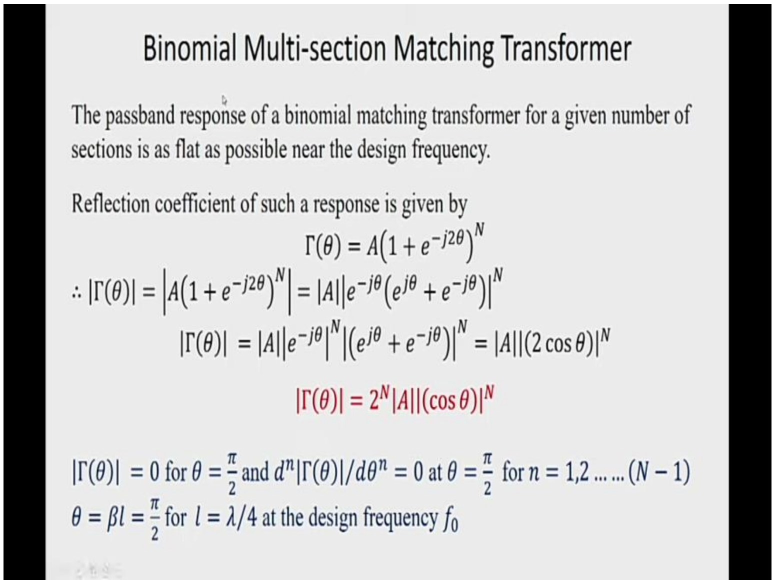
$$\therefore |\Gamma(\theta)| = \left|A\left(1 + e^{-j2\theta}\right)^N\right| = |A|\left|e^{-j\theta}\left(e^{j\theta} + e^{-j\theta}\right)\right|^N$$

$$|\Gamma(\theta)| = |A|\left|e^{-j\theta}\right|^N\left|\left(e^{j\theta} + e^{-j\theta}\right)\right|^N = |A||(2\cos\theta)|^N$$

$$|\Gamma(\theta)| = 2^N|A||(\cos\theta)|^N$$

$|\Gamma(\theta)| = 0$ for $\theta = \frac{\pi}{2}$ and $d^n|\Gamma(\theta)|/d\theta^n = 0$ at $\theta = \frac{\pi}{2}$ for $n = 1,2 \ldots \ldots (N-1)$

$\theta = \beta l = \frac{\pi}{2}$ for $l = \lambda/4$ at the design frequency f_0



So, now we have discussed how we can find out the reflection coefficient for a multi-section transformer in terms of the electrical length of the individual section theta and now let us see how we can design a Binomial multi-section matching transformer? Now the passband response of a binomial matching transformer for a given number of sections is as flat as possible near the design frequency. So that is why it is also called maximally flat response.

And in this type of design what we do the reflection coefficient that means gamma theta can be represented as gamma theta is equal to A 1 plus e to the power minus j2 theta raised to the power N. So we already have a Fourier expansion of gamma theta, so we are trying to now map this response so if we write modulus of gamma theta then we can write its mod of A 2 cos theta mod raised to the power N and this can be written as modulus of gamma theta is equal to 2 to the power N modulus of A modulus of cos theta raised to the power N.

Now we can see that modulus of gamma theta equal to zero for theta equal to pi by 2 and nth derivative d n modulus gamma theta n is equal to zero at theta is equal to Pi by 2 for n equal to 1, 2 capital n minus 1. And we find that theta equal to Pi by 2 corresponds to l is equal to lambda by 4 that means now we are considering a binomial multi-section transformer where each element, each transmission line sections are lambda by 4 or quarter wave long at the design frequency.

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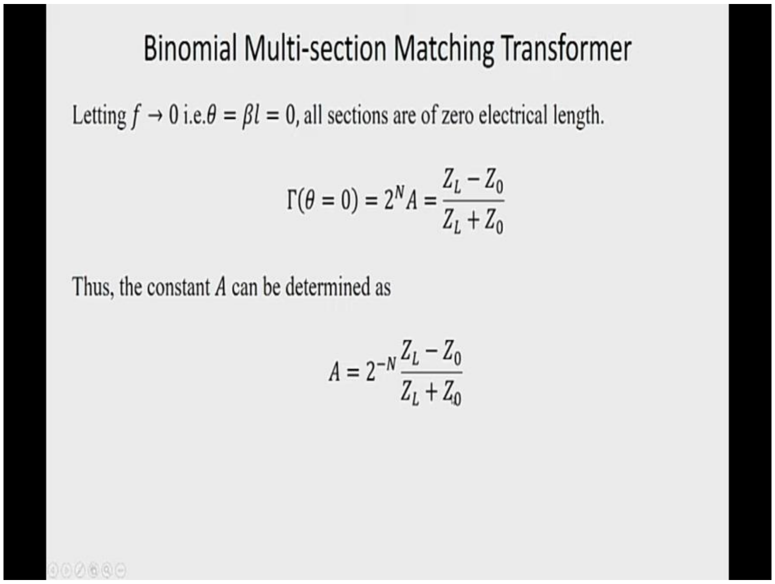
Letting $f \to 0$ i.e.$\theta = \beta l = 0,$ all sections are of zero electrical length.

$$\Gamma(\theta = 0) = 2^N A = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Thus, the constant A can be determined as

$$A = 2^{-N} \frac{Z_L - Z_0}{Z_L + Z_0}$$



Now if you consider f equal to 0 then theta becomes zero and all sections have zero electrical length. What does it mean that the line directly connected to the load impedance and therefore from the expression of gamma theta evaluated at theta equal to zero, we can write 2 to the power N A and N is equal to Z_L minus Z_0 divided by Z_L plus Z_0 and the constant A can be determined as A is equal to 2 to the power minus N Z_L minus Z_0 divided by Z_L plus Z_0. Please note that we have already mentioned that this matching transformer we are considering for a real load Z_L.

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On expanding $\Gamma(\theta) = A\left(1 + e^{-j2\theta}\right)^N$ using binomial expansion, we get

$$\Gamma(\theta) = A\left(1 + e^{-j2\theta}\right)^N = A\sum_{n=0}^{N} C_n^N e^{-j2n\theta}$$

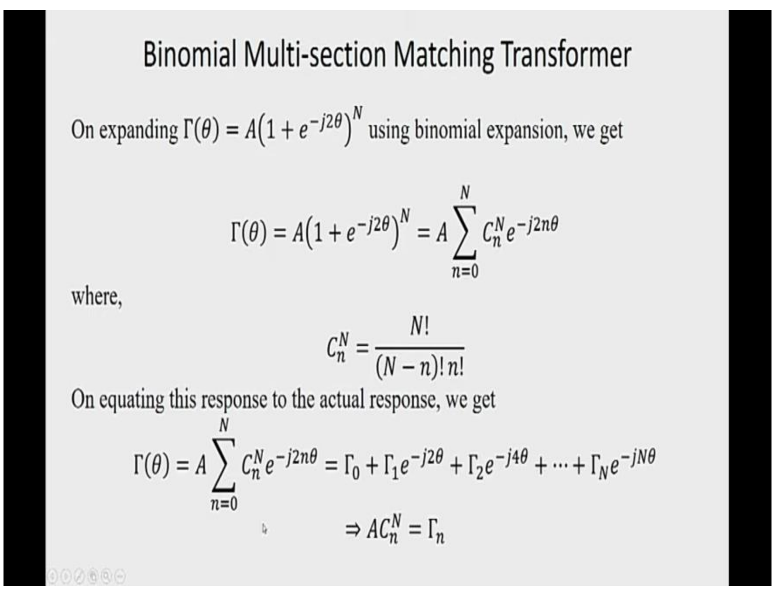
where,

$$C_n^N = \frac{N!}{(N-n)!\,n!}$$

On equating this response to the actual response, we get

$$\Gamma(\theta) = A\sum_{n=0}^{N} C_n^N e^{-j2n\theta} = \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \cdots + \Gamma_N e^{-jN\theta}$$

$$\Rightarrow AC_n^N = \Gamma_n$$



Now we can expand this expression 1 plus e to the power minus j 2 theta raise to the power N using binomial expansion and we can write gamma theta is equal to A sum of n equal to 0 to capital N and C_n^N e to the power minus j2n theta, where we are defining C_n^N as factorial N divided by factorial capital N minus n into factorial small n. And what we can do, as we said that now we equate it to the response of a multi section transformer which is given by gamma nought plus gamma 1 e to the power minus j2 theta plus gamma 2 e to the power minus j4 theta. And what we can do, we can equate on a term by term basis then we find A, C_n^N is equal to gamma n.

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Now

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n}$$

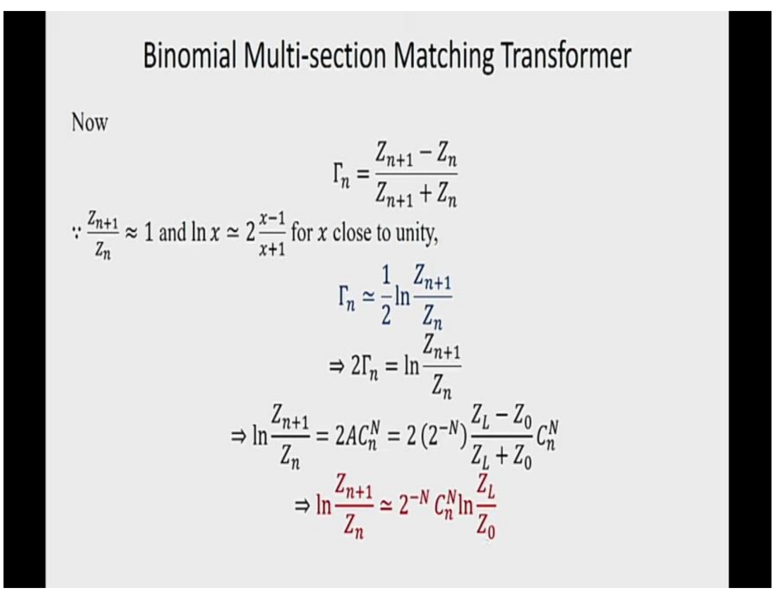
$\because \frac{Z_{n+1}}{Z_n} \approx 1$ and $\ln x \simeq 2\frac{x-1}{x+1}$ for x close to unity,

$$\Gamma_n \simeq \frac{1}{2}\ln\frac{Z_{n+1}}{Z_n}$$

$$\Rightarrow 2\Gamma_n = \ln\frac{Z_{n+1}}{Z_n}$$

$$\Rightarrow \ln\frac{Z_{n+1}}{Z_n} = 2AC_n^N = 2\,(2^{-N})\frac{Z_L - Z_0}{Z_L + Z_0}C_n^N$$

$$\Rightarrow \ln\frac{Z_{n+1}}{Z_n} \simeq 2^{-N}\,C_n^N \ln\frac{Z_L}{Z_0}$$



Now we have gamma n is Z_n plus 1 minus Z_n divided by Z_n plus 1 plus Z_n. What we can do as we have said that the change in characteristic impedance from one section to the next section is very small. So we can use this equation log of x is approximately equal to 2 x minus 1 divided by x plus 1 when x is close to unity. So our Z_n plus 1 by Z_n is close to unity, and using this expression for log x we can write gamma n to be equal to half log Z_n plus 1 divided by Z_n. Therefore log Z_n plus 1 by Z_n we can now substitute gamma n which is 2A $C_n^{\,N}$ and then substitute the value of A, which is 2 to the power minus N Z $_l$ minus Z_0 divided by Z_L plus Z_0 what we have derived earlier.

And then we can further simplify it into log of Z_n plus 1 by Z_n is approximately 2 to the power minus N, $C_n^{\,N}$ log Zl by Z nought. Now this gives us a relationship where we can calculate Z_n, Z small n alternatively where we can calculate Z small n recursively. So we start with Z_0, please note that right hand side is known so we can calculate Z_1 then once Z_1 is known we can calculate Z2 and in that way we can keep calculating till Z capital N. We can find the characteristic impedance of all the lambda by four sections that are used in that multi-section transformer design.

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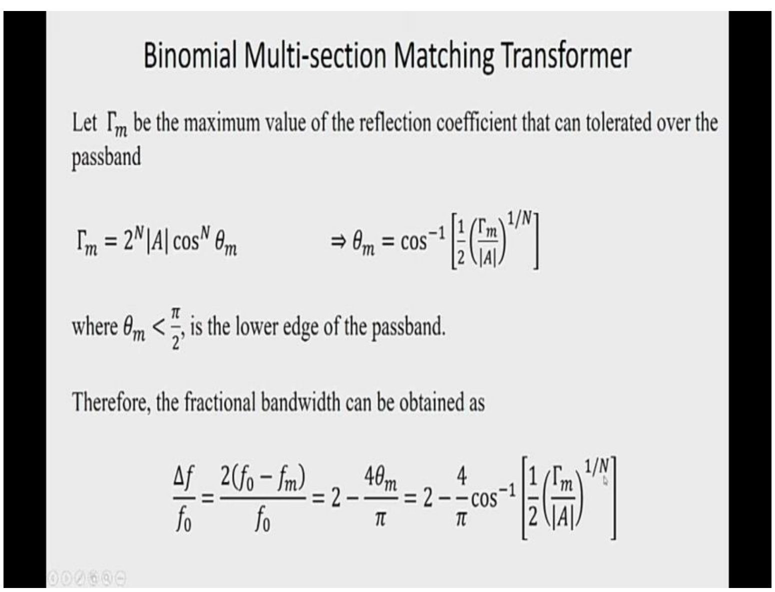
Let Γ_m be the maximum value of the reflection coefficient that can tolerated over the passband

$\Gamma_m = 2^N |A| \cos^N \theta_m \qquad\qquad \Rightarrow \theta_m = \cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right]$

where $\theta_m < \frac{\pi}{2}$, is the lower edge of the passband.

Therefore, the fractional bandwidth can be obtained as

$$\frac{\Delta f}{f_0} = \frac{2(f_0 - f_m)}{f_0} = 2 - \frac{4\theta_m}{\pi} = 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right]$$



We can also have an estimate of the bandwidth that we can have for this type of a multi-section quarter-wave matching transformer. Let gamma m be the maximum reflection coefficient that we can tolerate over the passband then we can write gamma m is equal to 2 to the power N mod A cos raised to the power N theta M and from there we can find out theta m corresponding to that value of gamma m, and we know that the reflection coefficient will increase on either side of the design frequency for which we have theta is equal to pi by 2.

So we take this theta m value less than Pi by 2 and now we can write fractional bandwidth delta f by f_0 is equal to two times f nought minus f_m divided by f nought, f_m is that frequency at which the reflection coefficient becomes gamma m and that can be written as 2 4 by pi cos inverse half gamma m divided by modulus of A raised to the power 1 by N. Here this N is the number of sections that are present in the transformer.

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Let the impedance $Z$of the line varies with z as shown

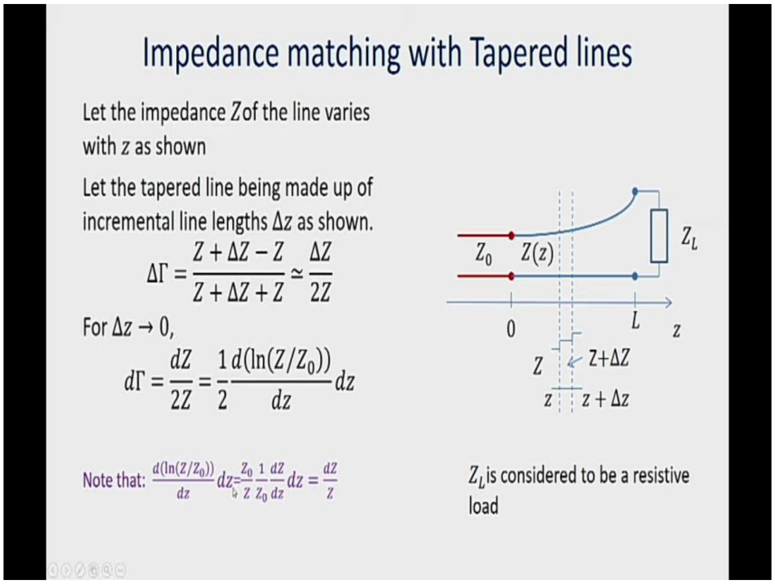
Let the tapered line being made up of incremental line lengths Δz as shown.

$$\Delta\Gamma = \frac{Z + \Delta Z - Z}{Z + \Delta Z + Z} \simeq \frac{\Delta Z}{2Z}$$

For $\Delta z \to 0$,

$$d\Gamma = \frac{dZ}{2Z} = \frac{1}{2}\frac{d(\ln(Z/Z_0))}{dz}dz$$

Note that: $\frac{d(\ln(Z/Z_0))}{dz}dz = \frac{Z_0}{Z}\frac{1}{Z_0}\frac{dZ}{dz}dz = \frac{dZ}{Z}$



Let us now move onto another method of impedance matching, the impedance matching with tapered lines. Here the impedance capital Z of the line varies with distance z and we are showing this in this figure. So at small z is equal to zero we have capital Z, Z is equal to Z_0 and over a length L this impedance capital Z changes to Z_L. Now what we can do, we can consider this tapered line being made up of incremental line lengths delta Z as shown.

So here you can see we have an incremental length delta Z, on the left-hand side we have impedance capital Z and over this length small delta z we assume the impedance to be capital Z plus delta capital Z and therefore at this interface the reflection coefficient the partial reflection coefficient delta gamma can be written as Z plus delta Z minus Z divided by Z plus delta Z plus Z. Now in the denominator we neglect this delta Z term and we can write the approximate expression as delta Z by 2Z. Now when this delta Z tends to zero this length tends to zero, we can write delta gamma as d gamma.

So d gamma becomes equal to d of capital Z divided by 2Z and this can be written in this form half d of log Capital Z by Z_0 divided by dz into dz. This can be seen from here because d of log of capital Z by Z_0 by dz into dz is essentially d of capital Z by capital Z.

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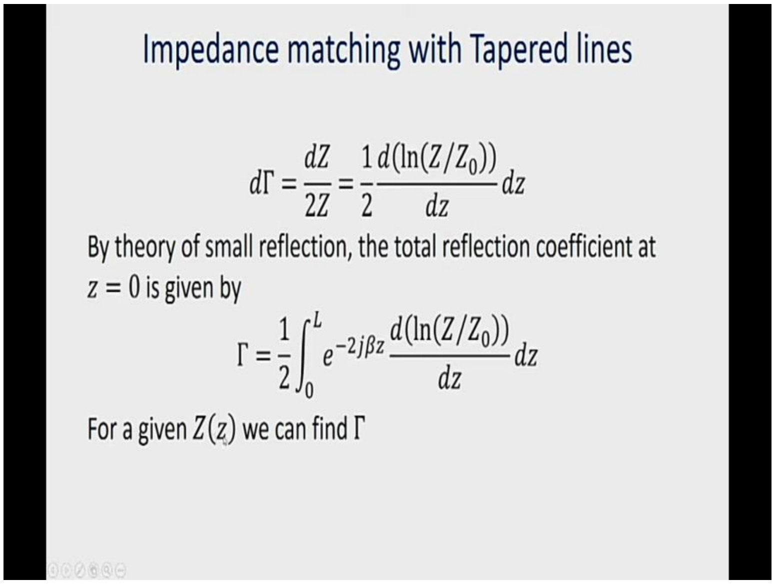
$$d\Gamma = \frac{dZ}{2Z} = \frac{1}{2}\frac{d(\ln(Z/Z_0))}{dz}dz$$

By theory of small reflection, the total reflection coefficient at $z = 0$ is given by

$$\Gamma = \frac{1}{2}\int_0^L e^{-2j\beta z}\frac{d(\ln(Z/Z_0))}{dz}dz$$

For a given $Z(z)$ we can find Γ



Next what we do, now we have this expression we apply again the theory of small reflection. Now the total reflection coefficient at Z is equal to zero. That means at the input of the tapered line is given by gamma is equal to half integration 0 to L, e to the power minus 2j beta z, this is the phase shift and d of log of capital Z by Z_0 by dz into dz. Now this gives the overall reflection coefficient at the input of the tapered line. Please note that if we know Z z that means the impedance variation as a function of distance Z we can find gamma.

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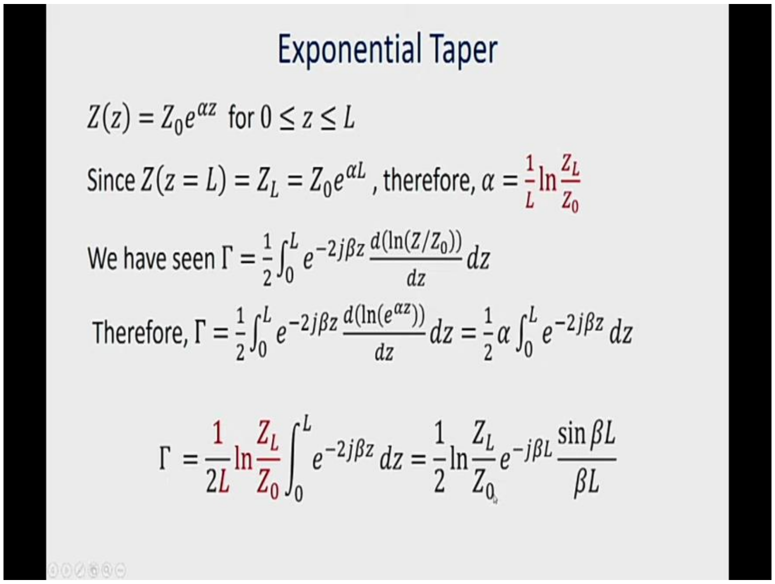
$Z(z) = Z_0 e^{\alpha z}$ for $0 \le z \le L$

Since $Z(z = L) = Z_L = Z_0 e^{\alpha L}$, therefore, $\alpha = \frac{1}{L}\ln\frac{Z_L}{Z_0}$

We have seen $\Gamma = \frac{1}{2}\int_0^L e^{-2j\beta z}\frac{d(\ln(Z/Z_0))}{dz}dz$

Therefore, $\Gamma = \frac{1}{2}\int_0^L e^{-2j\beta z}\frac{d(\ln(e^{\alpha z}))}{dz}dz = \frac{1}{2}\alpha\int_0^L e^{-2j\beta z}\,dz$

$$\Gamma = \frac{1}{2L}\ln\frac{Z_L}{Z_0}\int_0^L e^{-2j\beta z}\,dz = \frac{1}{2}\ln\frac{Z_L}{Z_0}e^{-j\beta L}\frac{\sin\beta L}{\beta L}$$



So let us see how we can do that, so we consider a very commonly used taper which is an exponential taper. Here capital Z as a function of distance Z is given by Z_0 e to the power alpha z, for zero less than z and Z less than capital L. Now since this capital Z at z is equal to L is Z_L and therefore we can find alpha to be equal to 1 by L log of ZL by Z_0. And we have the expression for gamma, now in this expression we have we can now substitute Z equal to Z_0 e to the power alpha z so this Z_0 and this Z_0 will get canceled, we will be left with log of e to the power alpha z and this can be log of e to the power alpha Z will give alpha z.

When differentiated we will be left with alpha and integration of 0 to capital L e to the power minus 2 j beta z dz. Now we substitute the expression for alpha from 1 by L log of capital Z_L by Z_0 and then carry out this integration, then we get the final expression for the reflection coefficient gamma to be half log of Z_L by Z_0 e to the power minus j beta L sine beta L by beta L.

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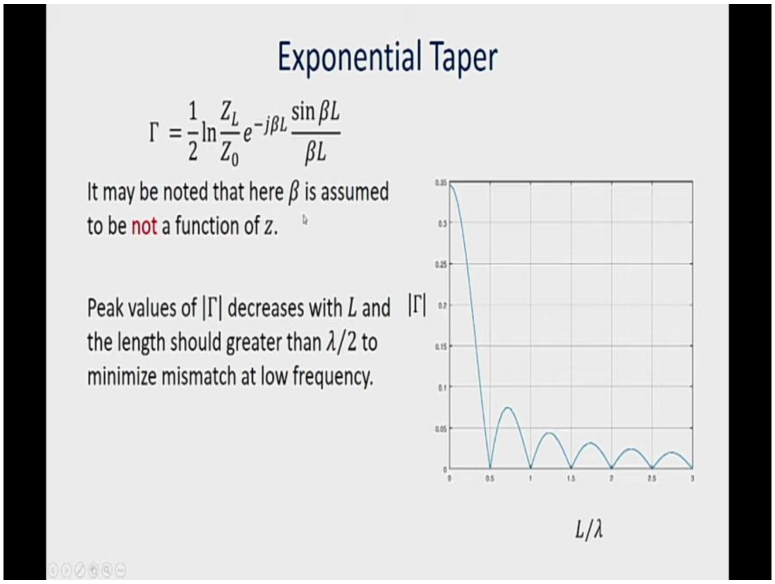
$$\Gamma = \frac{1}{2}\ln\frac{Z_L}{Z_0}e^{-j\beta L}\frac{\sin\beta L}{\beta L}$$

It may be noted that here β is assumed to be not a function of z.

Peak values of $|\Gamma|$ decreases with L and the length should greater than $\lambda/2$ to minimize mismatch at low frequency



The magnitude of the reflection coefficient is plotted in this figure as a function of L by lambda, and we can see that as L by lambda increases the peak value of mod gamma decreases, and the length should be greater than lambda by 2, at least lambda by two to minimize mismatch at low frequency. This is because this exponential taper is designed to operate for broadband of frequencies and we want the reflection coefficient to be less than some specified value, say if it is to be less than 0.1 even at the lowest operating frequency then we should have L by lambda at that frequency greater than 0.5.

So, the peak values of mod gamma decrease with Land length should be greater than lambda by 2 to minimize mismatch at low frequency, and it is to be noted that we have considered beta independent of z that means it is not a function of z and such assumptions are valid for TEM lines. There are other forms of tapers like triangular tapers, and the impedance matching can be performed with such tapers. This actually brings to the end of this module.

In this module we have discussed in detail some of the impedance matching techniques starting with the lambda elements then we have seen matching with stubs. We have seen how impedance matching can be done using a quarter-wave transformer, we have seen how we can design a multi-section quarter-wave transformer, and finally we have seen how we can use taper line for doing impedance matching. Our next module will be on microwave resonators where we will study first series and parallel resonant circuits, and we will have discussion on Q factor bandwidth, and then we will discuss transmission line resonators as well as web guide resonators.