Microwave Engineering
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Lecture 05
Rectangular and Circular Waveguides

In module one, we introduced a brief history of microwave and then we discussed different microwave frequency bands and also introduced the concept of microwave transmission lines. Then we have seen the lumped element representation of microwave transmission line. Thereafter, we developed the telegrapher's equation then we discussed the wave propagation on a transmission line.

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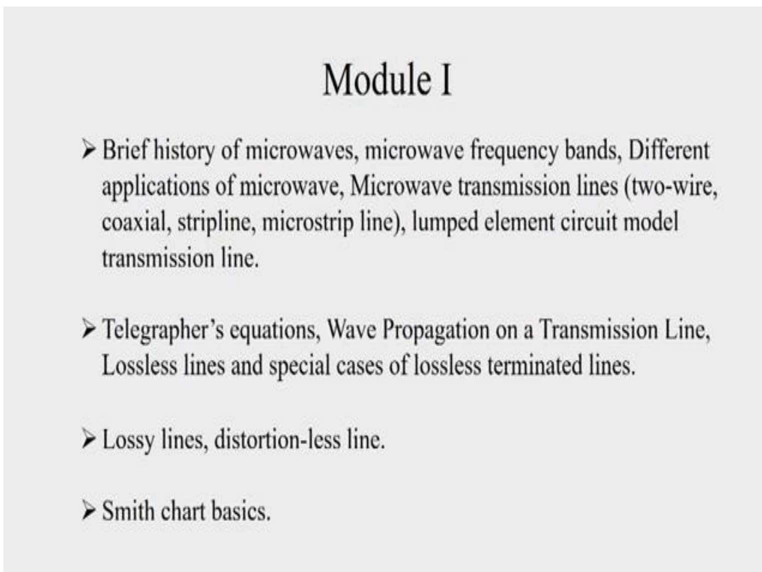

Then we discussed the lossless lines and special cases of lossless terminated lines. So then we discussed lossy transmission lines and also discussed the distortionless line. Thereafter, we introduced a graphical tool which is called Smith chart, and we discussed the basics of Smith charts and how this Smith chart can be utilized in solving transmission line problems.

In module two, we discussed waveguides, we briefly described the different modes of wave propagation, TEM and TE and TM and then we discussed how we could have TE and TM mode of propagation in rectangular waveguide, their cutoff frequencies. Next we discussed TE and TM modes in a circular waveguide, and we derived expression for cut off frequency of such modes in a circular waveguide, and finally in this module we discuss the attenuation of waves in rectangular and circular waveguides.

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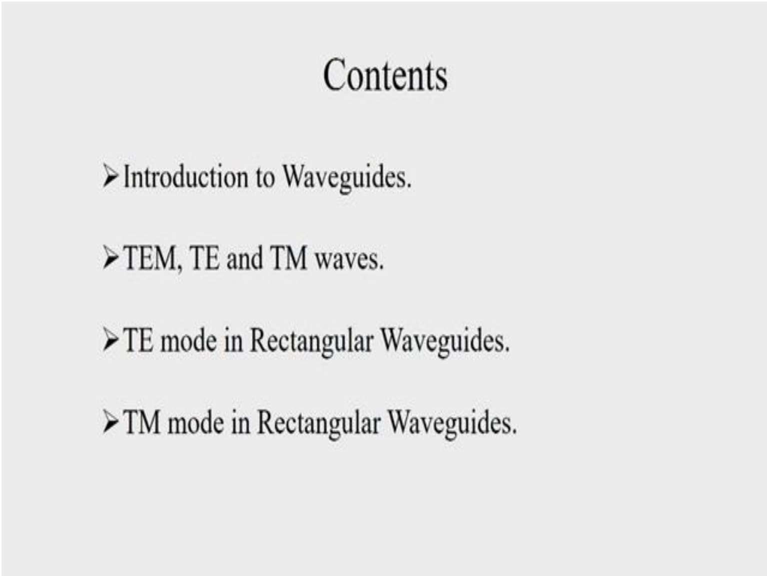

In this lecture, we will cover the following contents. a brief introduction to waveguides. This will be followed by discussion on TEM, TE and TM waves. Then we discussed the TE mode in rectangular waveguides then we consider the propagation of TM mode in rectangular waveguide. So a waveguide is a metallic tube which is used to guide electromagnetic waves. This is in the most basic form. Of course, there are other forms of waveguides, also there like dielectric waveguides.

But we will discuss only the hollow metallic waveguides. In a waveguide, electric and magnetic fields are confined to space within the guide by the surrounding conducting wall. It is possible to propagate several modes of EM waves within a waveguide. These modes correspond to solution of Maxwell's equation for particular waveguide geometry.

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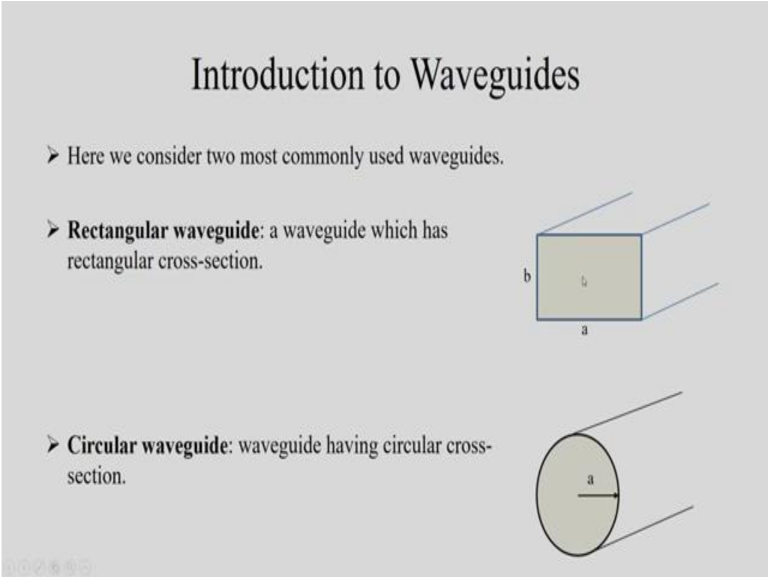

Now let us see the most commonly used form of waveguides, which are rectangular waveguide, having a cross-section that is rectangular and circular waveguide, having a cross-section circular. Please note that we are only showing the outline of the waveguide. In a practical waveguide, these metallic walls will have finite thickness and these dimensions a and b or the radius a, what we are showing here, these are the inner dimensions.

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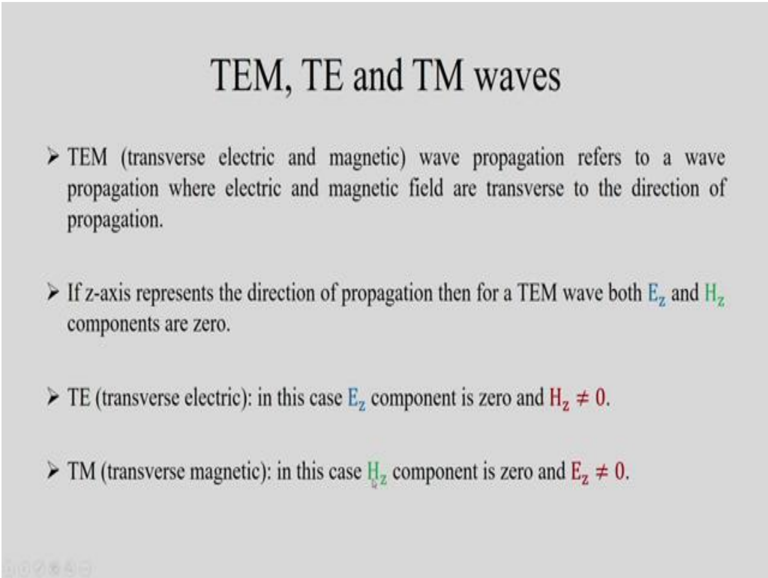

Now, TEM or transverse electromagnetic wave, it refers to a wave propagation where electric and magnetic fields are transverse to the direction of propagation. For example, if you consider, z-axis representing the direction of propagation then for a TEM wave both E_z and H_z components will be 0. So, the electric and magnetic field will be confined on the XY plane and may have E_x E_y or H_x H_y component.

For example, if you consider a coaxial transmission line, there the propagation mode is TEM. We have E row component and H phi component, and there is no E_z or H_z component. In the same manner, transverse electric or TE, in that case, E_z component is 0 and H_z is not 0. So, in a transverse electric wave or a TE wave, E_z component will always be 0. So electric field is transverse to the direction of propagation and the axial component of the magnetic field H_z will not be 0.

Similarly, in transverse magnetic case, a $H_z$component will be 0 (())(inaudible: 06:31) and the E_z is not equal to 0.

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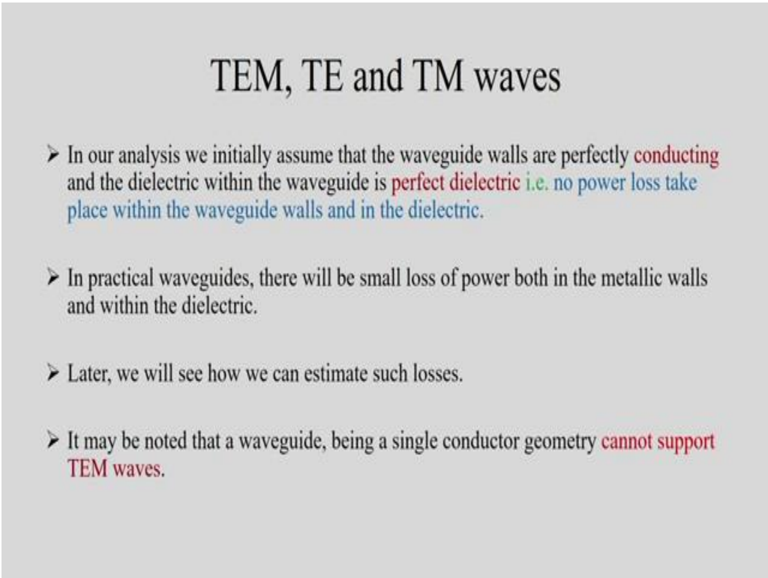

Now, while doing the waveguide analysis, we assume that the waveguide boundaries are perfectly conducting and the dielectric region that is enclosed within this metallic boundary, they are perfect dielectric. That means we are not considering initially, any power loss that takes place, either at the walls of the waveguide or within the dielectric. When we consider practical waveguides, there will be small loss of power both in the metallic walls and within the dielectric, and later we will see how we can estimate such losses.

It should be emphasized here that, in a hollow waveguide, it is a single conductor geometry, it cannot support a TEM wave. This is because suppose E_z is equal to 0, then in order to support a transverse magnetic field, you require current in the longitudinal direction. Waveguides being single conductor does not have such current components, and therefore this guide of waveguiding structures cannot support TEM waves. Coaxial cable is a two-conductor system, and that is why it can support TEM wave propagation.

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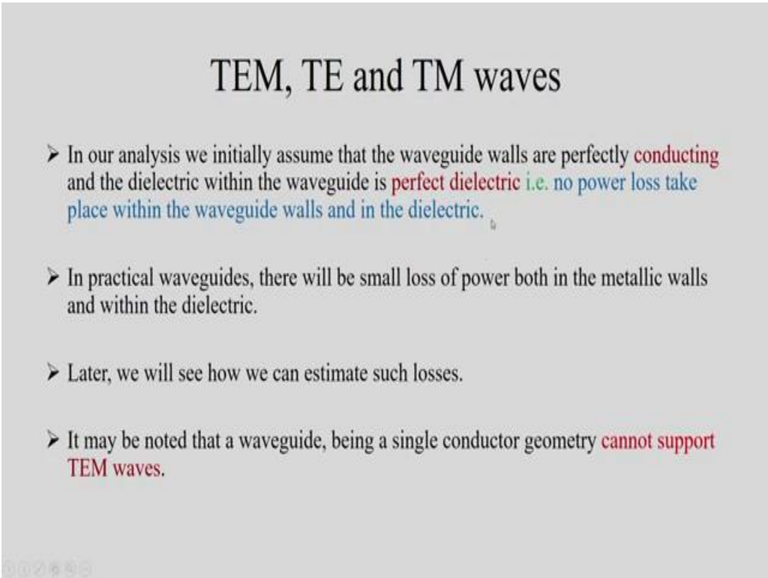
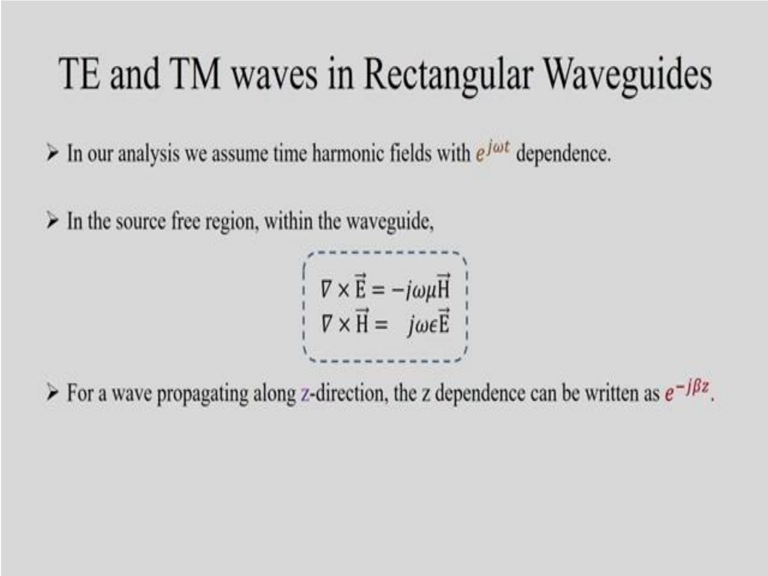


- In our analysis we assume time harmonic fields with $e^{j\omega t}$ dependence.
- In the source free region, within the waveguide,

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\nabla \times \vec{H} = j\omega\epsilon\vec{E}$$

- For a wave propagating along z-direction, the z dependence can be written as $e^{-j\beta z}$.
- For such dependence $\frac{\partial}{\partial z}$ results in to a multiplication by a factor $(-j\beta)$.



So, in a metallic waveguide, our solution for the EM waves will be either TE or TM. So in order to find the solutions, we assume that the fields are time-harmonic, that means time dependence is given in the form $e^{j\omega t}$ and if you consider the source-free region within the waveguide, we can write the following Maxwell's equation, Karl equations. We further assume that for a wave propagating in the z-direction, the z dependence can be written of the form $e^{-j\beta z}$.

So we have seen when this type of z dependence, it gives rise to a wave traveling in the plus z-direction. And for such dependence, this derivative, $\partial\ \partial_z$ essentially results in multiplication by a factor of minus jβ.

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From $\nabla \times \vec{E} = -j\omega\mu\vec{H}$, we can have

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -j\omega\mu\left[H_x\hat{a}_x + H_y\hat{a}_y + H_z\hat{a}_z\right]$$

Considering only the x-component from both sides, we have

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x$$

For a z dependence of the form $e^{-j\beta z}$

$$\frac{\partial E_z}{\partial y} + j\beta E_y = -j\omega\mu H_x$$

We evaluate the y and z components in the same manner,

$$-j\beta E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y$$

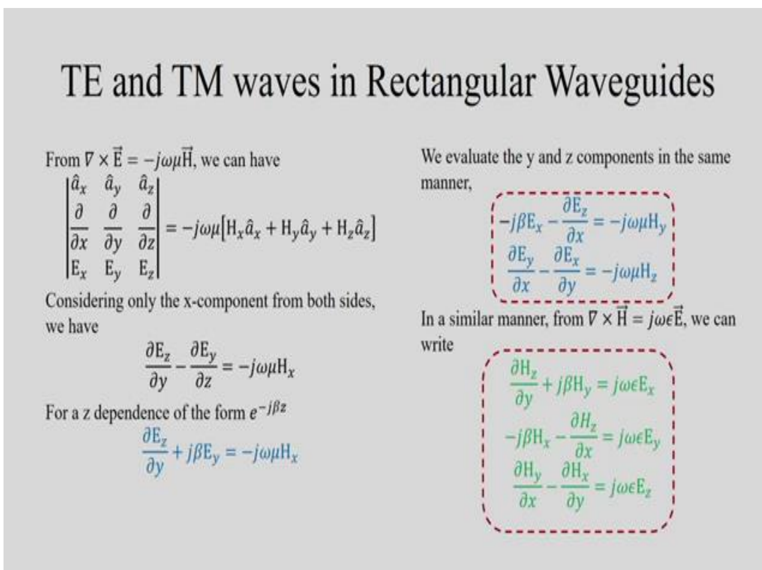
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z$$

In a similar manner, from $\nabla \times \vec{H} = j\omega\epsilon\vec{E}$, we can write

$$\frac{\partial H_z}{\partial y} + j\beta H_y = j\omega\epsilon E_x$$

$$-j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z$$



So keeping in mind these, we expand Maxwell's equations, then we get considering only the x – component from both sides, we have $\partial E_z\ \partial_y$ minus $\partial E_y\ \partial_z$ is equals to minus j$\omega\mu H_x$ and this $\partial\partial_z$ can now be replaced by $e^{-j\beta z}$ as we have discussed, and therefore when the z dependence is in the form $e^{-j\beta z}$ we could re-write the equation of this form, $\partial E_z\ \partial_y$, jβE_y minus j$\omega\mu H_x$.

If you take the other two components, H_y and H_z and their corresponding equations, then we can get these two sets of the equation. And if you consider$\nabla \times H = j\omega\epsilon\vec{E}$, then we get this set of equations.

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$$\frac{\partial E_z}{\partial y} + j\beta E_y = -j\omega\mu H_x$$

$$-j\beta E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z$$

$$\frac{\partial H_z}{\partial y} + j\beta H_y = j\omega\epsilon E_x$$

$$-j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y$$

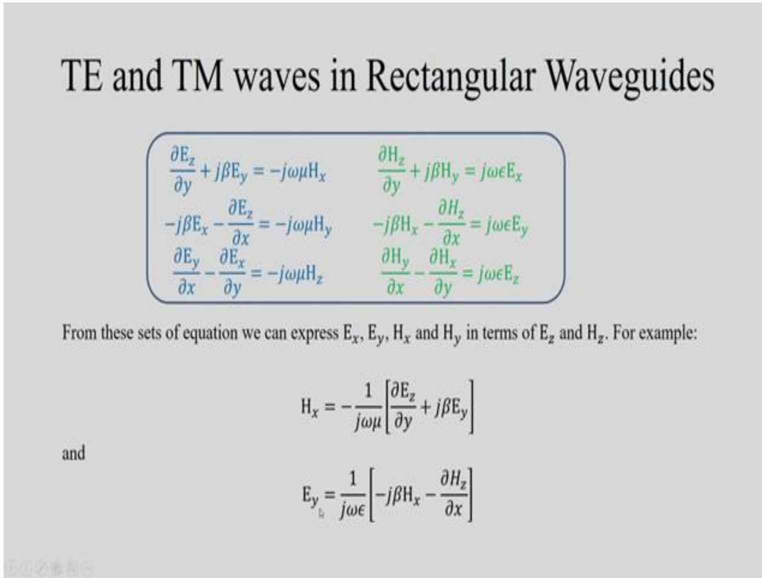
$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z$$

From these sets of equation we can express E_x, E_y, H_x and H_y in terms of E_z and H_z. For example:

$$H_x = -\frac{1}{j\omega\mu}\left[\frac{\partial E_z}{\partial y} + j\beta E_y\right]$$

and

$$E_y = \frac{1}{j\omega\epsilon}\left[-j\beta H_x - \frac{\partial H_z}{\partial x}\right]$$



Now the complete set of the equation is shown here. From this set of equation, what we can do, now we can see that different fill components, how they are related to each other. What we can do, we can express E_x, E_y, $H_{x,}$ H_y, please note that these are the transverse components in terms of E_z and H_z which are the longitudinal components?

So how we can have this relationship, if you consider, for example, the first equation, then we can write $H_x = -\frac{1}{j\omega\mu}\left[\frac{\partial E_z}{\partial_y} + j\beta E_y\right]$, from this equation, and please note that this equation involves E_y so it has E_z and E_y. E_z is the component which we want to retain, E_y is the component we want to substitute. Now from this equation, we find that this involves E_y, $H_{x,}$ and H_z. So if we substitute, E_y from this equation, we can write , E_y of this form $\frac{1}{j\omega\epsilon}$ $\left[-j\beta H_x - \frac{\partial H_z}{\partial_x}\right]$.

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Substituting for E_y, we get

$$H_x = -\frac{1}{j\omega\mu}\left[\frac{\partial E_z}{\partial y} - \frac{\beta}{\omega\epsilon}\left(-j\beta H_x - \frac{\partial H_z}{\partial x}\right)\right]$$

$$\Rightarrow H_x = -\frac{1}{j\omega\mu}\frac{\partial E_z}{\partial y} + \frac{\beta^2}{\omega^2\mu\epsilon}H_x - \frac{j\beta}{\omega^2\mu\epsilon}\frac{\partial H_z}{\partial x}$$

$$\therefore H_x\left(\frac{\omega^2\mu\epsilon - \beta^2}{\omega^2\mu\epsilon}\right) = -\frac{1}{j\omega\mu}\frac{\partial E_z}{\partial y} - \frac{j\beta}{\omega^2\mu\epsilon}\frac{\partial H_z}{\partial x}$$

Let

$$k^2 = \omega^2\mu\epsilon$$

and

$$k_c^2 = k^2 - \beta^2$$

$$\therefore H_x = \frac{j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right)$$

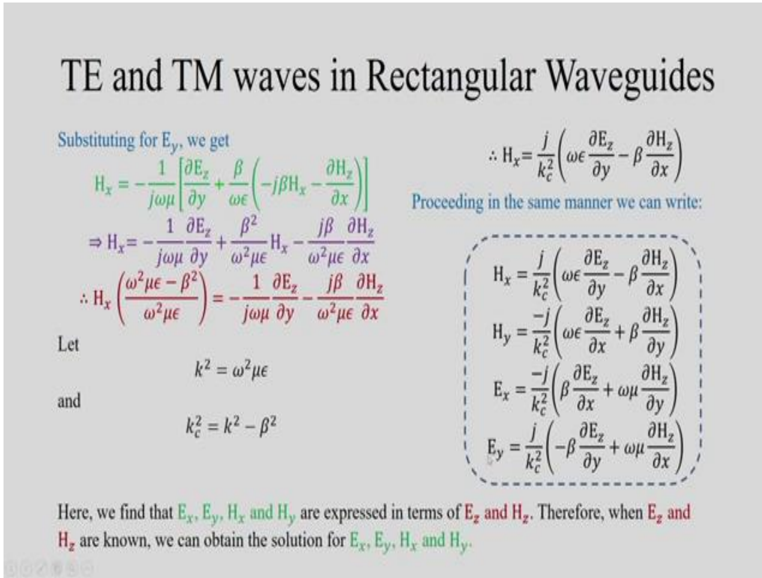
Proceeding in the same manner we can write:

$$H_x = \frac{j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right)$$

$$H_y = \frac{-j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y}\right)$$

$$E_x = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right)$$

$$E_y = \frac{j}{k_c^2}\left(-\beta\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right)$$



Now, this E_y can be substituted in the first equation of $H_{x,}$ once we do that $H_{x,}$ can be written in the function and once we rearrange the terms, once we expand and rearrange the terms we can express $H_{x,}$ in terms of E_z and H_z. So we introduced K^2 equal to $\omega^2\mu\epsilon$ and K_c^2 is equal to $K^2 - \beta^2$. So once we introduced, this is the wavenumber and this is the, K_c is the cut off wavenumber, we will see.

And in terms of k and K_c we can now write $H_x = \frac{j}{k_c^2}\left(\omega\epsilon\,\frac{\partial E_z}{\partial_y} - \beta\,\frac{\partial H_z}{\partial_x}\right)$. Please note that here we have now been able to express transverse component $H_{x,}$ in terms of E_z and H_z. If we proceed in the same manner, for other field components, H_y, E_x and E_y, then we get a set of four equations, which is shown here. Here we find that H_x , H_y, E_x and E_y all are related to E_z and H_z and therefore once we know, E_z and H_z we can always find out the trend transverse field component, H_x , H_y, E_x and E_y.

So solving the field equation within the waveguide region, essentially now becomes finding out the solution for E_z and H_z and depending upon whether E_z is 0 or H_z is 0 we will get the solutions which will be when E_z is 0 we will get TE solution and when H_z is 0 we get a TM solution. So one of this component, E_z or H_z needs to be there to support the transverse field component.

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With z dependence given by $e^{-j\beta z}$, we write

$$H_z(x,y,z) = h_z(x,y)e^{-j\beta z}$$

Further,

$$\nabla \times \vec{H} = j\omega\epsilon\vec{E}$$

$$\therefore \nabla \times \nabla \times \vec{H} = j\omega\epsilon\nabla \times \vec{E}$$

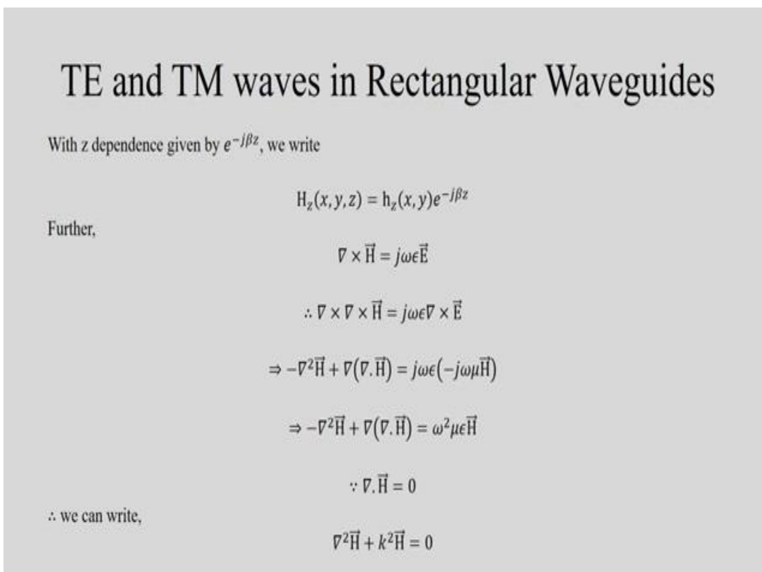
$$\Rightarrow -\nabla^2\vec{H} + \nabla(\nabla.\vec{H}) = j\omega\epsilon(-j\omega\mu\vec{H})$$

$$\Rightarrow -\nabla^2\vec{H} + \nabla(\nabla.\vec{H}) = \omega^2\mu\epsilon\vec{H}$$

$$\because \nabla.\vec{H} = 0$$

∴ we can write,

$$\nabla^2\vec{H} + k^2\vec{H} = 0$$



Now suppose we want to write the field variation H_z component, which is a function of x, y and z in this form. So we can write H_z in terms of h_z x, y a function of transverse co-ordinate and multiplied by $e^{-j\beta z}$. Now, starting from curl of H equal to $j\omega\epsilon\vec{E}$ if we take curl on both sides, and then expanding curl of curl of H as minus del square H plus gradient of divergence of H and substituting curl of E to be equal to minus j$\omega\mu\vec{H}$, we get an equation of this form, and since we have divergence of H equals to 0 , so this equation, can be now written as $\nabla^2\overrightarrow{H} + k^2 \vec{H} = 0$.

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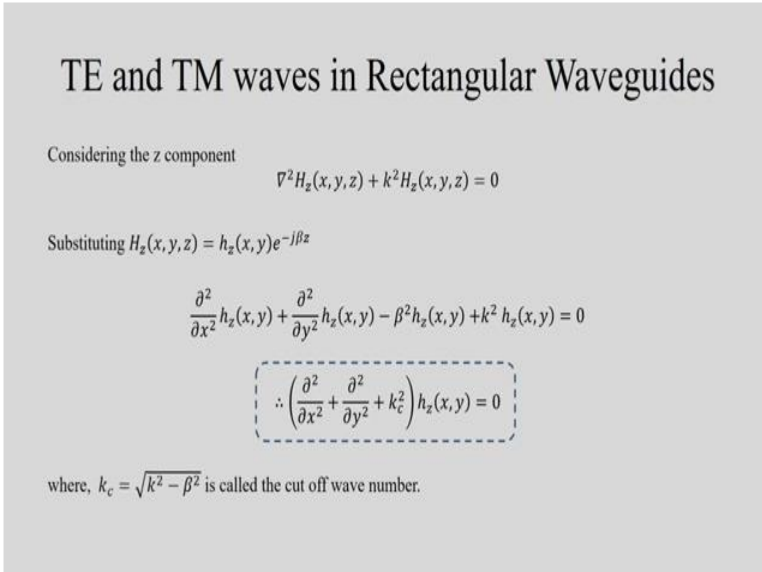

So considering z component of the magnetic field, we can write $\nabla^2 H_z\,(x,y,z) + \ K^2 H_z(x,y,z) = 0$ and substituting H_z as a function of small h_z as a function of x, y and $e^{-j\beta z}$. Then we can write this form of the equation and writing k square minus beta square is equal to k_c^2 we can write, $\left(\frac{\partial^2}{\partial x^2} + \ \frac{\partial^2}{\partial y^2} + \ k_c^2\right) = h_z(x,y) = 0$. Now, here $k_c which\ equal\ to\ \sqrt{k^2 - \beta^2}$ is called the cut off wave number.

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Considering the z component

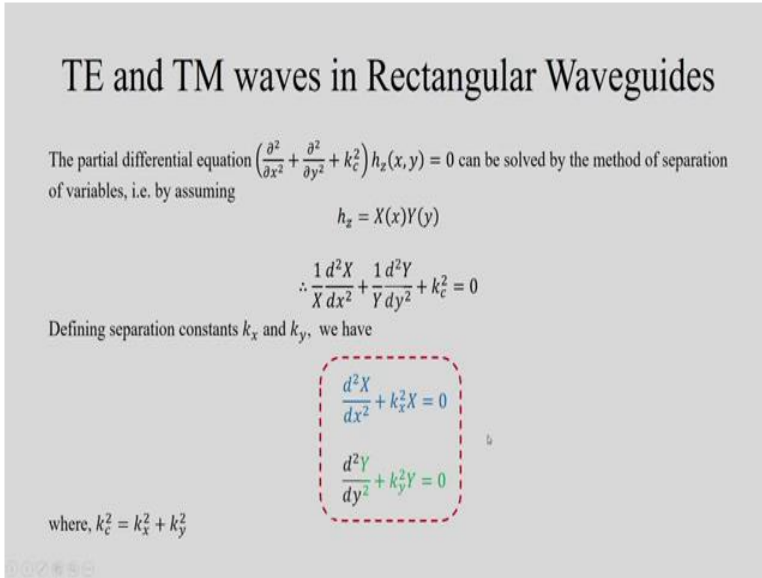
$$\nabla^2 H_z(x,y,z) + k^2 H_z(x,y,z) = 0$$

Substituting $H_z(x,y,z) = h_z(x,y)e^{-j\beta z}$

$$\frac{\partial^2}{\partial x^2} h_z(x,y) + \frac{\partial^2}{\partial y^2} h_z(x,y) - \beta^2 h_z(x,y) + k^2 h_z(x,y) = 0$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) h_z(x,y) = 0$$

where, $k_c = \sqrt{k^2 - \beta^2}$ is called the cut off wave number.



The partial differential equation is given by $\left(\frac{\partial^2}{\partial x^2}+\ \frac{\partial^2}{\partial y^2}+\ k_c^2\right)h_z(x,y)=0$ this can be solved by the method of separation of variables, that is, we can assume that h_z can be express of the form X(x)Y(y) and then substituting for h_z and then dividing both the sides by xy, we get an equation of this form like, $\frac{1}{x}\frac{d^2X}{dx^2}+\ \frac{1}{Y}\ \frac{d^2Y}{dy^2}+\ k_c^2$ =0. Now we define two separation constants $k_x\ and\ k_y$ And if we write $\frac{1}{x}\frac{d^2X}{dx^2}+\ \frac{1}{Y}\ \frac{d^2Y}{dy^2}+\ k_c^2$ =0, in that case, we can write this equation of this form, $\frac{d^2X}{dx^2}$ + k_x^2X= 0. Similarly, the other equation can be written as $\frac{d^2Y}{dy^2}$ + k_y^2Y =0.

Where we find that k_c^2 =$k_x^2+k_y^2$. So once we have these equations separated, now these equations are standard solutions.

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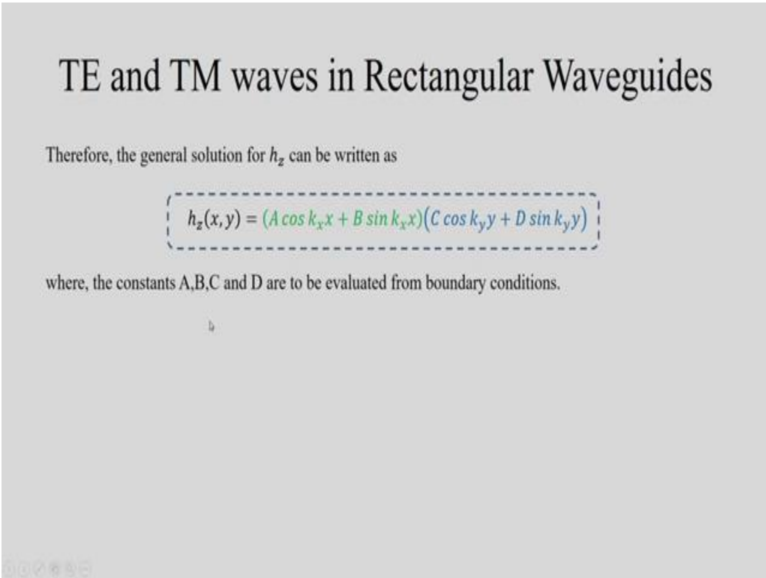
Therefore, the general solution for h_z can be written as

$h_z(x, y) = (A\,cos\,k_x x + B\,sin\,k_x x)(C\,cos\,k_y y + D\,sin\,k_y y)$

where, the constants A, B, C and D are to be evaluated from boundary conditions.



And therefore, we can now write a general solution h_z we can write, A $k_x x$ plus B sin $k_x x$ into C cos $K_y y$ plus D sin $K_y y$ and these parameters A, B, C and D, these constants are to be evaluated from the boundary condition.

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$$H_x = -\frac{j\beta}{k_c^2}\frac{\partial H_z}{\partial x}$$

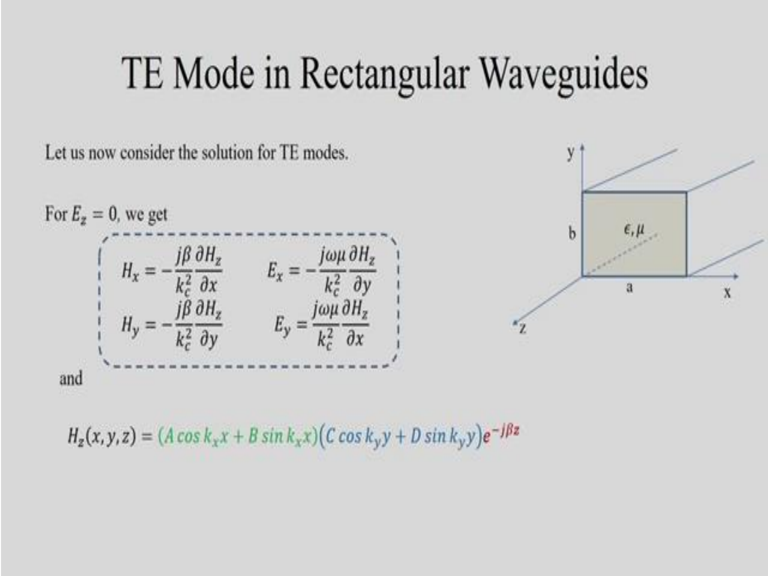
$$H_y = -\frac{j\beta}{k_c^2}\frac{\partial H_z}{\partial y}$$

$$E_x = -\frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y}$$

$$E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x}$$

and

$$H_z(x, y, z) = (A\,cos\,k_x x + B\,sin\,k_x x)(C\,cos\,k_y y + D\,sin\,k_y y)e^{-j\beta z}$$



So let us now see what are the boundary conditions, in a rectangular waveguide, when TE modes are being propagated. So for TE modes, we can write E_z is equal to 0 and H_x and H_y can be written in this form. Similarly, E_x and E_y component also can be written, please note that here, E_x and E_y, H_x and H_y all expressed in terms of H_z and H_z and we can write A cos $k_x x$ plus B sin $k_x x$ into C cos $K_y y$ plus D sin $K_y y$ into $e^{-j\beta z}$.

Now, we have to find out $H_z$component, such that this is the general form of H_z. We have to write H_z in such a way that this tangential electrical component E_x and E_y, they satisfy the boundary condition, on the surface of the waveguide.

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The boundary condition to be satisfied by the tangential field components in the walls of the waveguide are:

$E_x = 0$ for $y = 0$ and $y = b$

$E_y = 0$ for $x = 0$ and $x = a$

The solutions of H_z satisfying the boundary conditions for E_x and E_y can be evaluated as:

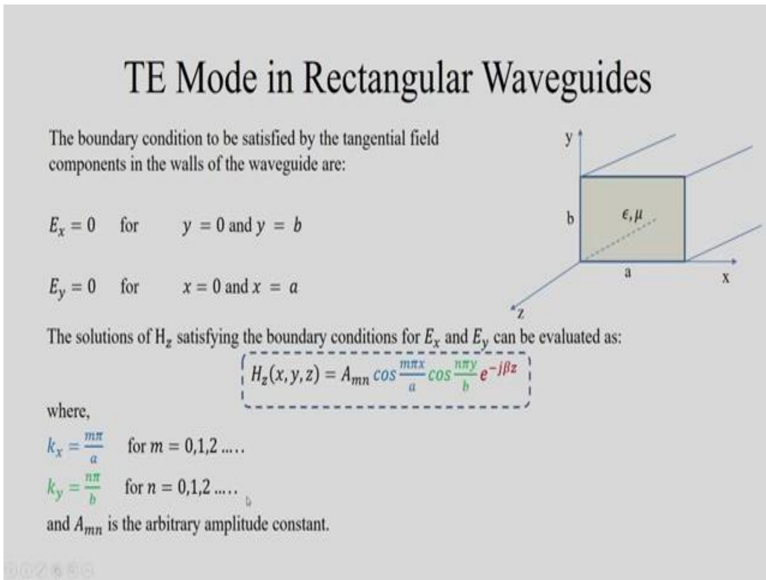
$$H_z(x,y,z) = A_{mn}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{-j\beta z}$$

where,

$k_x = \frac{m\pi}{a}$ for $m = 0,1,2 \dots ..$

$k_y = \frac{n\pi}{b}$ for $n = 0,1,2 \dots ..$

and A_{mn} is the arbitrary amplitude constant.



And these boundary conditions can be written as E_x component is 0 for y equal to 0, y equal to b and E_y component is 0 for x is equal to 0 and x is equal to a. So once we apply this boundary condition, we have to find H_z which satisfies this boundary condition for E_x and $E_{y,}$ and the solution of H_z, that satisfies this type of boundary condition for E_x and $E_{y,}$ will be of this form, $A_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}\ e^{-j\beta z}$.

Here we find that k_x is actually $\frac{m\pi}{a}$ and K_y is $\frac{n\pi}{b}$ and it can take values, m equal to 0, 1 2. Similarly n is equal to 0, 1 , 2, like that, and , A_{mn} s an arbitrary amplitude constant. Note that this sin terms in this general equation H_z they are dropped, so that we can satisfy the boundary condition, for E_x and E_y.

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We find that propagation constant,

$$\beta = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

is real when $k > k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$

Each combination of m and n will give a mode and each mode has a cut off frequency

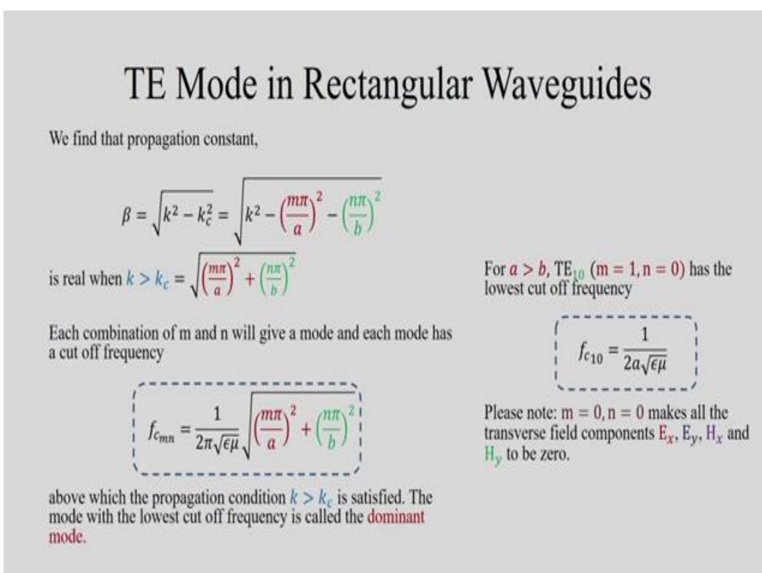
$$f_{c_{mn}} = \frac{1}{2\pi\sqrt{\epsilon\mu}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

above which the propagation condition $k > k_c$ is satisfied. The mode with the lowest cut off frequency is called the dominant mode.

For $a > b$, TE_{10} $(\mathrm{m} = 1,\ \mathrm{n} = 0)$ has the lowest cut off frequency

$$f_{c_{10}} = \frac{1}{2a\sqrt{\epsilon\mu}}$$

Please note: $\mathrm{m} = 0,\ \mathrm{n} = 0$ makes all the transverse field components E_x, E_y, H_x and H_y to be zero.



Now we can write the propagation constant β to be equal to the root of k^2 minus k_c^2 which is k^2 minus mπ by a whole square minus nπ by b whole square. And this beta becomes a real quantity when k is greater than k_c and when k is greater than k_c we get a propagating wave. Each combination of m and n will give a mode, and each mode has a cut off frequency.

Now the cut-off frequency for the mn^{th} mode is given by 1 by 2 pi root epsilon mu, under root m pie by a whole square plus n pi by b whole square, above which the propagation constant k greater than k_c is satisfied. The mode with the lowest cut off frequency is called the dominant mode. Now if we have a rectangular waveguide for which a is greater than b, then we see that the cut-off frequency, the least cut off frequency will be when m equal to 1 and n equal to 0. Similarly if b is greater than a, in that case the lowest cut off frequency will be for m equal to 0, n equal to 1.

So this particular mode, TE_{10} it has the lowest cut off frequency, and we will see that the dominant mode for a rectangular waveguide. Now, once you substitute m equal to 1 n equal to 0 here, we get, $f_{c_{10}}$ to be equal to 1 by 2a root mu epsilon. So this is the cut off frequency for the dominant TE_{10} mode in a rectangular waveguide. Please note that depending upon the direction a and the values of mu and epsilon for the dielectric media inside a waveguide, for example it is air-filled waveguide, it will be ϵ_0 μ_0 we will get a value of the cut off frequency $f_{c_{10}}$ and wave propagation inside such waveguide will only be possible to evolve this cut off frequency. Also note that m equal to 0, n equal to 0 makes all the transverse field components $E_x, E_y, H_{x,} H_y$ to be 0.

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So, for $\mathrm{m} = 1,\ \mathrm{n} = 0$, TE_{10} mode field components can be written as:

$$H_z = A_{10}\,cos\frac{\pi x}{a}\,e^{-j\beta z}$$

$$E_y = \frac{-j\omega\mu a}{\pi}A_{10}\,sin\frac{\pi x}{a}\,e^{-j\beta z}$$

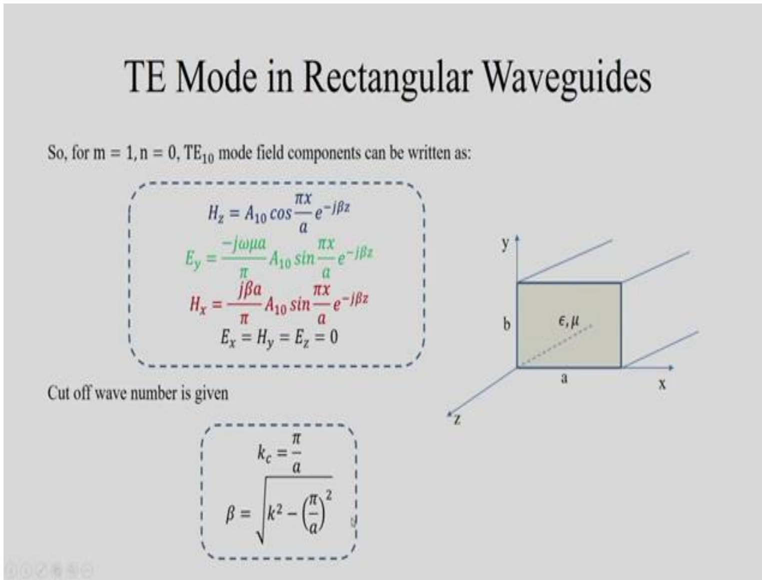
$$H_x = \frac{j\beta a}{\pi}A_{10}\,sin\frac{\pi x}{a}\,e^{-j\beta z}$$

$$E_x = H_y = E_z = 0$$

Cut off wave number is given

$$k_c = \frac{\pi}{a}$$

$$\beta = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}$$



So for the TE_{10} mode we can write the field components using the previous equations H_z to be A 1 0 cos pi x by a, e to the power of minus j beta z. And , E_y is minus j omega mu a by pi a 1 0 sin pi x by a e power minus j beta z. Please note that we have only , E_y component and it will have a sinusoidal distribution over the guide cross-section and it is a function of x, so we will have the maximum value of , E_y x is equal to a by 2, and we will see that for x is equal to 0, and x is equal to a, E_y becomes 0.

So this is the boundary condition, which is satisfied by E_y and apart from H_z the longitudinal magnetic field component, we have another transverse magnetic field component which is given by minus j beta a by pi A 1 0 sin pi x by a e to the power minus j beta z . And $E_x, H_y,$ E_z all are zero. So, here we have also the three field components, one electric field component, and two magnetic field component. So we can find out the cut off wave number, so k_c becomes pie by a and beta becomes k square minus pi by a whole square.

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- At a given operating frequency f, only those modes having $f > f_c$ will propagate.
- Modes with $f < f_c$ will attenuate (as this leads to an imaginary β) exponentially and such modes are called evanescent modes.
- When more than one modes propagate in the waveguide, the waveguide is called overmoded.

We have

$$E_x = -\frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y}$$

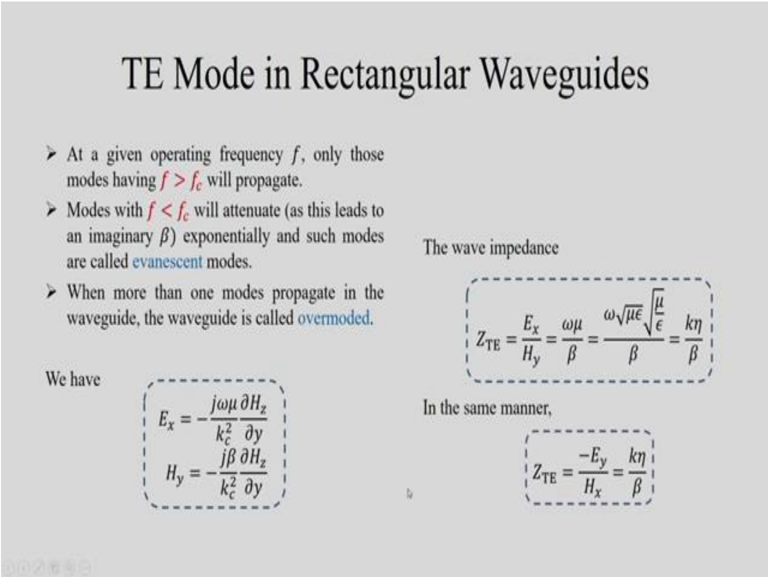
$$H_y = -\frac{j\beta}{k_c^2}\frac{\partial H_z}{\partial y}$$

The wave impedance

$$Z_{\text{TE}} = \frac{E_x}{H_y} = \frac{\omega\mu}{\beta} = \frac{\omega\sqrt{\mu\epsilon}\sqrt{\frac{\mu}{\epsilon}}}{\beta} = \frac{k\eta}{\beta}$$

In the same manner,

$$Z_{\text{TE}} = \frac{-E_y}{H_x} = \frac{k\eta}{\beta}$$



So at a given operating frequency f, only those modes which are having f greater than f_c propagates in a waveguide. And modes with an operating frequency f is less than f_c they attenuate because beta becomes imaginary and the attenuate exponentially and such modes are called evanescent modes. So even if these modes are excited, they will not propagate, they will eventually die out. And when we have more than one mode, propagating in a waveguide we say that the waveguide is overmoded.

So we have the E_x and $,H_y$ component, in a for TE wave mode of propagation, can be written in this form and therefore, the wave impedance Z_{TE} which is defined as $E_x\ by\ H_y$ and this becomes equal to k eta by beta where this eta is root mu by epsilon, the intrinsic impedance of dielectric media and also omega root mu epsilon is equal to k. Now, we also find Z_{TE} can be defined as ratio of minus E_y by H_x which is equal to k eta by beta. Please note that E_x and $,H_y$ component when we consider it gives the wave which is traveling in the z-direction, and Z_{TE} is the wave impedance as seen by this wave.

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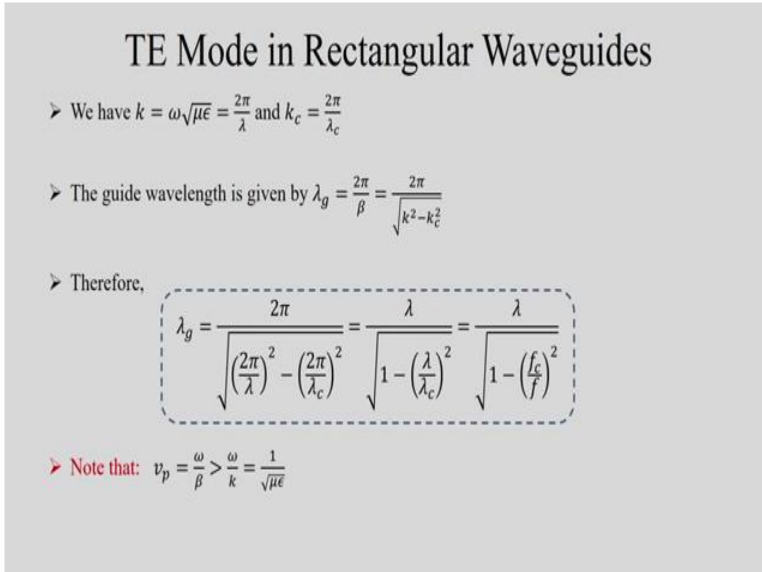
- We have $k = \omega\sqrt{\mu\epsilon} = \frac{2\pi}{\lambda}$ and $k_c = \frac{2\pi}{\lambda_c}$
- The guide wavelength is given by $\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{k^2 - k_c^2}}$
- Therefore,

$$\lambda_g = \frac{2\pi}{\sqrt{\left(\frac{2\pi}{\lambda}\right)^2 - \left(\frac{2\pi}{\lambda_c}\right)^2}} = \frac{\lambda}{\sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2}} = \frac{\lambda}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

- Note that: $v_p = \frac{\omega}{\beta} > \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}$



Now we have k is equal to omega root mu epsilon and which can be written as 2 pi by lambda and also the cut off wavelength k_c can be written as 2 pi by λ_c. Now the wavelength inside the waveguide will be different than the wavelength in the dielectric media at the same operating frequency. So this can be seen by considering the fact that, guide wavelength λ_g inside a waveguide for the propagation constant beta will be given by 2 pi by beta, whereas if you substitute beta in terms of the root of k square minus k c square.

And therefore we can write $\lambda_g = \frac{2\pi}{\sqrt{\frac{2\pi^2}{\lambda} - \frac{2\pi^2}{\lambda_c}}}$ which after rearranging we can write as lambda by 1 minus lambda by lambda c whole square, and in terms of frequency it can be written as lambda divided by root of 1 minus fc by f whole square, where f_c is the cutoff frequency. Please note that f_c will be less than f for a wave which is propagating within the waveguide and therefore one minus f_c by f whole square this will become a fraction and λ_g will be greater than lambda, where lambda is the wavelength in the dielectric media.

Further v_p the face velocity is given by omega by beta and this is greater than omega by k, omega by k is the velocity in the dielectric media which is given by 1 by root mu epsilon. It may be noted that although the face velocity within a waveguide becomes greater than the velocity of the wave propagation in the dielectric media and if it is free space then greater than 1 by root mu epsilon naught which is equal to c, so when we consider actual propagation of energy within the waveguide, it remains less than equal to c.

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For the TM modes we have,

$H_z = 0$ and the $E_z(x, y, z) = e_z(x, y)e^{-j\beta z}$

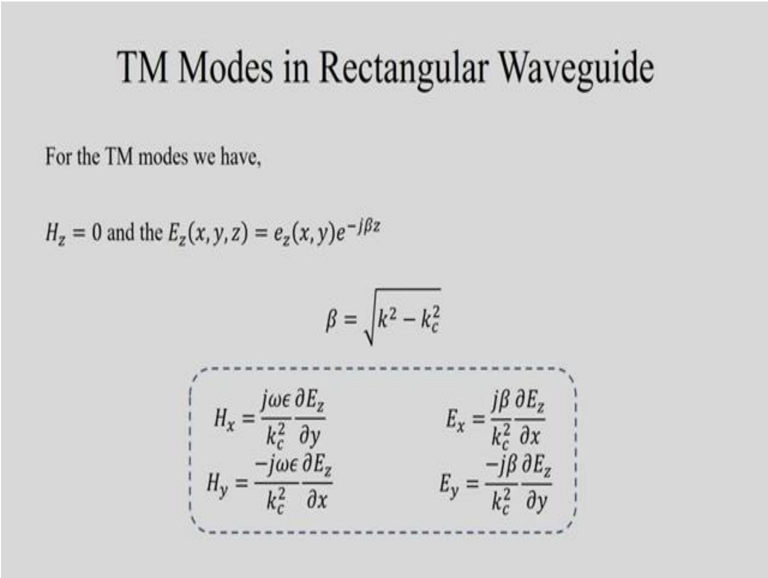
$$\beta = \sqrt{k^2 - k_c^2}$$

$$H_x = \frac{j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial y}$$

$$H_y = \frac{-j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial x}$$

$$E_x = \frac{j\beta}{k_c^2}\frac{\partial E_z}{\partial x}$$

$$E_y = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial y}$$



Now let us come to the TM modes of rectangular waveguide, for the TM modes, we have H_z the longitudinal component of the magnetic field to be 0, only the transverse components of the magnetic fields are present and E_z the longitudinal component of the electric field by our previous argument we can write in this form, $e_z(x, y)$a function of x and y and z dependance is given by $e^{-j\beta z}$

Now we have beta is given by root of k square minus kc square and H_x, H_y, similarly $E_x and\ E_y$ in terms of E_z can be expressed in the function.

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Similar to TE case, considering the z component of the electric field we can write

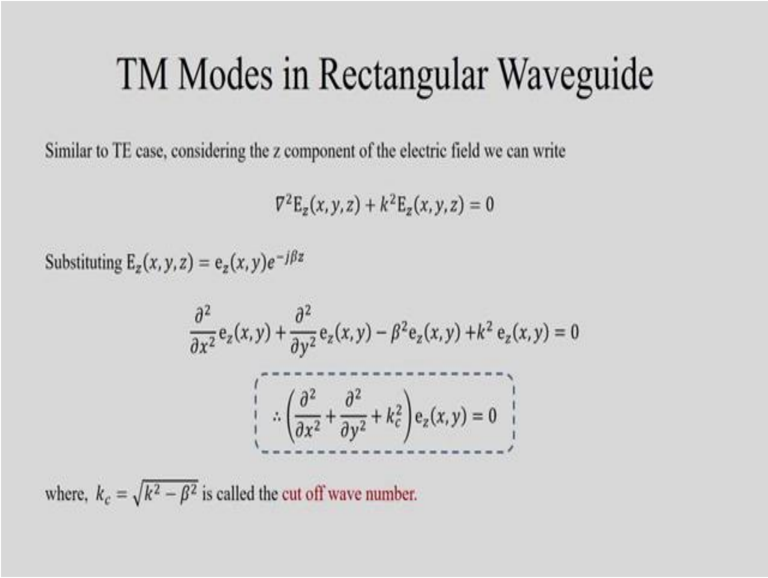
$$\nabla^2 E_z(x, y, z) + k^2 E_z(x, y, z) = 0$$

Substituting $E_z(x, y, z) = \mathrm{e}_z(x, y)e^{-j\beta z}$

$$\frac{\partial^2}{\partial x^2}\mathrm{e}_z(x, y) + \frac{\partial^2}{\partial y^2}\mathrm{e}_z(x, y) - \beta^2\mathrm{e}_z(x, y) + k^2\mathrm{e}_z(x, y) = 0$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right)\mathrm{e}_z(x, y) = 0$$

where, $k_c = \sqrt{k^2 - \beta^2}$ is called the cut off wave number.



And similar to TE case, considering the z component of the electric field, we can write the wave equation $\nabla^2 E_z + k^2 E_z = 0$. And substituting E_z in terms of $e_z(x, y)$ which is the transverse variation due to the transverse coordinate multiplied by $e^{-j\beta z}$ we can write del square del x square plus del square del y square plus kc square e z x, y is equal to 0. And k_c as before is the cut off wave number.

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The partial differential equation $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_c^2\right)\mathrm{e}_z(x,y)=0$ can be solved by the method of separation of variables, i.e. by assuming

$$\mathrm{e}_z = \mathrm{X}(x)\mathrm{Y}(y)$$

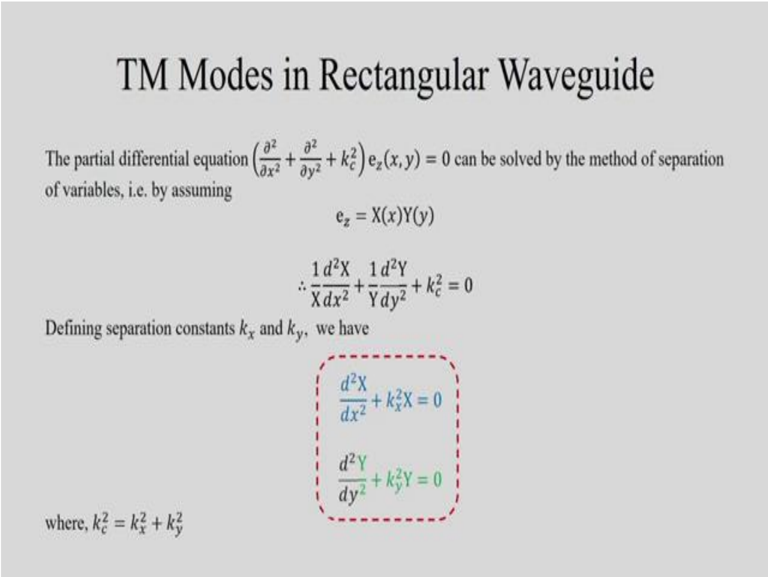
$$\therefore \frac{1}{\mathrm{X}}\frac{d^2\mathrm{X}}{dx^2}+\frac{1}{\mathrm{Y}}\frac{d^2\mathrm{Y}}{dy^2}+k_c^2=0$$

Defining separation constants k_x and k_y, we have

$$\frac{1}{\mathrm{X}}\frac{d^2\mathrm{X}}{dx^2}+k_x^2\mathrm{X}=0$$

$$\frac{1}{\mathrm{Y}}\frac{d^2\mathrm{Y}}{dy^2}+k_y^2\mathrm{Y}=0$$

where, $k_c^2 = k_x^2 + k_y^2$



Now as before this partial differential equation can be solved by the method of separation of variables assuming e_z to be XY product of two functions, Where this capital X is a function of x and capital Y is a function of y coordinate only. And then substituting e_z is equal to XY here and dividing throughout by XY we get a equation of this form. Once again we introduce the separation constants $k_x and\ k_y$ and we can separate these two equations.

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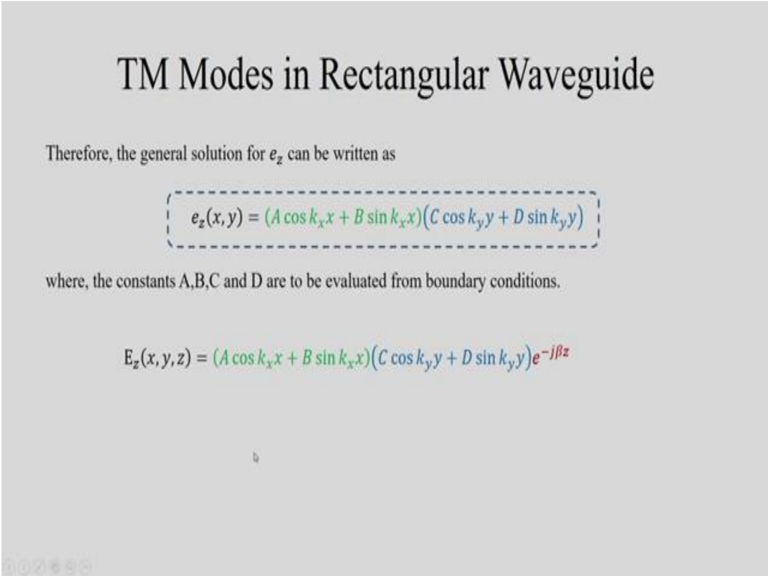
Therefore, the general solution for e_z can be written as

$e_z(x, y) = (A \cos k_x x + B \sin k_x x)(C \cos k_y y + D \sin k_y y)$

where, the constants A, B, C and D are to be evaluated from boundary conditions.

$$E_z(x, y, z) = (A \cos k_x x + B \sin k_x x)(C \cos k_y y + D \sin k_y y)e^{-j\beta z}$$



So from these two equations, we can write the z components of electric field as a function of x and y in this form (A cos k_xx + B sin $k_x x$)(C cos k_yy + D sin $k_y y$). Now this A, B, C, D are to be determined from the boundary conditions. And this shows that E_z component, z component of the electric field in the general form.

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The boundary condition to be satisfied by the E_z are

$E_z = 0$ for $y = 0$ and $y = b$

$E_z = 0$ for $x = 0$ and $x = a$

The solutions of E_z satisfying the boundary conditions can be evaluated as:

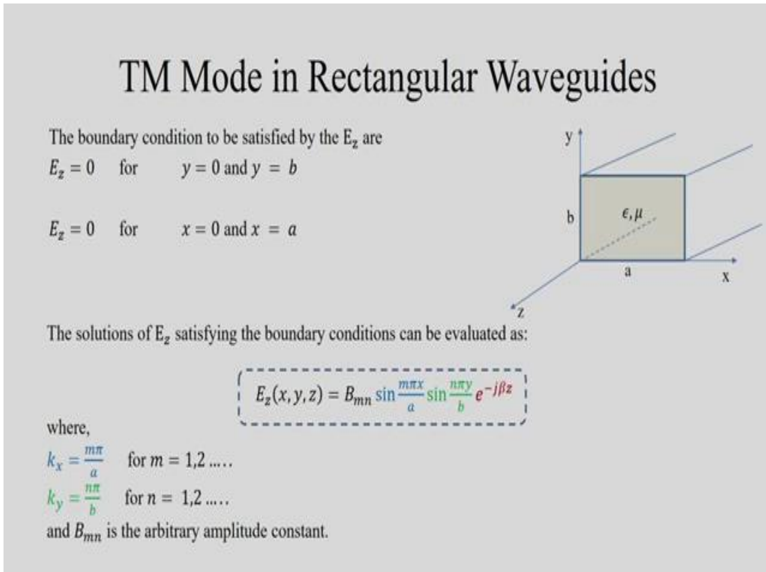
$$E_z(x, y, z) = B_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-j\beta z}$$

where,

$k_x = \frac{m\pi}{a}$ for $m = 1,2 \ldots..$

$k_y = \frac{n\pi}{b}$ for $n = 1,2 \ldots..$

and B_{mn} is the arbitrary amplitude constant.



Now when it comes to applying boundary condition, here we see that we can directly apply the boundary condition on the E_z component and the conditions are E_z component it is 0 for y equal to 0, y equal to b and similarly, E_z is equal to 0 for x equal to 0 and x equal to a.

So the solution, from the general solution of E_z which satisfy these boundary condition can be obtained as E_z x, y, z is equal to some B mn sin m pi x by a sin n pi y by b $e^{-j\beta z}$. Please note that here, k_x evaluates to be m pi by a but we can take the m for as 1, 2, 3, etc., not 0. Similarly, k_y is given by n pi by b and the values of n that we can take again 1, 2, 3, etc., not n equal to 0 because if you put either m or n equal to 0, ez will become zero. B mn here is the arbitrary amplitude constant and each combination of m and n will give a mode.

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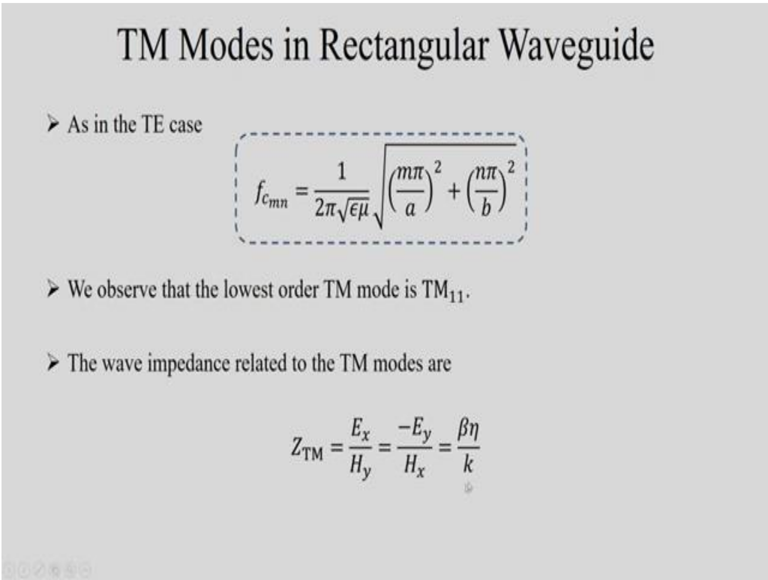
- As in the TE case

$$f_{c_{mn}} = \frac{1}{2\pi\sqrt{\epsilon\mu}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

- We observe that the lowest order TM mode is TM_{11}.
- The wave impedance related to the TM modes are

$$Z_{\mathrm{TM}} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\beta\eta}{k}$$



As in the TE case we have $f_{c_{mn}}$ the cut off frequency is given by 1 by 2 pi root mu epsilon m pi by a whole square plus n pi by b whole square whole under root and we observe that the lowest order mode that is possible for TM case is TM_{11}. The wave impedance related to the TM modes can be found out in the same manner as in TE it is E_x by H_y is equal to minus Ey by Hx now here in this case it is beta eta by k.

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Example

Let us consider a rectangular waveguide WR-90 for which a = 2.286 cm and b = 1.016 cm. For such air-filled waveguide, the cut-off frequencies for different modes of wave propagation are:

Mode	Cut-off frequency (in GHz)	Mode	Cut-off frequency (in GHz)
TE_{10}	6.5617	TE_{01}	14.764
TE_{20}	13.123	TE_{11} and TM_{11}	16.156

- It can be seen that the dominant mode is TE_{10} and there is no wave propagation below 6.5617 GHz.
- Two modes having the same cutoff frequency are called degenerate mode. Here TE_{11} and TM_{11} are degenerate modes

Now to illustrate a particular case of a rectangular waveguide, let us consider the example of a very popular rectangular waveguide, which is known as the WR-90. Here WR is waveguide rectangular and 90 stands for point nine inches that means the larger dimension a is point nine-inch. And therefore in centimeter a is 2.286 cm and for WR-90 b is 1.016 cm. For such air-filled waveguide, the cut-off frequencies for different modes of wave propagation, we can calculate and we find that the cut off frequency for TE_{10} modes are 6.5617 gigahertz.

Similarly, the next mode is TE_{20} having a cut off frequency of 13.123 GHz. The next mode is TE_{01} which is having a cut off frequency of 14.764 GHz. Now please note that below 6.5617 GHZ, with this guide direction, the propagation inside the waveguide is not possible. So above 6.5617 GHz if we operate the waveguide, we will have TE_{10} mode launched first, and then if we operate the waveguide above 13.123 GHz, then we have both the propagation possible. So, 6.5617 to 13.123, in this frequency range, the dominant mode, TE_{10} the mode only propagates in such waveguide.

Therefore, WR-90 is often used in the x band of frequencies, which means from 8 to 12 GHz when we require single-mode propagation. We find that if you compute the cut off frequencies TE_{11} and TM_{11} for this waveguide, it comes out to be, 16.156 and both TE_{11} and TM_{11}, they have the same cut off frequencies. And when we have this type of scenario, that two modes have the same set of frequency, these types of modes are called degenerate modes. For example here TE_{11} and TM_{11} are degenerate modes.

So in this lecture, we have studied the TE and TM mode propagation in a rectangular waveguide, and we have derived mathematical equations, which describe the propagation of such modes. In the next lecture, we will consider the wave propagation in a circular waveguide, and we will consider the TE and TM mode of wave propagation in a circular waveguide, and we will find expression for the cut-off frequencies for TE and TM mode, in a circular waveguide.