Microwave Engineering
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Lecture 06
Rectangular and Circular Waveguides

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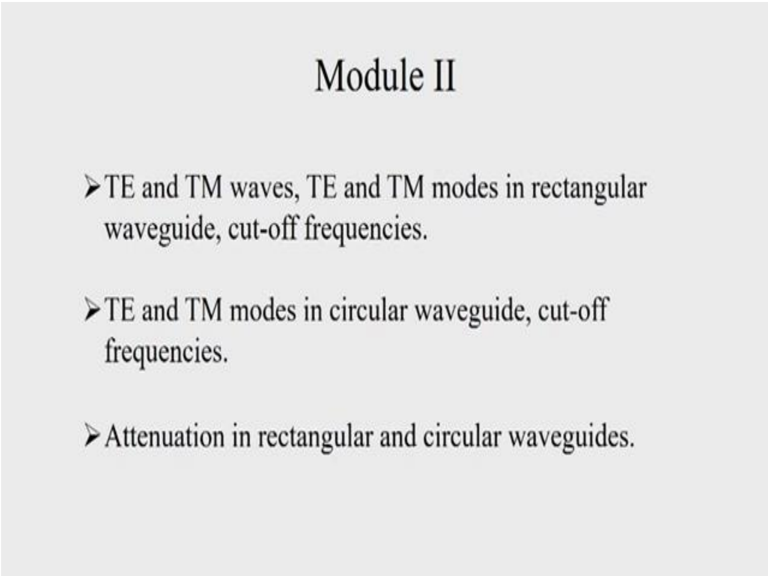

In the previous lecture, we have seen the wave propagation in a rectangular waveguide, we have seen how TE and TM modes propagate, we have also derived expressions for the electric and magnetic field components of such waves. We have also derived the expressions for the cut-off frequencies of such modes in a rectangular waveguide. In this lecture, we consider a different type of waveguide, which is a circular waveguide.

A circular waveguide has a circular cross-section for such waveguides we will see how wave propagation takes place. We will derive the expression for the electric and magnetic field components of TE and TM modes in such circular waveguides. We will also derive the expression for the cut-off frequencies for the TE and TM modes in such waveguide.

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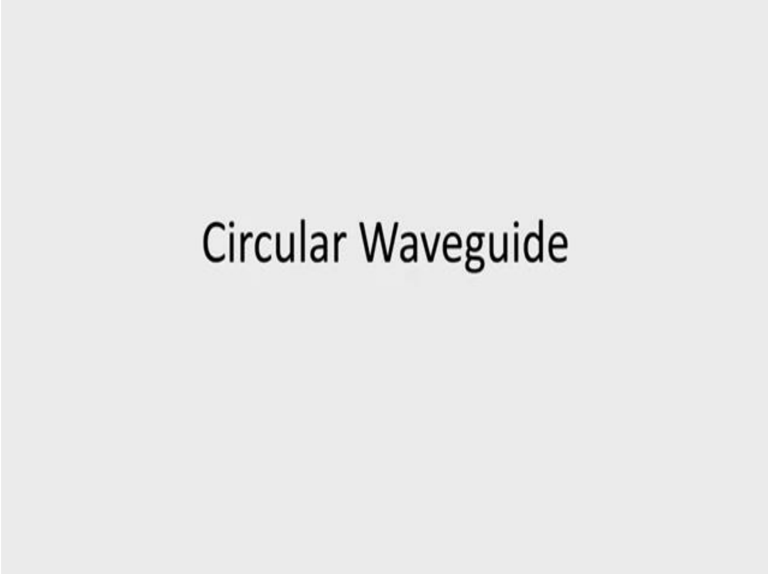

So, we start our discussion on the circular waveguide.

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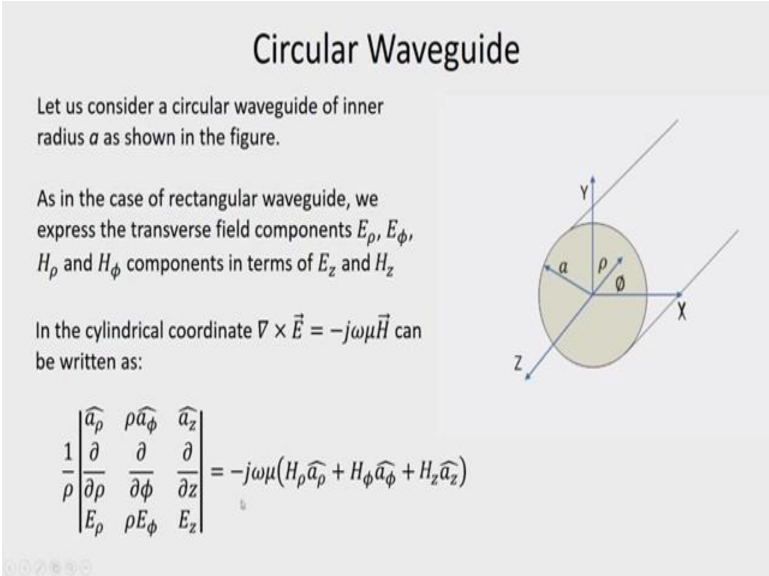
Let us consider a circular waveguide of inner radius *a* as shown in the figure.

As in the case of rectangular waveguide, we express the transverse field components E_ρ, E_ϕ, H_ρ and H_ϕ components in terms of E_z and E_z

In the cylindrical coordinate $\nabla \times \vec{E} = -j\omega\mu\vec{H}$ can be written as:

$$\frac{1}{\rho}\begin{vmatrix} \widehat{a_\rho} & \rho\widehat{a_\phi} & \widehat{a_z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ E_\rho & \rho E_\phi & E_z \end{vmatrix} = -j\omega\mu\left(H_\rho\widehat{a_\rho} + H_\phi\widehat{a_\phi} + H_z\widehat{a_z}\right)$$



And let us consider a circular waveguide of inner radius a as shown in the figure, here you can see that the wave propagation is assumed to take place along Z direction the waveguide will guide the energy in the Z direction and since this is a cylindrical waveguide of circular cross-section we will have to use the cylindrical coordinate and rho, phi and Z these three coordinates will have to use for developing the equations for this type of circular waveguide.

And as in the case of a rectangular waveguide, here also what we do the procedure is similar. We express the field components transverse field components, which are E rho E phi H rho and H phi components. In terms of E z and H z and then we will try to solve for E z and H z by applying the boundary condition. In the cylindrical coordinate, this equation curl of E is equal to minus j omega mu H can be written as 1 by rho arrow rho phi a z del del rho; del del phi; del del Z; E rho, rho E phi, E z and this is the left-hand side and right-hand side we expand it in terms of H rho H phi and H z.

Now, we can evaluate the components from the left-hand side and equate them. For example, this H rho component will be given as 1 by rho arrow del E z del phi minus rho del E phi del Z, so this will be equal to minus j omega mu H rho.

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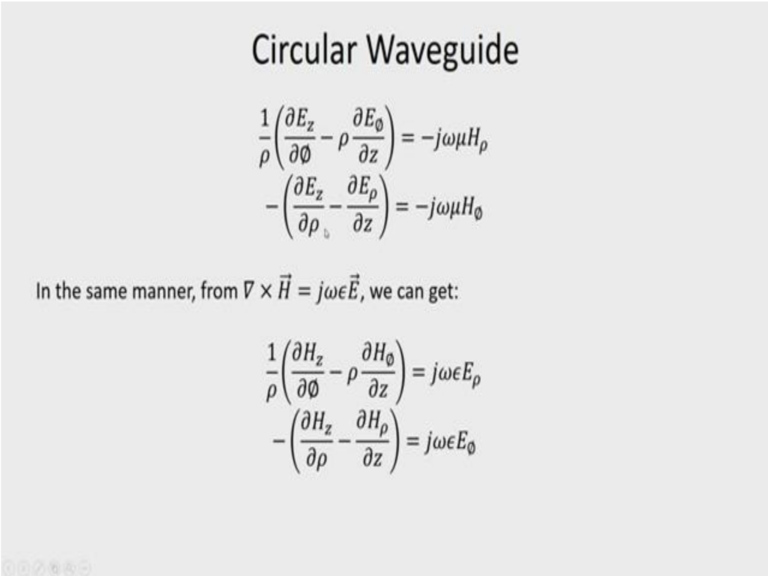
$$\frac{1}{\rho}\left(\frac{\partial E_z}{\partial \emptyset} - \rho\frac{\partial E_\emptyset}{\partial z}\right) = -j\omega\mu H_\rho$$

$$-\left(\frac{\partial E_z}{\partial \rho} - \frac{\partial E_\rho}{\partial z}\right) = -j\omega\mu H_\emptyset$$

In the same manner, from $\nabla \times \vec{H} = j\omega\epsilon\vec{E}$, we can get:

$$\frac{1}{\rho}\left(\frac{\partial H_z}{\partial \emptyset} - \rho\frac{\partial H_\emptyset}{\partial z}\right) = j\omega\epsilon E_\rho$$

$$-\left(\frac{\partial H_z}{\partial \rho} - \frac{\partial H_\rho}{\partial z}\right) = j\omega\epsilon E_\emptyset$$



So, if you equate the rho components, we get this equation. Similarly, if we equate the phi components we get the expression relating H phi E z and E rho. Now, from the other curl equation, curl of H is equal to j omega epsilon E we get two more equations 1 by rho del H z del phi minus rho del H phi del z is equal to j omega epsilon E rho. And similarly minus del H z del rho minus del H rho del Z is equal to j omega epsilon E phi.

So we use these equations, this set of four equations to express here E rho E phi H rho and H phi in terms of E z and H z.

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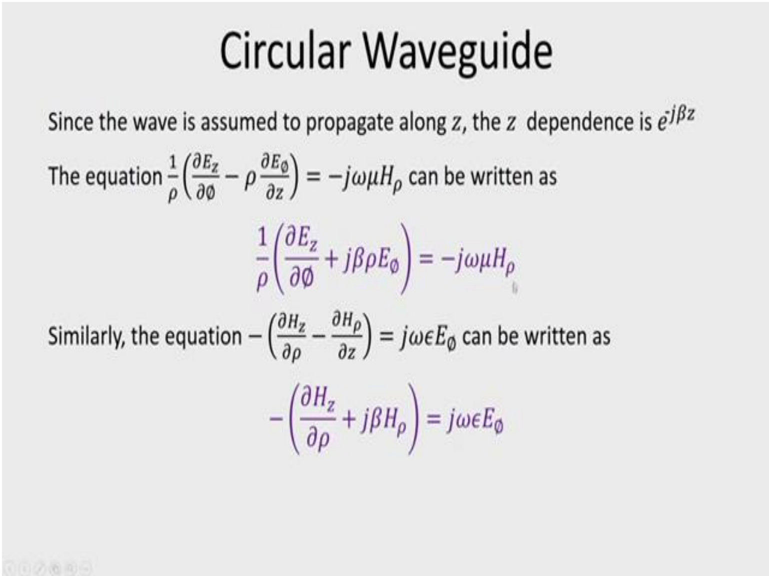
Since the wave is assumed to propagate along z, the z dependence is $e^{j\beta z}$

The equation $\frac{1}{\rho}\left(\frac{\partial E_z}{\partial \emptyset} - \rho\frac{\partial E_\emptyset}{\partial z}\right) = -j\omega\mu H_\rho$ can be written as

$$\frac{1}{\rho}\left(\frac{\partial E_z}{\partial \emptyset} + j\beta\rho E_\emptyset\right) = -j\omega\mu H_\rho$$

Similarly, the equation $-\left(\frac{\partial H_z}{\partial \rho} - \frac{\partial H_\rho}{\partial z}\right) = j\omega\epsilon E_\emptyset$ can be written as

$$-\left(\frac{\partial H_z}{\partial \rho} + j\beta H_\rho\right) = j\omega\epsilon E_\emptyset$$



So, as before we assume the z dependence for the propagating wave to be given by e to the power minus j beta z and this equation 1 by rho del E z del phi minus rho del E phi del z is equal to minus j omega mu H rho. In this equation this del del z now we can replace del del z we can evaluate, and that will result in multiplication by minus j beta. So, the equation is rewritten in this form.

Similarly, now we find that this equation contains E z H rho and E phi. If we can eliminate E phi in terms of a H z and H rho then we will get an equation where H rho will be explicitly expressed in terms of E z and H z. So, we start with this equation minus del H z del rho minus del H rho del z is equal to j omega epsilon e phi. Now, here also, this del del z will result in to minus j beta, and therefore this equation can be written in this form.

Now, what we can do, we can substitute E phi in this equation and evaluate for H rho, which is shown next.

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$$H_\rho = -\frac{1}{j\omega\mu\,\rho}\left(\frac{\partial E_z}{\partial \emptyset} + j\beta\rho E_\emptyset\right)$$

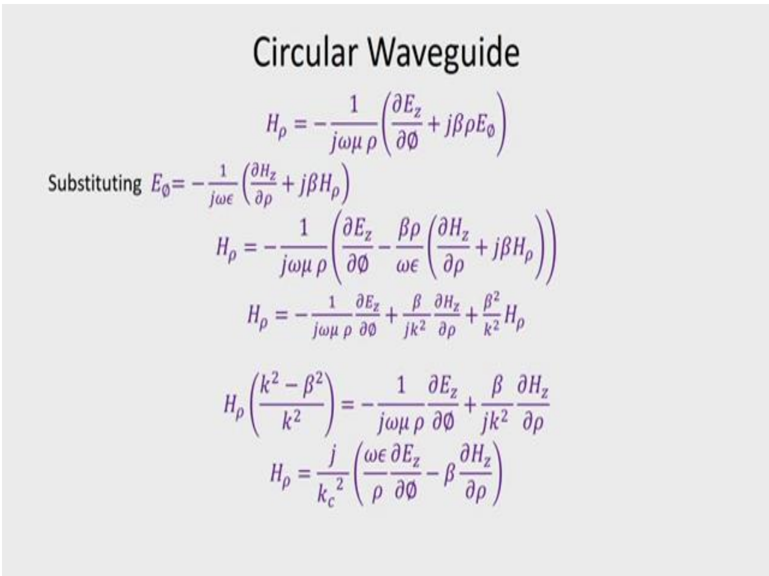
Substituting $E_\emptyset = -\frac{1}{j\omega\epsilon}\left(\frac{\partial H_z}{\partial \rho} + j\beta H_\rho\right)$

$$H_\rho = -\frac{1}{j\omega\mu\,\rho}\left(\frac{\partial E_z}{\partial \emptyset} - \frac{\beta\rho}{\omega\epsilon}\left(\frac{\partial H_z}{\partial \rho} + j\beta H_\rho\right)\right)$$

$H_\rho = -\frac{1}{j\omega\mu\,\rho}\frac{\partial E_z}{\partial \emptyset} + \frac{\beta}{jk^2}\frac{\partial H_z}{\partial \rho} + \frac{\beta^2}{k^2}H_\rho$

$$H_\rho\left(\frac{k^2-\beta^2}{k^2}\right) = -\frac{1}{j\omega\mu\,\rho}\frac{\partial E_z}{\partial \emptyset} + \frac{\beta}{jk^2}\frac{\partial H_z}{\partial \rho}$$

$$H_\rho = \frac{j}{{k_c}^2}\left(\frac{\omega\epsilon}{\rho}\frac{\partial E_z}{\partial \emptyset} - \beta\frac{\partial H_z}{\partial \rho}\right)$$



So, we write h rho is equal to 1 by j omega mu rho into del E z del phi plus j beta rho e phi and substituting E phi we have seen the expression for E phi then we can re-write the expression for H rho in this manner where you can see this term has been substituted here and we will find that omega square mu epsilon can be written as k square and once we simplify H rho in this form then we can write, we can take the last time involving H rho to the left-hand side and then k square minus beta square can be replaced by k c square and then we can write H rho as j by k c square omega epsilon by rho del E z del phi minus beta del H z del rho.

So, this gives us the expression for the rho component of the magnetic field H rho. Now, similar exercise can be carried out for finding the other three field components, namely H phi, E rho, and E phi.

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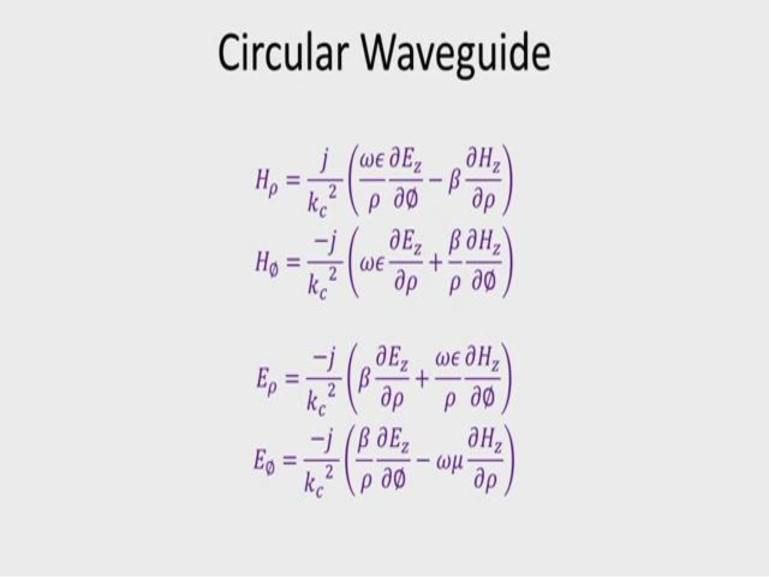
We have seen that



So, once we do that we get a set of four equations, and we can see that we have expressed H rho, H phi, E rho, and E phi in terms of E z and H z. Now, that means once we can solve E z and H z we can get the solution of field components for the transverse fields, and this we can further simplify. For example, if we consider TM propagation, then H z will be 0, and if you consider TE mode of propagation then E z will be 0.

So, our next target will be to solve this set of equations for TE and TM modes for which we will have to find out the longitudinal components, either E z or H z, by applying appropriate boundary conditions.

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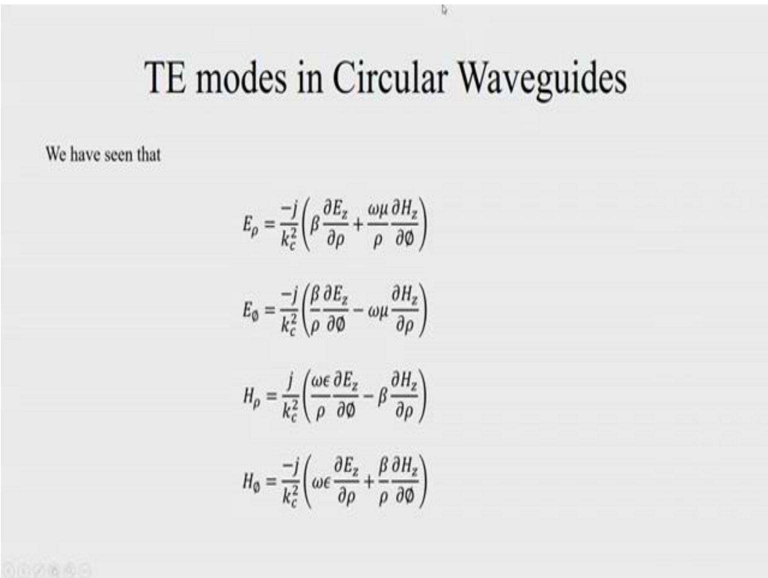
$$E_\rho = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial \rho} + \frac{\omega\mu}{\rho}\frac{\partial H_z}{\partial \emptyset}\right)$$

$$E_\emptyset = \frac{-j}{k_c^2}\left(\frac{\beta}{\rho}\frac{\partial E_z}{\partial \emptyset} - \omega\mu\frac{\partial H_z}{\partial \rho}\right)$$

$$H_\rho = \frac{j}{k_c^2}\left(\frac{\omega\epsilon}{\rho}\frac{\partial E_z}{\partial \emptyset} - \beta\frac{\partial H_z}{\partial \rho}\right)$$

$$H_\emptyset = \frac{-j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial \rho} + \frac{\beta}{\rho}\frac{\partial H_z}{\partial \emptyset}\right)$$



Let us now consider transverse electric or TE modes in circular waveguides. We have seen that the transverse field component E rho, E phi, H rho, and H phi these components can be related to the longitudinal components E z and H z as shown.

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For TE case, $E_z = 0$ and the wave equation involving H_z component can be written as:

$$\nabla^2 H_z + k^2 H_z = 0$$

Now,

$H_z(\rho, \emptyset, z) = h_z(\rho, \emptyset)e^{-j\beta z}$

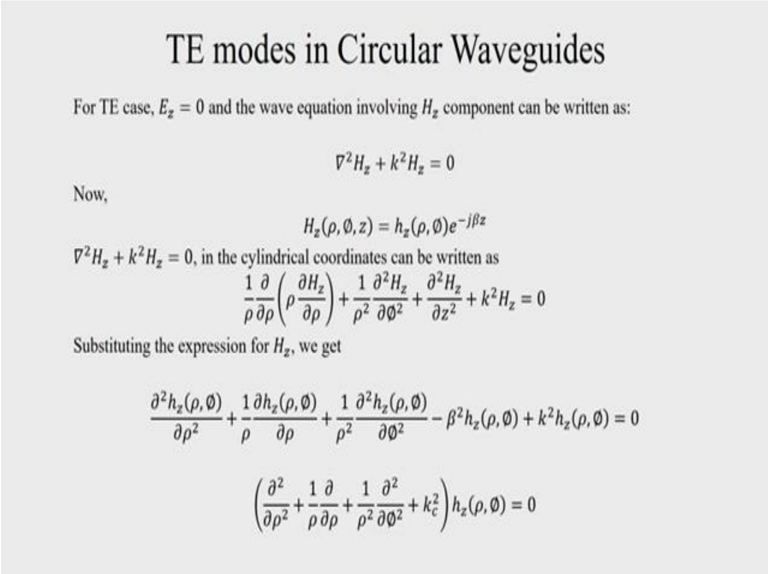
$\nabla^2 H_z + k^2 H_z = 0$, in the cylindrical coordinates can be written as

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial H_z}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial \emptyset^2} + \frac{\partial^2 H_z}{\partial z^2} + k^2 H_z = 0$$

Substituting the expression for H_z, we get

$$\frac{\partial^2 h_z(\rho,\emptyset)}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial h_z(\rho,\emptyset)}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 h_z(\rho,\emptyset)}{\partial\emptyset^2}-\beta^2 h_z(\rho,\emptyset)+k^2 h_z(\rho,\emptyset)=0$$

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\emptyset^2}+k_c^2\right)h_z(\rho,\emptyset)=0$$



So, when we consider TE modes, we have E z component equal to 0, and we will have only the H z components, and the wave equation involving H z component can be written as the square H z plus k square H z equal to 0. Now, this H z in terms of rho, phi, and z can be written as small h z rho, phi that means this is a function of transverse coordinates rho and phi, and z variation is given by e to the power minus j beta z.

So, if we consider the wave equation del square H z plus k square H z is equal to 0 and expand it in cylindrical coordinates then we can write 1 by rho del del rho, rho del phi, del H z, del rho plus 1 by rho square del square H z, del phi square plus del square H z, del z square plus k square H z equal to 0. Now, if we substitute the expression for a H z in this equation and we find that because the z dependence is given by e to the power minus j beta z, the derivative with respect to z will result in a term minus j beta.

And therefore we can write del square h z del rho square 1 by rho del h z del rho plus 1 by rho square del square h z del phi square minus beta square h z plus k square h z equal to 0 and this equation can be further simplified by substituting k square minus beta square to be equal to k c square where the k c as we have seen represents the cut-off wavenumber.

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We apply the method of separation of variables and write

$$h_z(\rho, \emptyset) = R(\rho)\Phi(\phi)$$

Substituting this in the previous equation and dividing both sides by $R\Phi$ we get

$$\frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{\rho R}\frac{dR}{d\rho} + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\emptyset^2} + k_c^2 = 0$$

The above equation can be written as

$$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} + k_c^2\rho = -\frac{1}{\Phi}\frac{d^2\Phi}{d\emptyset^2}$$

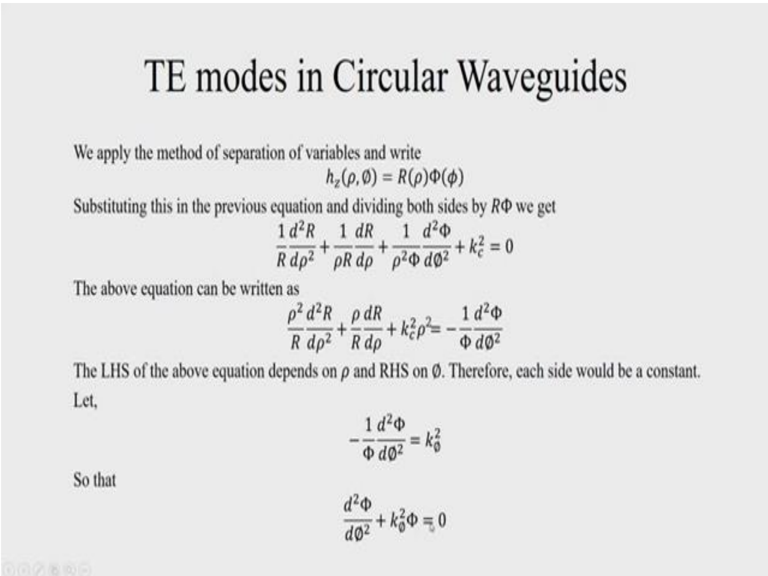
The LHS of the above equation depends on ρ and RHS on $\emptyset$. Therefore, each side would be a constant.

Let,

$$-\frac{1}{\Phi}\frac{d^2\Phi}{d\emptyset^2} = k_\emptyset^2$$

So that

$$\frac{d^2\Phi}{d\emptyset^2} + k_\emptyset^2\Phi = 0$$



Now, to solve this equation we apply the method of separation of variables and we write h z rho phi to be a product of two function R of rho and phi of phi, here note that R is only a function of rho and phi is only a function of small phi and once we substitute this in the previous equation and divide both sides by R and phi we get, so we substitute this R phi h z is equal to

R phi and then divide throughout by R phi then we get 1 by R d square r, d rho square plus 1 by rho R dR d rho plus 1 by rho square phi d square phi d phi square plus k c square equal to 0.

Now, what we can do? We can rearrange the terms, we can take the terms involving phi on the right-hand side, then the left-hand side becomes only a function of rho, and the right-hand side becomes only a function of phi and therefore each side has to be a constant and we write minus 1 by phi d square phi d phi square is equal to k phi square. And therefore we get the equation d square phi by d phi square plus k phi square phi equal to 0.

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- Also we get,

$$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2}+\frac{\rho}{R}\frac{dR}{d\rho}+k_c^2\rho=k_\emptyset^2$$

$$\therefore \frac{\rho^2}{R}\frac{d^2R}{d\rho^2}+\frac{\rho}{R}\frac{dR}{d\rho}+\left(k_c^2\rho-k_\emptyset^2\right)=0$$

The solution of the equation, $\frac{d^2\Phi}{d\emptyset^2}+k_\emptyset^2\Phi=0$, is given by

$$\Phi(\phi)=A\sin k_\emptyset\emptyset+B\cos k_\emptyset\emptyset$$

Since, $h_z(\rho,\emptyset)=h_z(\rho,\emptyset\pm 2m\pi)$,$k_\emptyset$ is an integer. Thus,

$$\Phi(\phi)=A\sin n\emptyset+B\cos n\emptyset$$

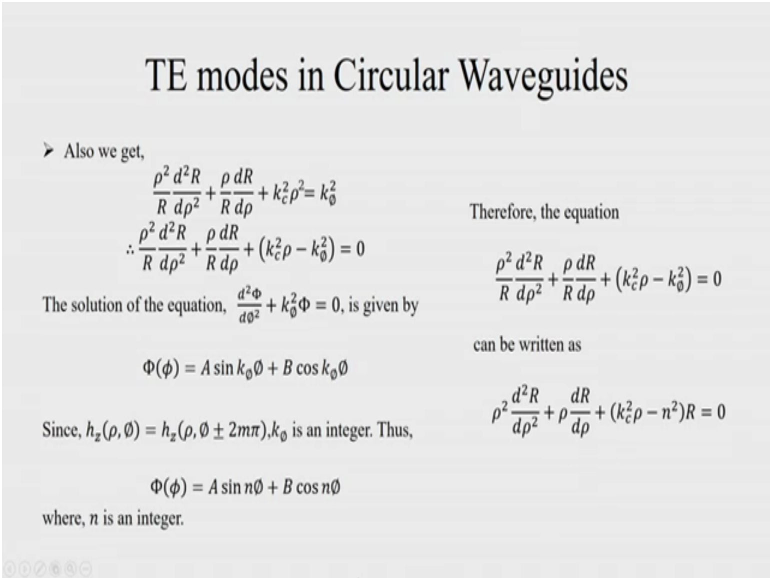
where, n is an integer.

Therefore, the equation

$$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2}+\frac{\rho}{R}\frac{dR}{d\rho}+\left(k_c^2\rho-k_\emptyset^2\right)=0$$

can be written as

$$\rho^2\frac{d^2R}{d\rho^2}+\rho\frac{dR}{d\rho}+(k_c^2\rho-n^2)R=0$$



Introducing k phi square, we can write the equation involving rho to be rho square by R, d square R d rho square plus rho by R dR d rho plus k c square rho square is equal to k phi square, and therefore we can rearrange it and write it in this form. Now, let us consider the equation involving phi d square phi, d phi square plus k phi square phi equal to 0. The solution to this equation, the general solution is given by phi phi is equal to A sin k phi phi plus B cos k phi phi.

Now, the h z rho, phi it can be seen that because of the circular symmetry is same as h z rho phi plus or minus 2m pi and therefore k phi will be an integer and we can write phi phi is equal to A sin n phi plus b cos n phi where n is an integer. So, essentially, we replace in this equation k phi by an integral value, and this is to take into account that h z it is periodic with respect to phi.

And therefore the equation here rho square by R d square R by d rho square plus rho by R dR d rho plus k c square rho minus k phi square equal to 0 can be written as replacing n in place of k phi and multiplying throughout by R we can put it in this form.

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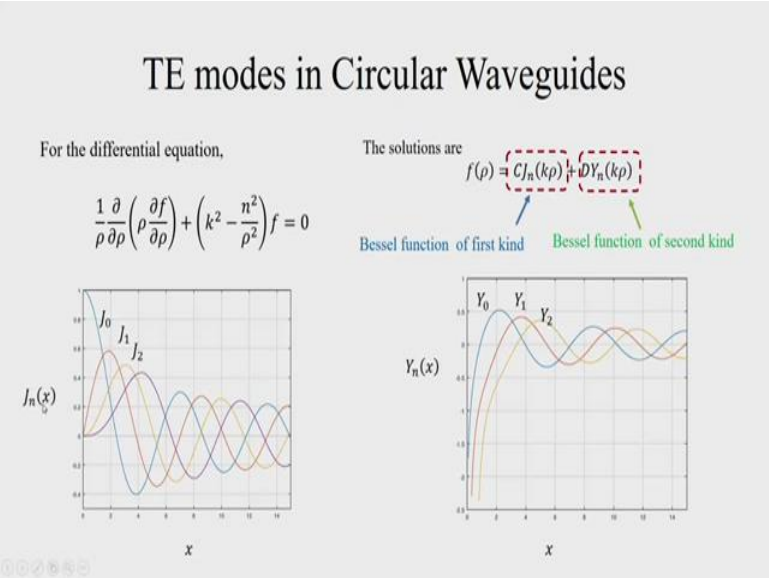
For the differential equation,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right)+\left(k^2-\frac{n^2}{\rho^2}\right)f=0$$

The solutions are

$$f(\rho)=CJ_n(k\rho)+DY_n(k\rho)$$



The differential equation 1 by rho; del del rho; rho del f del rho plus k square minus n square by rho square f equal to 0, this equation is known as Bessel equation, and the solution to this equation are the Bessel function. So, the solution f of rho is the general solution is given by CJ n; k rho plus DY n k rho. Here J n these are the Bessel function of the first kind and Y n Bessel function of the second kind and CJ n plus DY n this gives the general solution.

If we plot the Bessel functions for say J n x for n is equal to 0, 1, 2 so we get this type of plot, and we note that for example J 0 it is oscillatory in nature, and it is amplitude decreases, and it also crosses the x-axis several times. Now, whenever J n x becomes 0 those values of x this will call the roots of the Bessel function. For example, this will be the first root of J naught, this will be the first root of J 1, and we will designate them by p, n, m, which means the mth root of the nth order Bessel function.

The Y n when it is plotted, Y n x we find that the value of Y n becomes tend to become minus infinity as x approaches towards 0. Now, please note that this solution CJ n k rho plus DY n k rho this is a general solution, we will consider only that solution, which is consistent with our waveguide. Now, in a circular waveguide system the field at the center has to remain finite that means when rho equal to 0, the field has to be finite, and therefore this Y n solution is not a feasible solution for our waveguide, and we will retain only the J n solutions.

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Therefore, the solution of

$$\rho^2 \frac{d^2R}{d\rho^2} + \rho \frac{dR}{d\rho} + (k_c^2\rho^2 - n^2)R = 0$$

can be written as

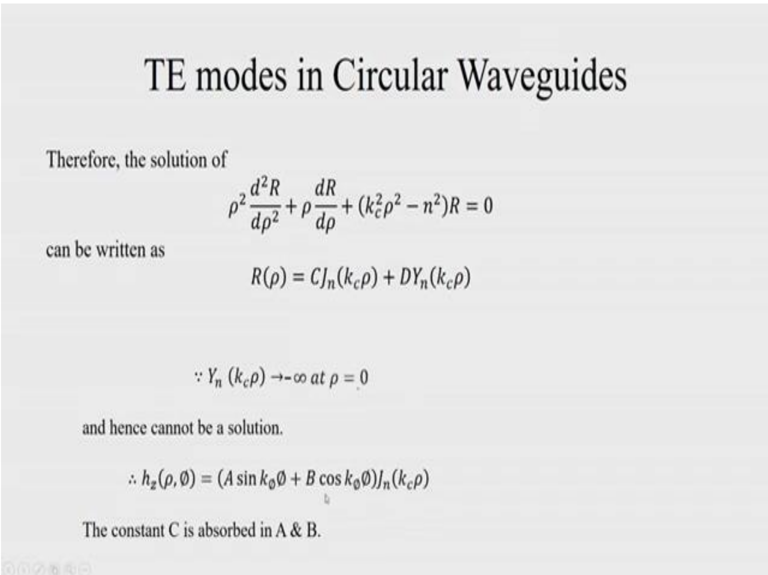
$$R(\rho) = CJ_n(k_c\rho) + DY_n(k_c\rho)$$

$$\because Y_n(k_c\rho) \rightarrow \infty \; at \; \rho = 0$$

and hence cannot be a solution.

$$\therefore h_z(\rho, \emptyset) = (A \sin k_\emptyset \emptyset + B \cos k_\emptyset \emptyset) J_n(k_c\rho)$$

The constant C is absorbed in A & B.



So, while writing the solution of this equation we can write R rho is equal to C n k c rho plus DY n k c rho as a general solution but as we have discussed Y n k c rho will tend to infinity as rho is equal to 0 and the field at the centre of the waveguide that means rho is equal to 0 has to become finite we cannot take it to be a solution and therefore we write h z rho phi as A sin k phi phi plus B cos k phi phi J n k c rho, so this is the solution of h z in terms of that transverse coordinate rho and phi.

Now, note that this constant c here is absorbed in the constants A and B.

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The cut-off wave number k_c is determined as follows:

We have

$$E_{\emptyset} = 0 \, for \, \rho = a$$

and

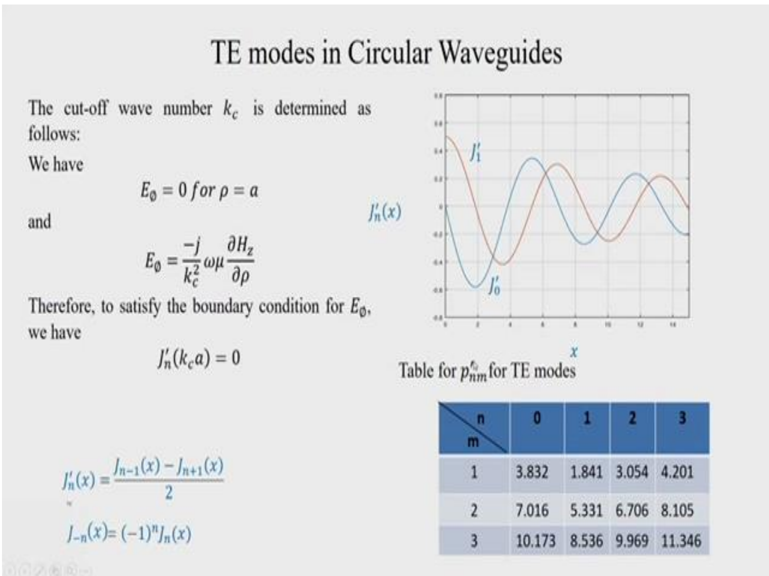
$$E_{\emptyset} = \frac{-j}{k_c^2} \omega\mu \frac{\partial H_z}{\partial \rho}$$

Therefore, to satisfy the boundary condition for $E_{\emptyset}$, we have

$$J_n'(k_c a) = 0$$

$$J_n'(x) = \frac{J_{n-1}(x) - J_{n+1}(x)}{2}$$

$$J_{-n}(x) = (-1)^n J_n(x)$$



The cut-off wave number k c we can determine as follows. We know that in a circular waveguide the E phi component it will be tangential to the waveguide wall, and E phi has to become 0 at rho equal to a, and therefore, we evaluate E phi equal to minus j by k c square omega mu del H z del rho. We have seen how we can express these field components in terms of the H z and E z components in our earlier discussion.

So, continuing from there we can write E phi is in this form, and therefore, to satisfy this boundary condition for E phi we must have J dash n which this dash indicates the derivative because we have del del rho k c rho into an equal to 0. This follows from the fact that H z has J n k c rho, and we take the derivative of the same and then substitute rho equal to a then J n dash k c a must be equal to 0 for satisfying the boundary condition E phi equal to 0.

Now, we can evaluate the derivative of Bessel function in terms of the Bessel functions themselves and J dash n x is given by J n minus 1 x minus J n plus 1 x by 2, and also we note that for example when n will become 0 it will be J minus 1, so J minus n is minus 1 to the power n J n x. So, using these two relationships, what we can do? We can find out J n dash this function.

For example, we have plotted here the derivative of the Bessel function for n is equal to 0 and n is equal to 1. So, we find that J 1 prime or J 1 dash it becomes 0 at x is equal to 1.84 around this value, and so this represents a root of the derivative of the Bessel function. Now, we call P dash nm as the root of the m^{th} root of the derivative of nth order Bessel function, and this is tabulated here.

So, let us verify, for example when n is equal to 1 and m is equal to 1, which means the first root of the J 1 dash or derivative of J 1 it occurs at 1.841, the next root is at 5.331. So, you can see this gives the order of the vessel function, and this gives the roots. Please note that for J naught dash the first 0 crossing occurs at 3.832, so this is the first root next root is at 7.016, so it should be here 7.016.

So this table it gives the roots of the derivative of nth order Bessel function, and it gives the mth root and we denote it by P dash nm.

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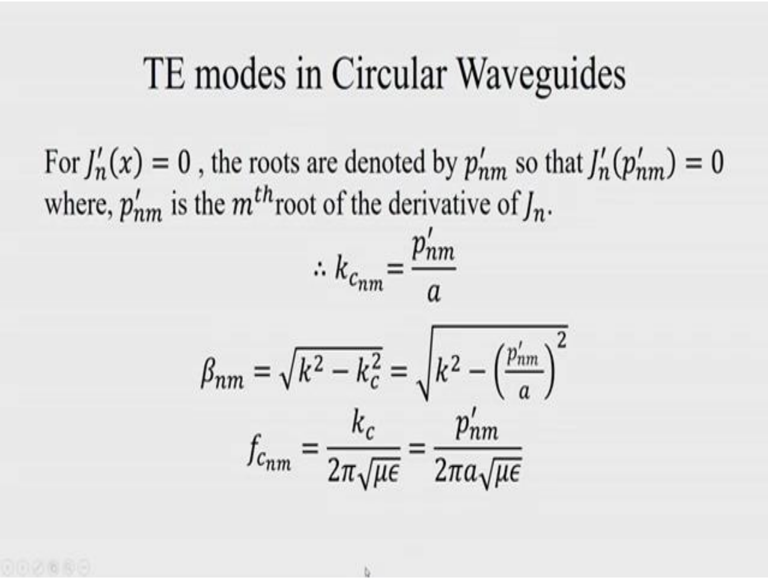
For $J_n'(x) = 0$, the roots are denoted by p_{nm}' so that $J_n'(p_{nm}') = 0$ where, p_{nm}' is the m^{th}root of the derivative of J_n.

$$\therefore k_{c_{nm}} = \frac{p_{nm}'}{a}$$

$$\beta_{nm} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{p_{nm}'}{a}\right)^2}$$

$$f_{c_{nm}} = \frac{k_c}{2\pi\sqrt{\mu\epsilon}} = \frac{p_{nm}'}{2\pi a\sqrt{\mu\epsilon}}$$



So, for J n dash x is equal to 0, the roots are denoted by P nm dash and as we have mentioned that it is the mth root of the derivative of J n. And now we find that k c nm can be written as P nm dash by a because we have a k c nm is equal to P nm dash. Now, beta for the nmth mode can be found out as k square minus P nm dash by a whole square, and in the same manner, the cut-off frequency f c nm can be found as P $_{nm}$ dash divided by 2 pi a root mu epsilon.

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- Therefore, for TE case we can write

$$\therefore H_z(\rho, \emptyset, z) = (A \sin k_\emptyset \emptyset + B \cos k_\emptyset \emptyset) J_n(k_c \rho) e^{-j\beta z}$$

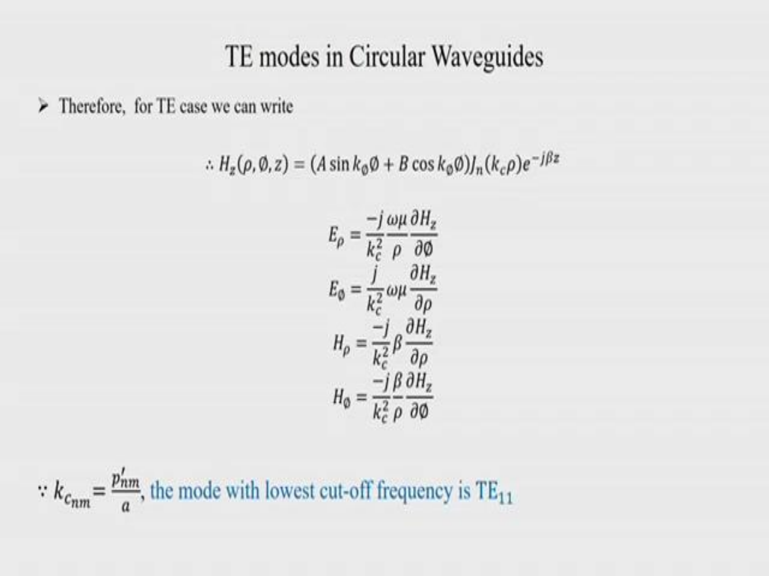
$$E_\rho = \frac{-j}{k_c^2} \frac{\omega\mu}{\rho} \frac{\partial H_z}{\partial \emptyset}$$

$$E_\emptyset = \frac{j}{k_c^2} \omega\mu \frac{\partial H_z}{\partial \rho}$$

$$H_\rho = \frac{-j}{k_c^2}\beta\frac{\partial H_z}{\partial \rho}$$

$$H_\emptyset = \frac{-j}{k_c^2}\frac{\beta}{\rho}\frac{\partial H_z}{\partial \emptyset}$$

$\because k_{c_{nm}} = \frac{p'_{nm}}{a}$, the mode with lowest cut-off frequency is TE_{11}



Now, if we summarize for the TE case we can write the complete expression for H z in this form and the transverse field components E rho, E phi, H rho, H phi they can be evaluated from the expressions given and we have already found out H z, we have found out k c so all the field components can be known. Now, we note that the cut-off frequency of the lowest mode is determined by this value P nm dash and the mode with the lowest cut-off frequency is TE 11 because P 11 dash has the lowest value in that table we have shown and it is equal to 1.841.

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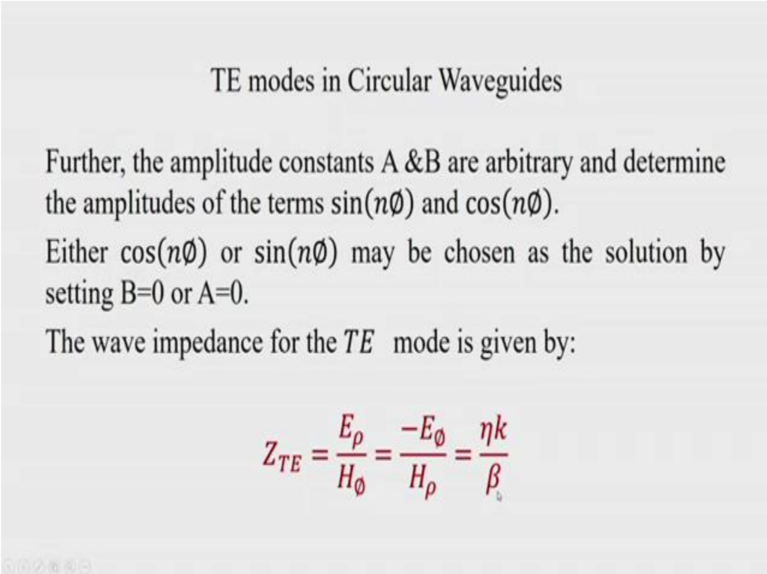
Further, the amplitude constants A &B are arbitrary and determine the amplitudes of the terms $\sin(n\emptyset)$ and $\cos(n\emptyset)$.

Either $\cos(n\emptyset)$ or $\sin(n\emptyset)$ may be chosen as the solution by setting $B = 0$ or $A = 0$.

The wave impedance for the TE mode is given by:

$$Z_{TE} = \frac{E_\rho}{H_\emptyset} = \frac{-E_\emptyset}{H_\rho} = \frac{\eta k}{\beta}$$



Further these amplitude constants A and B they are arbitrary and determine the amplitude in terms of sin n phi and cos n phi. Now, A sin n phi plus b cos n phi this is a general solution either cos n phi or sin n phi may also be chosen as a solution by setting either B equal to 0 or A equal to 0. The wave impedance for the TE mode is given by Z TE is equal to E rho by H phi which is equal to minus of E phi by H rho and it can be found out to be eta k by beta.

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For TM case, $H_z = 0$ and $E_z(\rho, \emptyset, z) = e_z(\rho, \emptyset)e^{-j\beta z}$

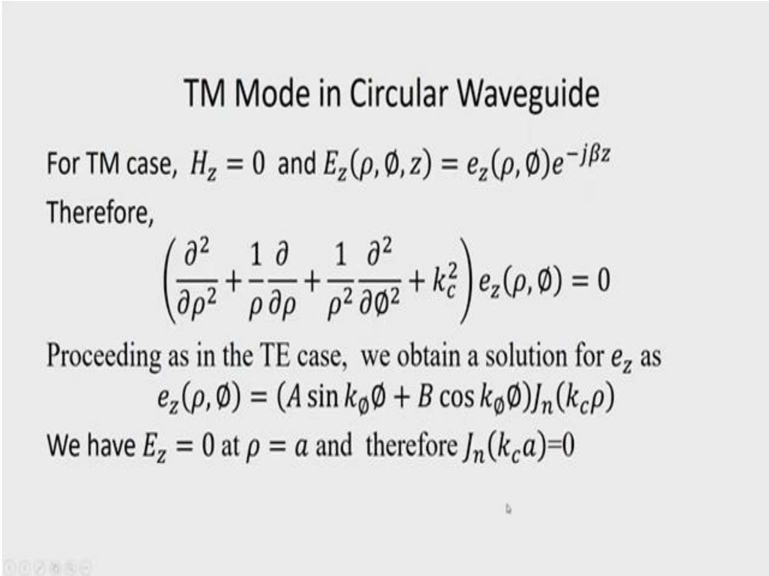
Therefore,

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \emptyset^2} + k_c^2\right) e_z(\rho, \emptyset) = 0$$

Proceeding as in the TE case, we obtain a solution for e_z as

$$e_z(\rho, \emptyset) = (A \sin k_\emptyset \emptyset + B \cos k_\emptyset \emptyset) J_n(k_c \rho)$$

We have $E_z = 0$ at $\rho = a$ and therefore $J_n(k_c a)$=0



Let us now consider how we find out the transverse magnetic or TM mode in circular waveguide. The treatment is identical to that used for analysing the TE mode, for this TM case we have H z equal to 0 and E z can be expressed as E z rho, phi which gives the variation of E z with respect to transverse coordinate rho and phi and the z variation is given by e to the power minus j beta z.

And therefore as in the previous case now we can write the wave equation in this form and if you proceed in the same manner as we did in the TE case we can obtain a solution for E z and E z rho phi it can be written as A sin k phi phi plus B cos k phi phi J and k 0. Please note that here again as in the case of as in TE case this becomes an integer n k phi and we have now is that itself is a tangential component and we can apply the boundary condition on E z directly that is E z becomes 0 at rho equal to a and therefore this term J n k ca is equal to 0.

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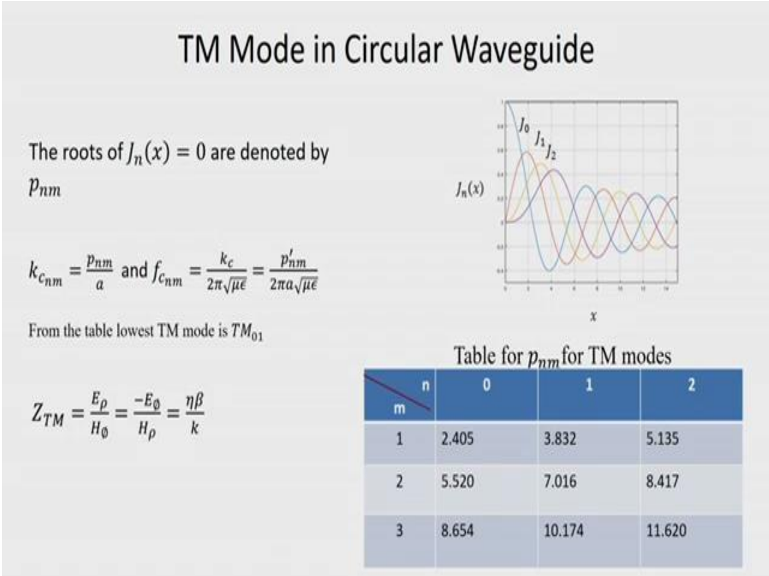
The roots of $J_n(x) = 0$ are denoted by p_{nm}

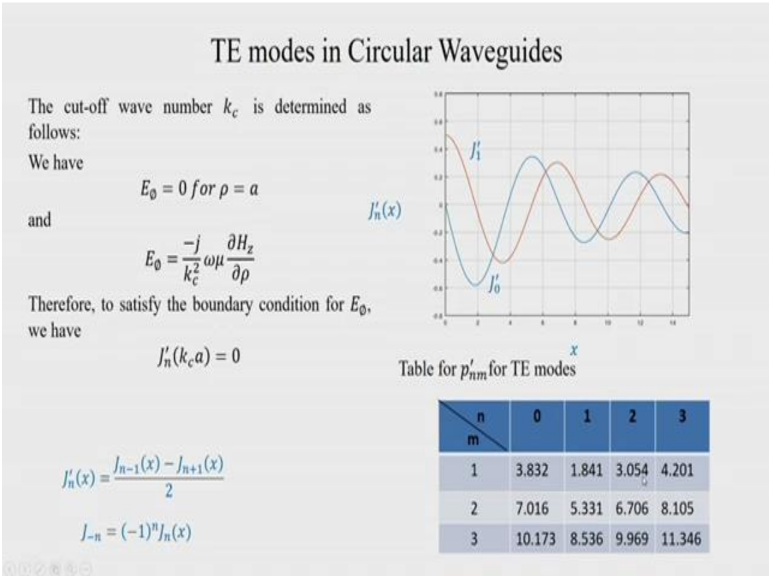
$k_{c_{nm}} = \frac{p_{nm}}{a}$ and $f_{c_{nm}} = \frac{k_c}{2\pi\sqrt{\mu\epsilon}} = \frac{p_{nm}}{2\pi a\sqrt{\mu\epsilon}}$

From the table lowest TM mode is TM_{01}

$Z_{TM} = \frac{E_\rho}{H_\emptyset} = \frac{-E_\emptyset}{H_\rho} = \frac{\eta\beta}{k}$



m \ n	0	1	2
1	2.405	3.832	5.135
2	5.520	7.016	8.417
3	8.654	10.174	11.620



m \ n	0	1	2	3
1	3.832	1.841	3.054	4.201
2	7.016	5.331	6.706	8.105
3	10.173	8.536	9.969	11.346

We have already plotted the Bessel functions and the roots of J n x equal to 0. They are denoted by P nm and if we consider J n x then we have seen that J 0 crosses this x-axis for a value of 2.4054 then followed by 5.520 like that. So we have a table for the values of P nm, which are essentially represents the mth root of J n x and once we have these values of P nm we can write cut-off wave number k c nm is equal to P nm by a.

And similarly cut-off frequency f c nm is equal to k c divided by 2 pi root mu epsilon, which we can write P nm divided by 2 pi a root mu epsilon. Please note that when it comes to TM mode the lowest mode is TM 01, we saw in the case of TE the lowest mode was be TE_{11} but if we compare the cut-off frequency of TM 01 and be TE_{11}, be $TE_{11}$is lower the value of P dash 11 was 1.841.

So, for a given radius of the circular waveguide, the mode with the overall if you consider both TM and TE cases the mode with the lowest cut-off frequency will be TE_{11}, and that is why this is called the dominant mode in a circular waveguide. We should also note that when we discussed a rectangular waveguide, the cut-off frequency could be changed by changing the parameters A and B the width and height of the guide.

So, we can have different sequences of occurrence of the mode cut-off frequencies of the mode but when it comes to a circular waveguide we have only the radius, and once the radius is set the sequence in which the modes will come becomes fixed. For example, the mode with the lowest cut-off frequency is $TE_{11,}$ and then the next mode that will come is $TM_{01,}$ and then if we consider the table, we find that the next mode that will come is TE_{21} because this is a value 3.054.

So, in that order the modes will keep on also coming there are certain values which are common in these two tables, and this will represent the degenerate modes as we have already discussed if we have two modes with same cut-off frequency they become degenerate. For example, if you consider TM 12 mode here the value for P nm is 7.016 and if we go back to our TE mode case, then we find that TE 02 has the same value 7.016, so TE 02 will become degenerate with TM 12 case.

The wave impedance Z_{TM} is given by E rho by H phi, which is equal to minus E phi by H rho and is equal to eta, beta by k here eta is the intrinsic impedance of the dielectric media within the guide and k is the wavenumber in that dielectric media.

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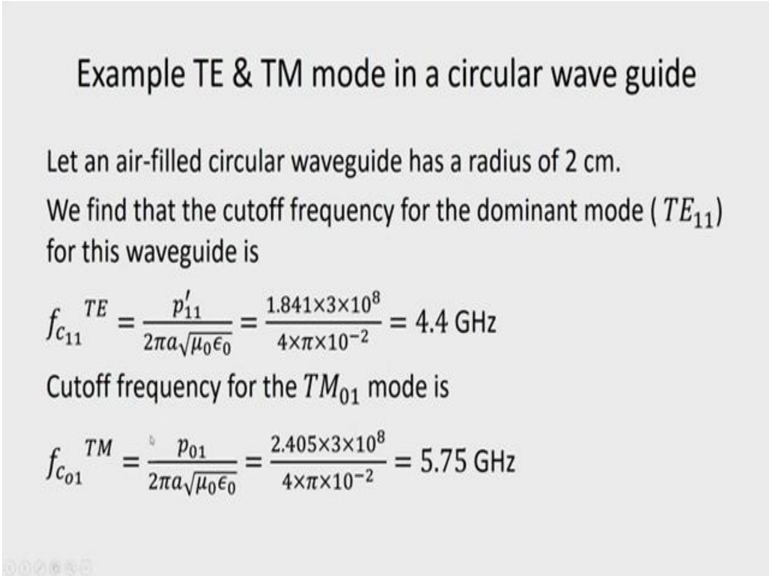
Let an air-filled circular waveguide has a radius of 2 cm.

We find that the cutoff frequency for the dominant mode (TE_{11}) for this waveguide is

$$f_{c_{11}}{}^{TE} = \frac{p'_{11}}{2\pi a\sqrt{\mu_0\epsilon_0}} = \frac{1.841\times3\times10^8}{4\times\pi\times10^{-2}} = 4.4 \text{ GHz}$$

Cutoff frequency for the TM_{01} mode is

$$f_{c_{o1}}{}^{TM} = \frac{p_{01}}{2\pi a\sqrt{\mu_0\epsilon_0}} = \frac{2.405\times3\times10^8}{4\times\pi\times10^{-2}} = 5.75 \text{ GHz}$$



Let us now consider a simple example to see how the order of the cut-off frequencies that we get in practical wave guides. Let us consider an air-filled circular waveguide and let us assume that the radius is 2 centimetre, and then we find that the cut-off frequency for the dominant mode TE_{11} is given by f c 11 TE is p 11 dash divided by 2 pi a root mu naught epsilon naught and this can be written as 3 into 10 to the power 8 and then this p 11 dash is 1.841 and substituting these values we get the cut-off frequency for TE_{11} mode as 4.4 gigahertz.

Similarly, if we consider the cut-off frequency for TM_{01} mode then everything remains same only this p 11 dash is now replaced by p 01, and therefore cut off frequency for the TM_{01} mode becomes 5.75 gigahertz. So, please note that this is the range in which we have a two-centimeter circular waveguide will give single-mode operation because beyond 5.75 you can have the second mode can be propagated, but from 4.4 to 5.75 for this particular waveguide with radius two centimeter, only TE_{11} mode can be propagated.

So, in these lectures, we have considered the propagation of different modes in the circular waveguide. We have not considered any attenuation of signals because we have assumed the waveguide walls to be perfectly conducting. Next, we will consider the attenuation of signal inside the waveguide walls because of the finite conductivity of the waveguide walls.