Microwave Engineering
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Lecture 14
Series and Parallel RLC Resonators

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We start a new module Microwave Resonators.

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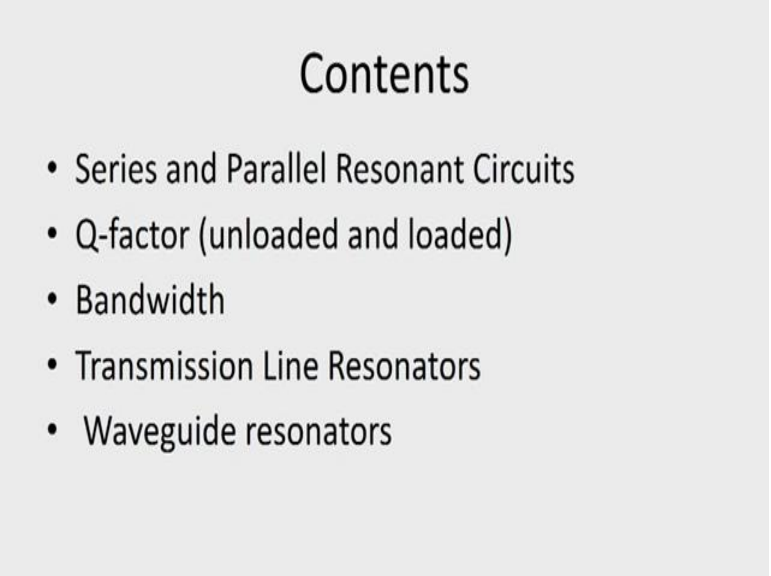

In this module, we cover the following contents. First, we discuss series and parallel resonant circuits, conventional circuits, then we discuss Q-factor both unloaded and loaded Q-factor.

We discuss the bandwidth of such circuits then we also cover transmission line resonators, and finally we discuss waveguide resonators.

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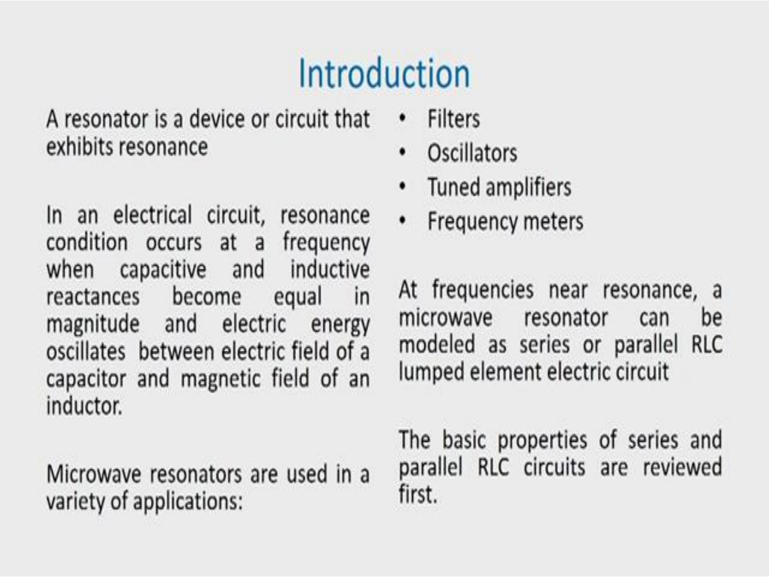

So, a resonator is a device or circuit that exhibits resonance. An electrical circuit resonance condition occurs at a frequency when the capacitive and inductive reactances become equal in magnitude, and electrical energy oscillates between electric field of a capacitor and magnetic field of an inductor. Microwave resonators are used in a variety of applications, for example, filters, oscillators, tuned amplifiers, frequency meters, these are some of the examples where microwave frequency resonators are used.

Now, at frequencies near resonance, a microwave resonator can be modeled as series or parallel RLC lumped-element circuit. Therefore, the basic properties of series and parallel RLC circuits are reviewed first.

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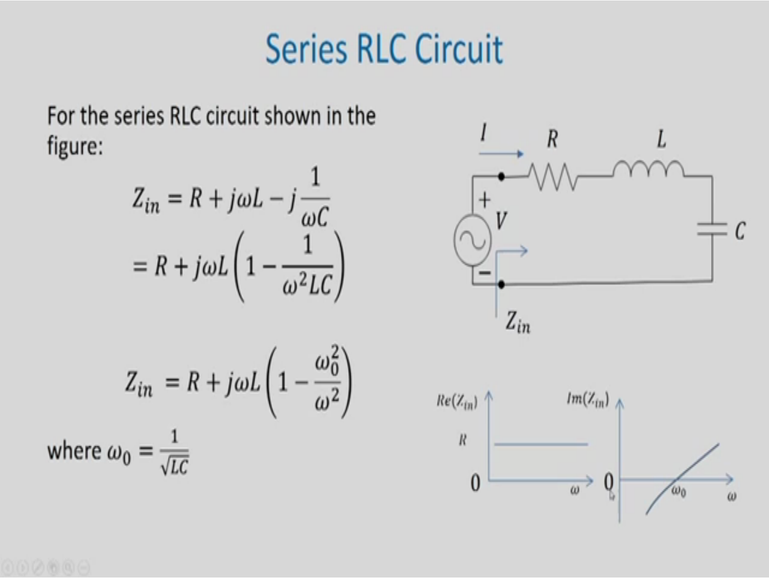
For the series RLC circuit shown in the figure:

$$Z_{in} = R + j\omega L - j\frac{1}{\omega C} = R + j\omega L\left(1 - \frac{1}{\omega^2 LC}\right)$$

$$Z_{in} = R + j\omega L\left(1 - \frac{\omega_0^2}{\omega^2}\right)$$

where $\omega_0 = \frac{1}{\sqrt{LC}}$



So, we consider series RLC circuit, and for the series, RLC circuit shown in the figure we can write Z in is equal to R plus j omega L minus j 1 by omega C, and then we can write Z_{in} equal to R plus j omega L into 1 minus 1 by omega square LC. Now, writing omega naught square is equal to 1 by LC. We find that Z in can be written as R plus j omega L 1 minus omega naught square divided by omega square.

So, omega naught is the resonant frequency because whenever omega is equal to omega naught the imaginary part of the input impedance become zero, and Z_{in} becomes purely (())(04:04). We should note that for a series RLC circuit, the real part of Z_{in}, this is constant R and the imaginary part of Z_{in} below resonant frequency omega naught, this is capacitive and above resonant frequency omega naught it becomes inductive. So, if we plot the imaginary part of Z_{in} and here at omega equal to omega naught the resonant frequency the imaginary part of Z_{in} becomes zero.

(Refer Slide Time: 04:58)

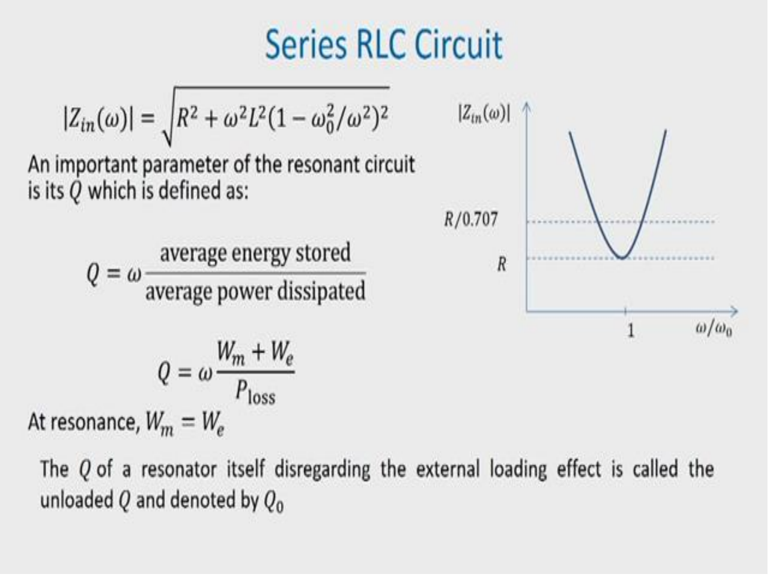
$$|Z_{in}(\omega)| = \sqrt{R^2 + \omega^2 L^2 (1 - \omega_0^2/\omega^2)^2}$$

An important parameter of the resonant circuit is its Q which is defined as:

$$Q = \omega \frac{\text{average energy stored}}{\text{average power dissipated}}$$

$$Q = \omega \frac{W_m + W_e}{P_{\text{loss}}}$$

At resonance, $W_m = W_e$



We can write magnitude of Z_{in} as a function of omega as under root R square plus omega square L square 1 minus omega naught square by omega square whole square and if we plot magnitude of Z_{in} omega then we find that from this expression at omega equal to omega naught, magnitude of Z_{in} omega becomes minimum and equal to R for other values of omega magnitude of Z_{in} omega is greater than R, and we have marked this horizontal line where it becomes R by 0.707 that means root to R.

These particular points see we will relate it to half-power bandwidth of the resonator. An important parameter of resonant circuit is it is Q which is defined as Q is equal to omega average energy stored divided by average power dissipated, this parameter Q is called the equality factor and therefore, we can write Q is equal to omega W_m the energy stored in magnetic field plus W_e the energy stored in the capacitor, this is the energy stored in the inductor W_m.

So, sum of these two energies it gives the total energy divided by P loss, the loss happens at the resistance R. Now, at resonance that means when omega is equal to omega naught W m becomes equal to W_e and therefore we can write Q it can be written as either omega naught $2W_m$ by P_{loss} or omega naught $2W_e$ by P_{loss} we will see and the Q of a resonator itself this regarding external loading.

That means we are assuming that no external load is connected to this resonator whatever Q that we are getting that is within the resonator itself that is because of the losses within the resonator itself and this is called unloaded Q and denoted by Q naught.

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Therefore, $Q_0 = \omega_0 \frac{2W_m}{P_{\text{loss}}}$

$$Q_0 = \omega_0 \frac{|I|^2 L}{|I|^2 R} = \omega_0 \frac{L}{R}$$

Since ${\omega_0}^2 = \frac{1}{LC}$

$$Q_0 = \frac{1}{\omega_0 RC}$$

Let us now study the behavior of the input impedance of a series RLC resonator near its resonance

$$Z_{in} = R + j\omega L \left(\frac{\omega^2 - \omega_0^2}{\omega^2} \right)$$

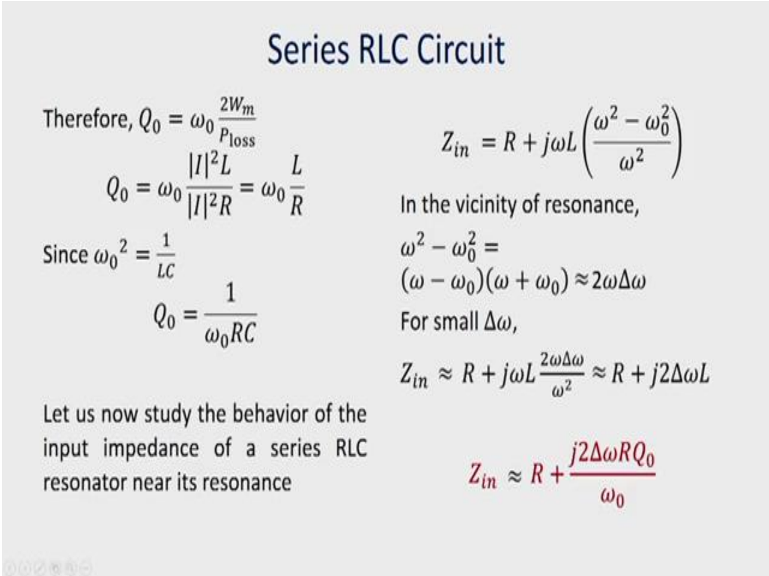
In the vicinity of resonance,

$\omega^2 - \omega_0^2 = (\omega - \omega_0)(\omega + \omega_0) \cong 2\omega\Delta\omega$

For small $\Delta\omega$,

$Z_{in} \cong R + j\omega L \frac{2\omega\Delta\omega}{\omega^2} \cong R + j2\Delta\omega L$

$$Z_{in} \cong R + \frac{j2\Delta\omega R Q_0}{\omega_0}$$



So, therefore Q naught is omega naught $2W_m$ divided by P_{loss}, and when we substitute the term related to the energy stored in the inductor and the power loss or dissipation in the resistor then we get Q naught equal to omega naught L by R and since omega naught square is equal to 1 by LC we can also write Q naught is equal to 1 by omega naught RC we can substitute L to be equal to 1 by omega naught square C and then we will get this term.

Now, let us study the behavior of the input impedance of a series RLC circuit or a series RLC resonator near it is resonance frequency. So, this expression Z_{in} is equal to R plus j omega L omega square minus omega naught square divided by omega square that we have already seen. So, what do we can do in the vicinity of the resonance? We can expand this term omega square minus omega naught square is equal to omega minus omega naught into omega plus omega naught.

Now, we write omega minus omega naught as delta omega and when we are close to the resonance omega plus omega naught can be written as 2 omega and therefore small delta omega we can write Z in is equal to R plus j omega L into 2 omega delta omega by omega square which becomes approximately equal to R plus j2 delta omega L and if we substitute from here L equal to R Q naught by omega naught then we get approximate expression for Z_{in} which is given by R plus j2 delta omega R Q naught by omega naught.

Now, this form of representation of the input impedance near it is resonant frequency this expression is a very useful expression, and we will make use of this expression in our later analysis of transmission line resonators.

(Refer Slide Time: 11:38)

Let us now consider half power fractional bandwidth of the resonator

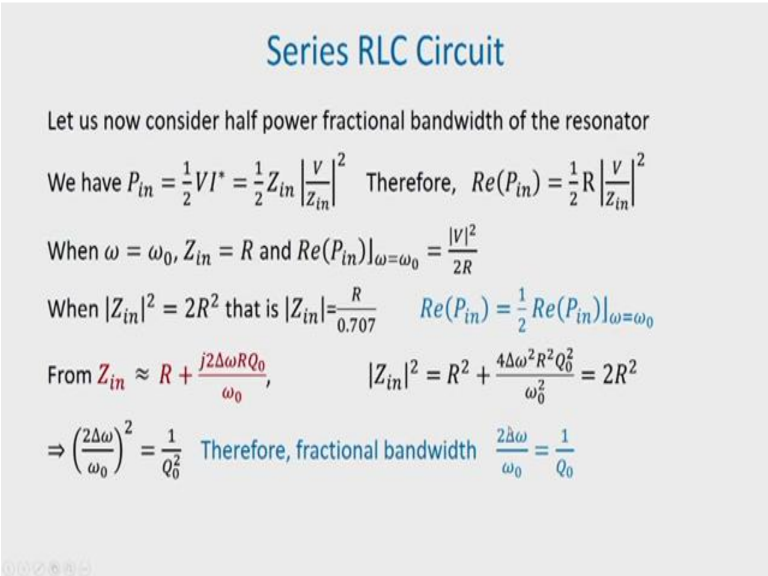
We have $P_{in} = \frac{1}{2} VI^* = \frac{1}{2} Z_{in} \left|\frac{V}{Z_{in}}\right|^2$ Therefore, $Re(P_{in}) = \frac{1}{2}\text{R}\left|\frac{V}{Z_{in}}\right|^2$

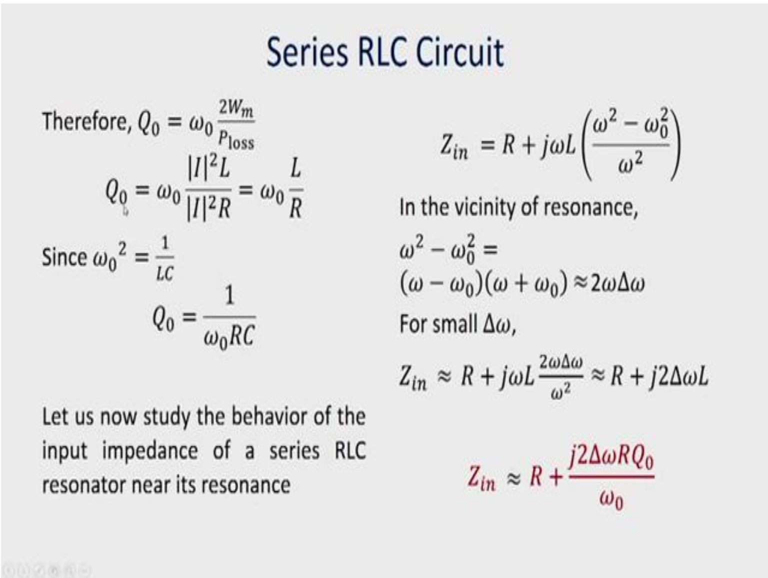
When $\omega = \omega_0$, $Z_{in} = R$ and $Re(P_{in})|_{\omega=\omega_0} = \frac{|V|^2}{2R}$

When $|Z_{in}|^2 = 2R^2$ that is $|Z_{in}|=\frac{R}{0.707}$ $Re(P_{in}) = \frac{1}{2} Re(P_{in})|_{\omega=\omega_0}$

From $Z_{in} \cong R + \frac{j2\Delta\omega R Q_0}{\omega_0}$, $|Z_{in}|^2 = R^2 + \frac{4\Delta\omega^2 R^2 Q_0^2}{\omega_0^2} = 2R^2$

$\Rightarrow \left(\frac{2\Delta\omega}{\omega_0}\right)^2 = \frac{1}{Q_0^2}$ Therefore, fractional bandwidth $\frac{2\Delta\omega}{\omega_0} = \frac{1}{Q_0}$





Let us now consider half-power fractional bandwidth of the resonator. Now, we have P_{in} is equal to half VI conjugates which can be written as half Z_{in} V by Z_{in} mod whole square and therefore the real part of P_{in} becomes real part of Z_{in} is R. So, half R V by Z_{in} mod square and when omega is equal to omega naught we have seen that Z_{in} is equal to R and therefore real part of P_{in} at omega equal to omega naught this becomes equal to mod V square by 2R.

And if you remember we have seen that mod Z_{in}, we mark the line when mod Z_{in} becomes equal to R by 0.707 that means root 2R. So, mod Z_{in} square at those frequencies become 2R square and when mod Z_{in} square becomes equal to 2R square, the real part of P_{in} becomes half of real part of P_{in} at omega equal to omega naught.

So, that is why this frequency points are called half-power points, and in the plot shown we see that we have two such points, one below omega naught and one is above omega naught and from this expression Z_{in} is equal to R plus j 2 delta omega R Q naught by omega naught we can find the modulus of Z_{in} square and this becomes R square 4 delta omega square R square Q naught square by omega naught square and we can equate this to 2R square.

So, once we do that we will get if you can take out R square common from here then R square will get cancel from both side, will be left with 1 here and when it is subtracted from here then we will get 4 delta omega square Q naught square by omega naught square is equal to 1, and therefore we can write 2 delta omega by omega naught whole square is equal to 1 by Q naught square and this quantity 2 delta omega by omega naught this quantity we defined as fractional bandwidth, and therefore we see that the fractional bandwidth is given by 1 by Q naught.

So, higher the value of Q the fractional bandwidth will be narrower and we have seen that Q naught is given by omega naught L by R. So, if R increases, the power loss in the circuit increases Q will decrease and we will have larger fractional bandwidth whereas if R is very small then Q naught will be large and fractional bandwidth will be very small.

(Refer Slide Time: 16:01)

Series RLC Circuit

Let us now consider half power fractional bandwidth of the resonator

We have $P_{in} = \frac{1}{2}VI^* = \frac{1}{2}Z_{in}\left|\frac{V}{Z_{in}}\right|^2$ Therefore, $Re(P_{in}) = \frac{1}{2}\mathrm{R}\left|\frac{V}{Z_{in}}\right|^2$

When $\omega = \omega_0, Z_{in} = R$ and $Re(P_{in})|_{\omega=\omega_0} = \frac{|V|^2}{2R}$

When $|Z_{in}|^2 = 2R^2$ that is $|Z_{in}| = \frac{R}{0.707}$ $\quad Re(P_{in}) = \frac{1}{2}Re(P_{in})|_{\omega=\omega_0}$

From $Z_{in} \approx R + \frac{j2\Delta\omega R Q_0}{\omega_0}$, $\quad |Z_{in}|^2 = R^2 + \frac{4\Delta\omega^2 R^2 Q_0^2}{\omega_0^2} = 2R^2$

$\Rightarrow \left(\frac{2\Delta\omega}{\omega_0}\right)^2 = \frac{1}{Q_0^2}$ Therefore, fractional bandwidth $\frac{2\Delta\omega}{\omega_0} = \frac{1}{Q_0}$

So, for a highly tune circuit or a very narrow band resonator, we will have large value of Q naught, and for that we will have R to be very small.

(Refer Slide Time: 16:21)

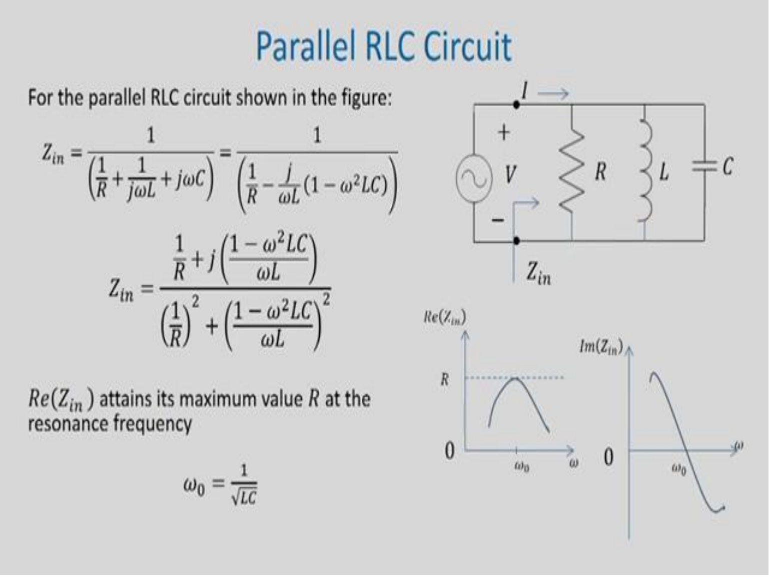
For the parallel RLC circuit shown in the figure:

$$Z_{in} = \frac{1}{\left(\frac{1}{R} + \frac{1}{j\omega L} + j\omega C\right)} = \frac{1}{\left(\frac{1}{R} - \frac{j}{\omega L}(1 - \omega^2 LC)\right)}$$

$$Z_{in} = \frac{\frac{1}{R} + j\left(\frac{1 - \omega^2 LC}{\omega L}\right)}{\left(\frac{1}{R}\right)^2 + \left(\frac{1 - \omega^2 LC}{\omega L}\right)^2}$$

$Re(Z_{in})$ attains its maximum value R at the resonance frequency

$\omega_0 = \frac{1}{\sqrt{LC}}$



Let us now consider the properties of parallel RLC circuit near its resonance. So, the parallel RLC circuit is shown in the figure. For this type of a circuit we can write Z_{in} is equal to 1 by, 1 by R plus 1 by j omega L plus j omega C and this can be written as 1 by 1 by R minus j by omega L 1 minus omega square LC. Now, we can find out the real and imaginary parts of Z_{in} if we multiply the numerator and the denominator by the complex conjugate of this term.

Now, real Z_{in} will attend it is maximum value R at the resonant frequency, and at the resonant frequency the imaginary Z_{in} will be zero and this resonant frequency is given by omega naught which is equal to 1 by root LC. Now this plot show the roughly the nature of variation of real part of Z_{in} and imaginary part of Z_{in} with frequency.

And we see that for a parallel RLC circuit below the resonant frequency omega naught it will be inductive in nature and above omega naught it will be capacitive in nature. In a series RLC circuit, we saw that below resonant frequency omega naught the circuit behaves like a capacitive circuit while above resonant frequency it is an inductive circuit.\

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For such parallel RLC circuit

$$Q_0 = \omega_0 RC = \frac{R}{\omega_0 L}$$

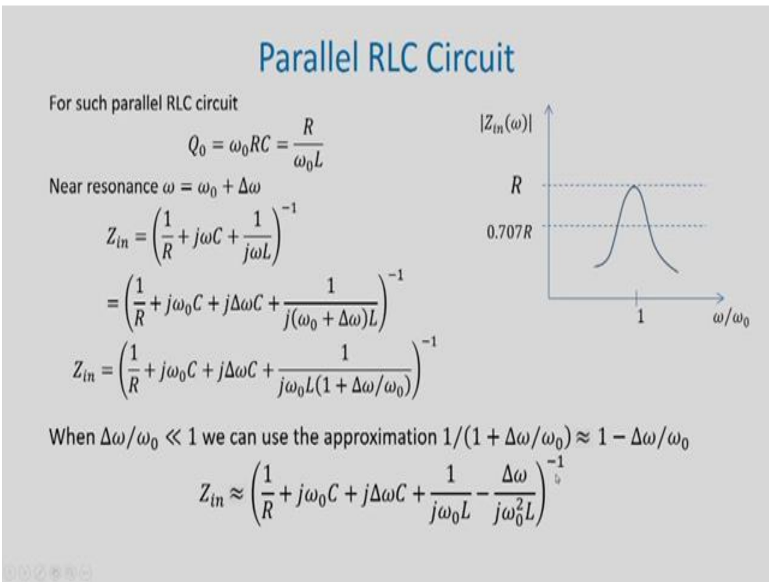
Near resonance $\omega = \omega_0 + \Delta\omega$

$$Z_{in} = \left(\frac{1}{R} + j\omega C + \frac{1}{j\omega L}\right)^{-1} = \left(\frac{1}{R} + j\omega_0 C + j\Delta\omega C + \frac{1}{j(\omega_0 + \Delta\omega)L}\right)^{-1}$$

$$Z_{in} = \left(\frac{1}{R} + j\omega_0 C + j\Delta\omega C + \frac{1}{j\omega_0 L(1 + \Delta\omega/\omega_0)}\right)^{-1}$$

When $\Delta\omega/\omega_0 \ll 1$ we can use the approximation $1/(1 + \Delta\omega/\omega_0) \cong 1 - \Delta\omega/\omega_0$

$$Z_{in} \cong \left(\frac{1}{R} + j\omega_0 C + j\Delta\omega C + \frac{1}{j\omega_0 L} - \frac{\Delta\omega}{j\omega_0^2 L}\right)^{-1}$$



Now, Q naught the unloaded Q factor for such resonant circuit can be found out and Q naught will be equal to omega naught RC and which is also R by omega naught L. Near it is resonance the frequency angular frequency omega can be written as omega naught plus delta omega and Z_{in} once we substitute omega is equal to omega naught plus delta omega can be expressed in this form.

And if we plot the variation of magnitude of Z_{in} with respect to omega by omega naught then we will have the maximum value of the magnitude of Z_{in} at omega equal to omega naught that means when this ratio is 1, and we will see that half-power points will be for magnitude of Z_{in} omega to be equal to 0.707 of it is maximum value that is R. Now, this term we can do some approximation particularly because of the fact that delta omega by omega naught is very, very less as compared to 1 when we are operating the circuit near the resonant frequency omega naught.

So, what we can do? This term 1 by 1 plus delta omega by omega naught, this can be approximated as 1 minus delta omega by omega naught. So, once we substitute this approximation then we get Z_{in} to be equal to 1 by R plus j omega naught C plus j delta omega C plus 1 by j omega naught L minus delta omega by j omega naught square L entire thing raised to the power minus 1.

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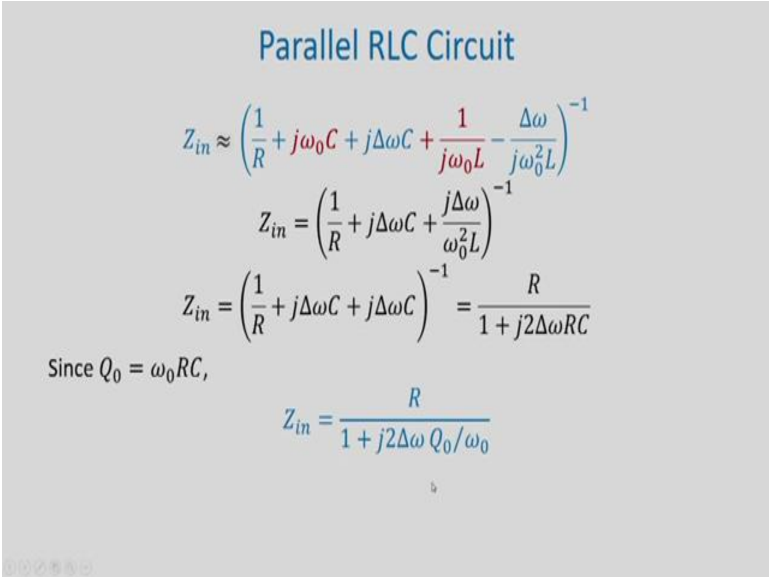
$$Z_{in} = \left(\frac{1}{R} + j\omega_0 C + j\Delta\omega C + \frac{1}{j\omega_0 L} - \frac{\Delta\omega}{j\omega_0^2 L}\right)^{-1}$$

$$Z_{in} = \left(\frac{1}{R} + j\Delta\omega C + \frac{j\Delta\omega}{\omega_0^2 L}\right)^{-1}$$

$$Z_{in} = \left(\frac{1}{R} + j\Delta\omega C + j\Delta\omega C\right)^{-1} = \frac{R}{1 + j2\Delta\omega RC}$$

Since $Q_0 = \omega_0 RC$,

$$Z_{in} = \frac{R}{1 + j2\Delta\omega\, Q_0/\omega_0}$$



Now, this is the form of Z_{in} we have seen, and these two terms j omega naught C plus 1 by j omega naught L when to combine it will give 1 minus omega naught square LC and therefore these two term will cancel out. So, we find that this j omega naught C and 1 by j omega naught L when combining they will give term like 1 minus omega square LC and therefore it will become sum of these two terms will become 0 and we will be left with Z_{in} equal to 1 by R plus j delta omega C plus j delta omega by omega naught square L whole raised to the power minus 1.

Now, once we substitute omega square is equal to 1 by LC here we will see that this term j delta omega by omega naught square L will be reduced to j delta omega C and therefore Z_{in} can be written as 1 by R plus j delta omega C from here and another j delta omega C from here whole raised to the power minus 1 and therefore it can be written as R divided by 1 plus j2 delta omega RC.

Since we have Q naught is equal to omega naught RC, therefore, we can now write Z_{in} equal to R divided by 1 plus j2 delta omega Q naught by omega naught. So, this expression of input impedance gives the input impedance of the parallel resonant circuit near it is resonance, and you can see that this expression also includes Q naught the unloaded Q factor and also omega naught the angular resonant frequency.

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$$Re(P_{in}) = \frac{1}{2} V \left(\frac{V}{R}\right)^* = \frac{1}{2}\frac{|V|^2}{R} = \frac{1}{2}|I|^2|Z_{in}|^2\frac{1}{R}$$

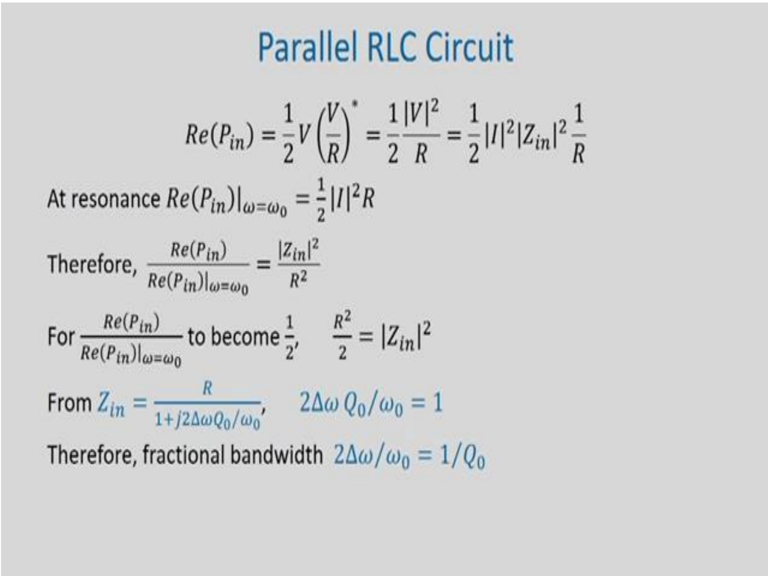
At resonance $Re(P_{in})|_{\omega=\omega_0} = \frac{1}{2}|I|^2 R$

Therefore, $\frac{Re(P_{in})}{Re(P_{in})|_{\omega=\omega_0}} = \frac{|Z_{in}|^2}{R^2}$

For $\frac{Re(P_{in})}{Re(P_{in})|_{\omega=\omega_0}}$ to become $\frac{1}{2}$, $\quad \frac{R^2}{2} = |Z_{in}|^2$

From $Z_{in} = \frac{R}{1+j2\Delta\omega Q_0/\omega_0}$, $\quad 2\Delta\omega\, Q_0/\omega_0 = 1$

Therefore, fractional bandwidth $2\Delta\omega/\omega_0 = 1/Q_0$



Now, let us develop the expression for the fractional bandwidth. From the parallel RLC circuit, we see that real part of P in input power is equal to half V the voltage it is a parallel RLC circuit so the voltage will be same across all the elements. So, it is half V, V by R conjugate and V by R is essentially the current through the resistance and therefore we can write it half mod V square by R and which is equal to half I square mod of Z_{in} square divided by R because V is written as I into Z_{in}.

Now, at resonance that means when omega equal to omega naught real part of P_{in} becomes equal to half mod of I square R because we have seen that at omega equal to omega naught mod Z_{in} become equal to R and therefore we can now write real part of P_{in} divided by real part of P_{in} at omega equal to omega naught this can be written as mod of Z_{in} square divided by R square and for this quantity to become half because we are talking of half-power bandwidth we must have R square by 2 equal to mod of Z_{in} square.

Now, we already have an expression for the approximate input impedance of the parallel RLC circuit near it is resonant frequency and therefore we can write from Z_{in} is equal to R divided by 1 plus j 2 delta omega Q naught by omega naught from this expression we can see that mod Z_{in} square will become R square by 2 when we have 2 delta omega Q naught by omega naught is equal to 1.

And the fractional bandwidth is given by 2 delta omega by omega naught, and therefore once again, we find that just as the case of series RLC circuit the fractional bandwidth 2 delta omega by omega naught is equal to 1 by Q naught. That means for higher Q the fractional bandwidth will be very less and for lower Q we will have larger fractional bandwidth.

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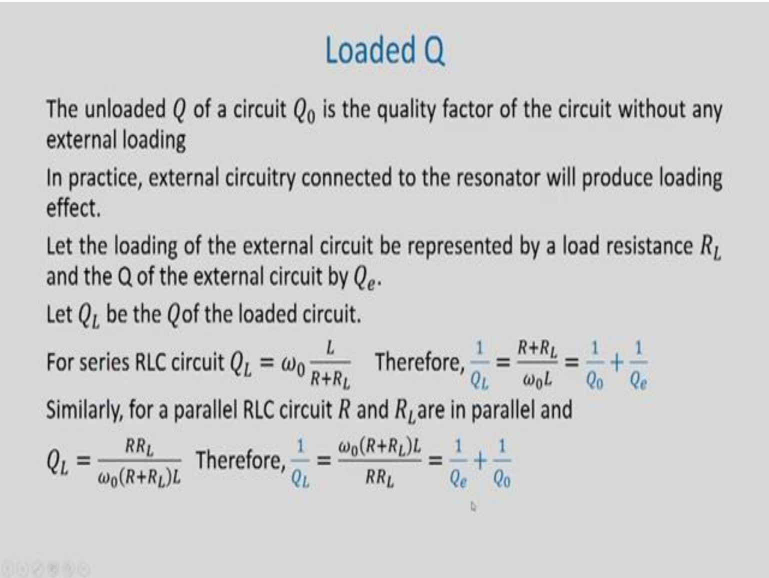
Let the loading of the external circuit be represented by a load resistance R_L and the Q of the external circuit by Q_e.

Let Q_L be the Q of the loaded circuit.

For series RLC circuit $Q_L = \omega_0 \frac{L}{R+R_L}$ Therefore, $\frac{1}{Q_L} = \frac{R+R_L}{\omega_0 L} = \frac{1}{Q_0} + \frac{1}{Q_e}$

Similarly, for a parallel RLC circuit R and R_L are in parallel and

$Q_L = \frac{RR_L}{\omega_0 (R+R_L) L}$ Therefore, $\frac{1}{Q_L} = \frac{\omega_0 (R+R_L) L}{RR_L} = \frac{1}{Q_e} + \frac{1}{Q_0}$



Now, so far we were discussing Q naught the unloaded Q. Let us now consider another form of quality factor, which is called the Loaded Q. The unloaded Q of a circuit Q naught is the quality factor of the circuit without any external loading. In practice, external circuitry that will be connected to the resonator will produce loading effect and let us see how we can take this loading effect into account.

Let us represent the loading of the external circuit by a load resistance R_L and Q of the external circuit by Q_e and Q_L with a Q of the loaded circuit. Now, when we consider series RLC circuit our external resistance R_L will come in series with the resistance of the RLC circuit R and therefore the effective resistance will be R plus R_L and therefore by definition of Q we can write Q_L that of the loaded circuit is equal to omega naught L divided by R plus R_L.

If you write 1 by Q_L then it becomes R plus R_L divided by omega naught L, and this term R divided by omega naught L can be written as 1 by Q naught and R_L divided by omega naught L can be written as 1 by Q_e. So, the Q of the loaded RLC series RLC circuit 1 by Q_L is 1 by Q naught the unloaded Q plus 1 by Q_e the Q of the external circuit. Let us now consider the parallel RLC circuit.

In case of parallel RLC circuit, the external resistance R_L will come in parallel with R and the effective resistance will now be R into R_L divided by R plus R_L and therefore for a parallel circuit Q_L can be written as RR L divided by omega naught R plus R_L into L and if we write 1 by Q_L can be written as omega naught R plus R_L into L divided by RR L and when this is separated then we will get omega naught L by R_L which is 1 by Q_e plus omega naught L by R which is 1 by Q naught.

So, once again we find that for a parallel RLC circuit also 1 by Q_L is equal to 1 by Q_e plus 1 by Q naught. So, once we know the unloaded Q of the series or parallel RLC circuit and we know the external loading that means R_L we can find out the Q of the loaded circuit. So, we have studied series RLC and parallel RLC circuits and particularly how such circuits behave near their resonant frequency.

Next we will see how transmission line sections either short or open can act as resonators and also near the resonant frequency of how we can model this type of transmission line resonators by equivalent RLC circuit.