Micro Wave Engineering
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Lecture 09
Scattering Matrix (S-Parameters) Part-1

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This forms the basis of scattering matrix formulation.

For a N port network, at the n^{th}port let us define

$$a_n = \frac{V_n^+}{\sqrt{z_{0n}}} \quad \& \quad b_n = \frac{V_n^-}{\sqrt{z_{0n}}}$$

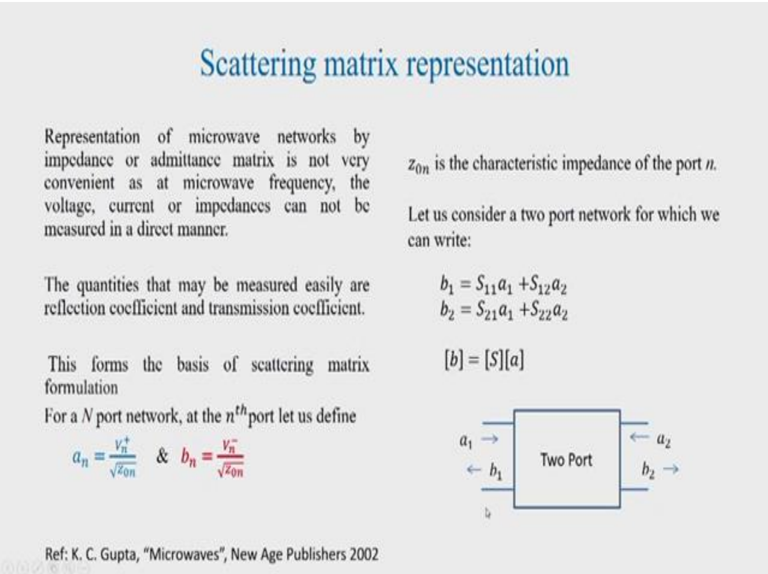
z_{0n} is the characteristic impedance of the port n.

Let us consider a two port network for which we can write:

$$b_1 = S_{11}a_1 + S_{12}a_2$$

$$b_2 = S_{21}a_1 + S_{22}a_2$$

$$[b] = [S][a]$$



We now consider the scattering matrix representation of microwave networks. Representation of microwave networks. By impedance or admittance matrix is not very convenient as at microwave frequency, the voltage, current, or impedance cannot be measured in a direct manner. The quantities that may be measured easily are the reflection coefficient and transmission coefficient, and this forms the basis of scattering matrix formulation.

The scattering parameters can be measured directly from instruments called network analyzers so, let us consider an N port network and at the N^{th} port let us define, the parameter an which is voltage Vn plus the incident voltage at the N^{th} port, normalized with respect to square root of Z_0^n the Z_0^n being the impedance of the N^{th} port, similarly we define bn is V_n minus divided by root Z_0^n. So to illustrate, let us consider a 2 port network.

So, these parameters a1 and b1 there the incident and reflected parameters, they represent the incident and reflected waves, in port 1 similarly a_2 b_2 represents the incident and reflected waves at port 2. And we can write the reflected waves b_1 as S_{11} a_1 plus S_{12} a_2; similarly b_2 S_{21} a_1 plus S_{22} a_2, and this can be written in the form of matrix b_s a now this s matrix it is the scattering matrix for this 2 port network.

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We find that

$$S_{11} = \frac{b_1}{a_1}\bigg|_{a_2 = 0}$$

$$S_{11} = \frac{V_1^-}{V_1^+}\bigg|_{V_2^+ = 0}$$

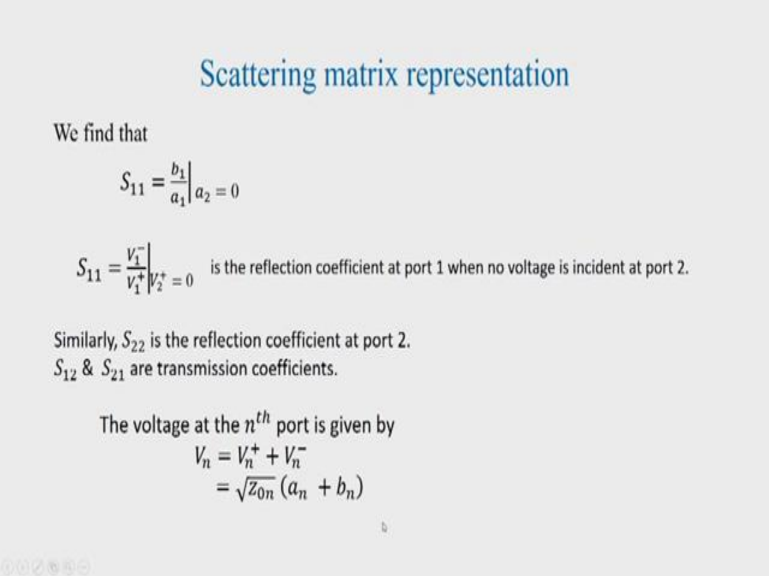
Similarly, S_{22} is the reflection coefficient at port 2.

S_{12} & S_{21} are transmission coefficients.

The voltage at the n^{th} port is given by

$$V_n = V_n^+ + V_n^-$$

$$= \sqrt{z_{0n}}\,(a_n + b_n)$$



Now we find that S_{11} is actually b_1 by a_1 when a_2 equal to 0. And therefore, it represents V_1 minus by V_1 plus at port 1 when V_2 plus is equal to 0. And this is essentially the reflection coefficient at port 1 when no voltage is incident at port 2. Similarly, S_{22} is the reflection coefficient at port 2, and S_{12} and S_{21} will be the transmission coefficients from port 2 to 1 and from 1 to 2. Now the voltage at the Nth port it is given by V_n is equal to V_n plus Vn minus, and in terms of the parameters an and bn we can write V_n to be equal to root Z_0^n a plus b_n.

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The current at the n^{th} port

$$I_n = \frac{1}{z_{0n}}\,(V_n^+ - V_n^-)$$

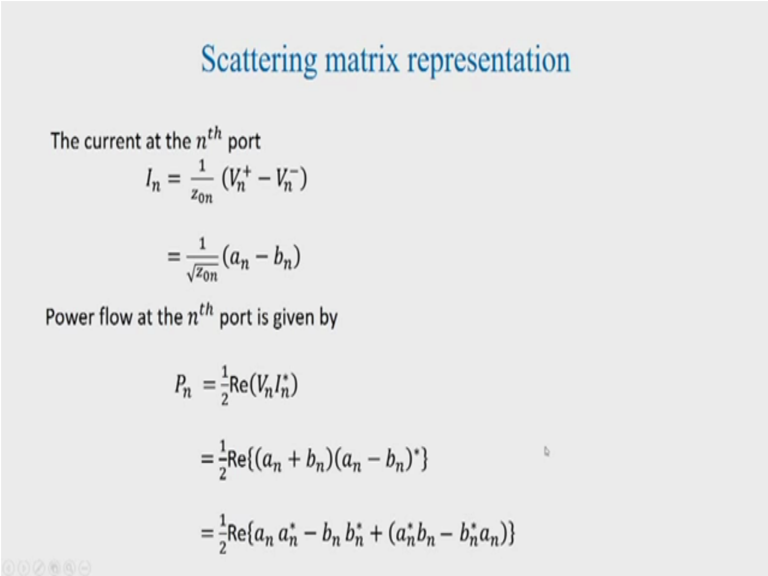
$$= \frac{1}{\sqrt{z_{0n}}}(a_n - b_n)$$

Power flow at the n^{th} port is given by

$$P_n = \frac{1}{2}\mathrm{Re}(V_n I_n^*)$$

$$= \frac{1}{2}\mathrm{Re}\{(a_n + b_n)(a_n - b_n)^*\}$$

$$= \frac{1}{2}\mathrm{Re}\{a_n\, a_n^* - b_n\, b_n^* + (a_n^* b_n - b_n^* a_n)\}$$



Similarly, the current at the N^{th} port is defined as 1 by Z_0^n V_n plus minus V_n minus, and therefore, we can write In to be equal to 1 by root Z_0^n, n minus V_n. And if we calculate the power flow at the N^{th} port, then it is half real part of V_n I_n conjugate and when we substitute V_n and I_n we get, P_n is half real part of an plus bn and an minus bn conjugate. When it is expanded, we can write in this form a conjugate minus bn conjugate. Plus a conjugate bn minus bn conjugate an.

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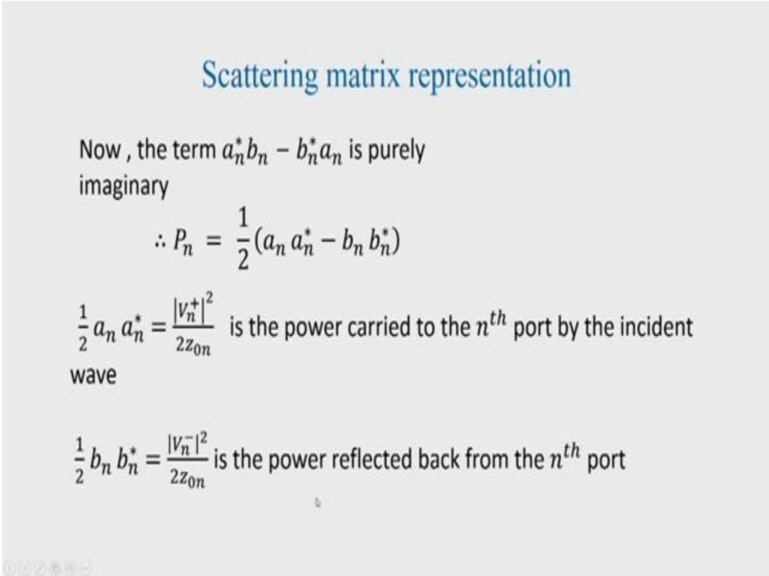
Now , the term $a_n^* b_n - b_n^* a_n$ is purely imaginary

$$\therefore P_n = \frac{1}{2}(a_n a_n^* - b_n b_n^*)$$

$\frac{1}{2} a_n a_n^* = \frac{|V_n^+|^2}{2z_{0n}}$ is the power carried to the n^{th} port by the incident wave

$\frac{1}{2} b_n b_n^* = \frac{|V_n^-|^2}{2z_{0n}}$ is the power reflected back from the n^{th} port



Now this term can be shown to be purely imaginary and therefore, we can write P_n is equal to half an an conjugate minus bn bn conjugate. Now this term half an an conjugate, which can be written as mod V_n plus square divided by Z_0^n. it is the power carried to the Nth port by the incident wave. Similarly, half bn conjugate is bn minus magnitude square divided by 2 Z_0^n it is the power reflected back from the N^{th} port.

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We have seen that for a reciprocal network Z-matrix is symmetric.

$$\therefore [V] = [Z][I]$$

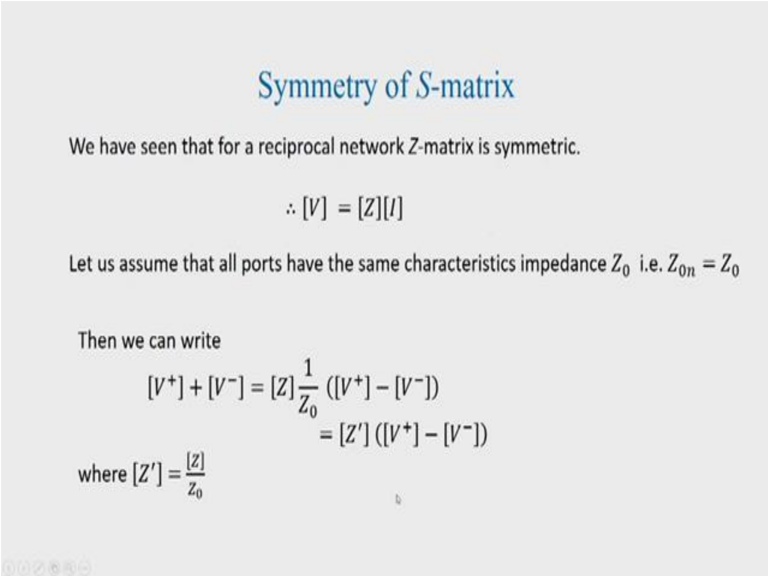
Let us assume that all ports have the same characteristics impedance Z_0 i.e. $Z_{0n} = Z_0$

Then we can write

$$[V^+] + [V^-] = [Z]\frac{1}{Z_0}([V^+] - [V^-])$$

$$= [Z'] ([V^+] - [V^-])$$

where $[Z'] = \frac{[Z]}{Z_0}$



Now let us discuss the symmetry property of s matrix. Now we have already seen that for a reciprocal network z matrix is symmetric. Now let us start over discussion, with the assumption that all the port impedances are same that means Z_0^n is equal to z Naught. So we can write V matrix as V plus and V matrix and I matrix and replace by 1 by z Naught V plus minus V minus. And that can be written as Z dash V plus minus V minus.

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Rearranging the terms we can write

$$([Z'] + [U])[V^-] = ([Z'] - [U])[V^+]$$

Here, $[U]$ is the unity matrix

when $Z_{0n} = Z_0$, $[b] = [S][a]$ can be written as

$$[V^-] = [S][V^+]$$

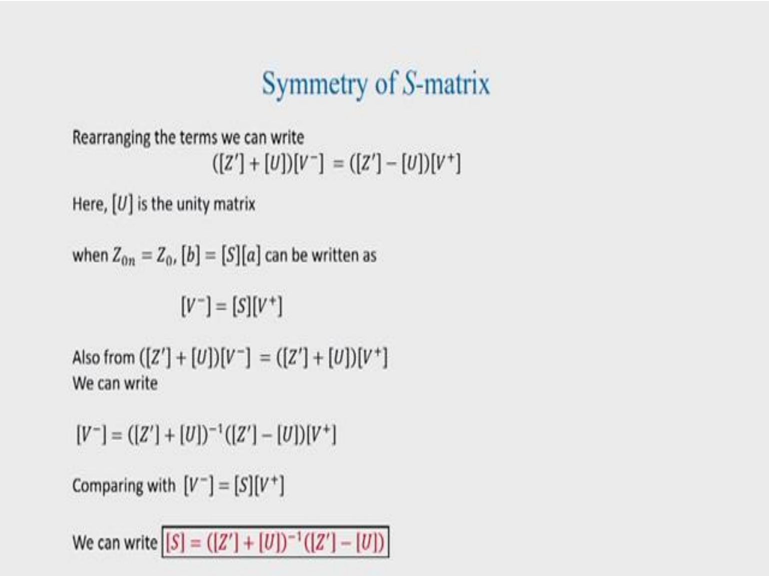
Also from $([Z'] + [U])[V^-] = ([Z'] + [U])[V^+]$

We can write

$$[V^-] = ([Z'] + [U])^{-1}([Z'] - [U])[V^+]$$

Comparing with $[V^-] = [S][V^+]$

We can write $\boxed{[S] = ([Z'] + [U])^{-1}([Z'] - [U])}$



So if we rearrange the terms that mean we take V minus, a terms in one side and V plus terms on the other side then, we can write, this u is the unity matrix and similarly when we have all the port impedances are same equal to z Naught the scattering matrix be is equal to S a this term can be written as V minus S, V plus and from this term above we can find V minus to be equal to Z dash plus U inverse into z dash minus U into V plus. If we compare these 2, then we get s equal to z dash plus u inverse into z dash minus u. so these expressions establish a relationship between the s matrix and the z matrix rather z dash matrix where we have normalized z matrix by z Naught. We can also have an alternative form of representation for the scattering matrix.

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An alternate form for the scattering matrix can be derived as follows:

At the n^{th}port

$$V_n = V_n^+ + V_n^-$$

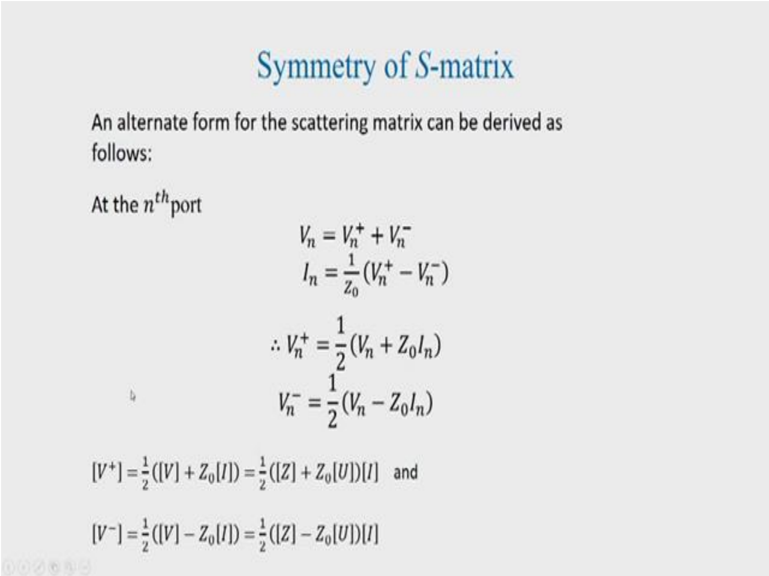
$$I_n = \frac{1}{Z_0}(V_n^+ - V_n^-)$$

$$\therefore V_n^+ = \frac{1}{2}(V_n + Z_0 I_n)$$

$$V_n^- = \frac{1}{2}(V_n - Z_0 I_n)$$

$[V^+] = \frac{1}{2}([V] + Z_0[I]) = \frac{1}{2}([Z] + Z_0[U])[I]$ and

$[V^-] = \frac{1}{2}([V] - Z_0[I]) = \frac{1}{2}([Z] - Z_0[U])[I]$



We can derive it like this at the Nth port, we have V_n equal to Vn plus plus Vn minus and In is 1 by z Naught, Vn plus minus Vn minus. So, we can write, we can find out the expression for Vn plus and Vn minus from these 2, and then we can write, Vn plus as half the voltage matrix for the incident wave, V plus is equal to half Z plus Z Naught U I and similarly, for the voltage matrix for the reflected waves we can write in this form.

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$[I] = 2([Z] + Z_0[U])^{-1}[V^+]$

$\therefore [V^-] = ([Z] - Z_0[U])([Z] + Z_0[U])^{-1}[V^+]$

$[V^-] = ([Z'] - [U])([Z'] + [U])^{-1}[V^+]$

$\therefore [S] = ([Z'] - [U])([Z'] + [U])^{-1}$

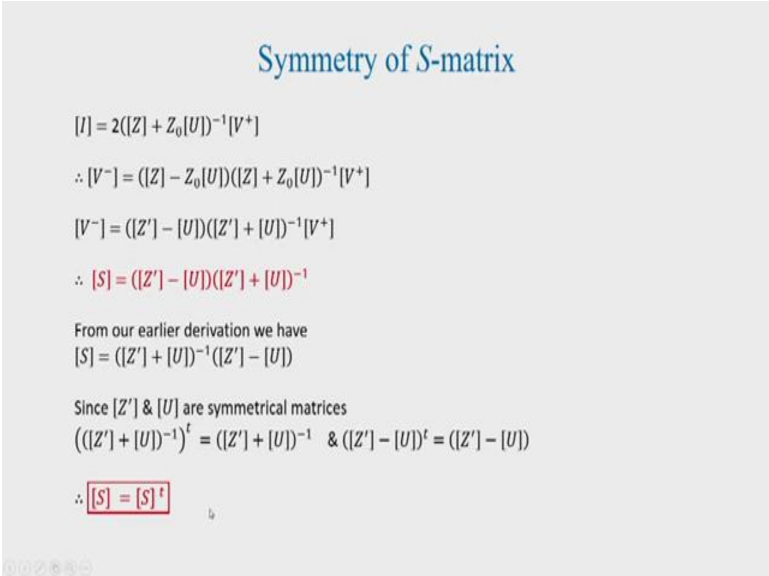
From our earlier derivation we have

$[S] = ([Z'] + [U])^{-1}([Z'] - [U])$

Since $[Z']$ & $[U]$ are symmetrical matrices

$(([Z'] + [U])^{-1})^t = ([Z'] + [U])^{-1}$ & $([Z'] - [U])^t = ([Z'] - [U])$

$\therefore \boxed{[S] = [S]^t}$



Now from these 2 relation we can find S scattering matrix, so once we make all these substitutions, we finally get the expression for s, which is s is equal to z dash minus U, multiplied by Z dash plus U inverse. So, this is another expression for scattering matrix S in terms of this normalized impedance matrix. And from our earlier derivation we got, this expression, now since Z dash and U that, these 2 matrix are symmetrical, so if we take transverse of these, then we can find S is equal to S transpose.

Because once, we take the transverse of this matrix, this inverse transpose will be same as the inverse for a symmetric matrix and Z dash minus U transpose will be equal to Z dash minus U. From this we can find that S is equal to S transpose, and this shows that the scattering matrix for a microwave network is symmetric.

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When the port impedances Z_{0n} are all different we can write

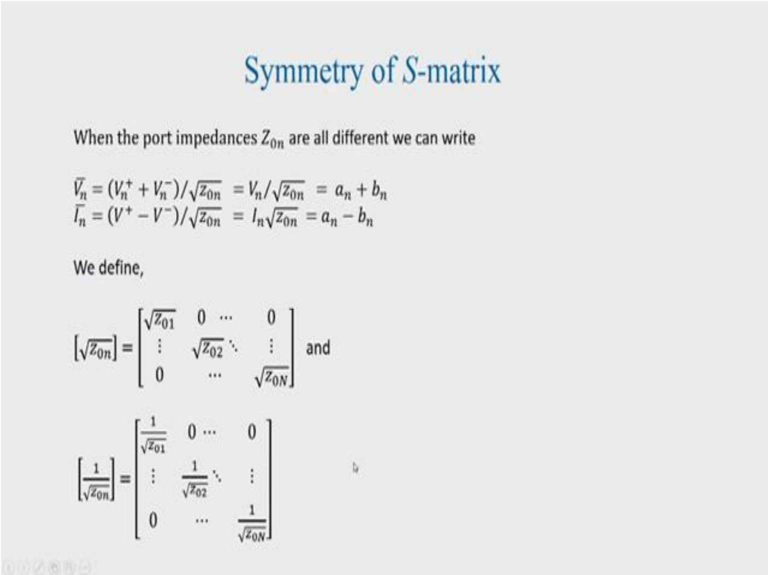
$$\bar{V_n} = (V_n^+ + V_n^-)/\sqrt{z_{0n}} \quad = V_n/\sqrt{z_{0n}} \quad = a_n + b_n$$

$$\bar{I_n} = (V^+ - V^-)/\sqrt{z_{0n}} \quad = \quad I_n\sqrt{z_{0n}} = a_n - b_n$$

We define,

$$[\sqrt{z_{0n}}] = \begin{bmatrix} \sqrt{z_{01}} & 0 & \cdots & 0 \\ \vdots & \sqrt{z_{02}} & \ddots & \vdots \\ 0 & & \cdots & \sqrt{z_{0N}} \end{bmatrix} \text{ and}$$

$$\left[\frac{1}{\sqrt{z_{0n}}}\right] = \begin{bmatrix} \frac{1}{\sqrt{z_{01}}} & 0 \cdots & 0 \\ \vdots & \frac{1}{\sqrt{z_{02}}} \ddots & \vdots \\ 0 & \cdots & \frac{1}{\sqrt{z_{0N}}} \end{bmatrix}$$



We can also show that when all the port impedances z0n are different then also this property holds. So in order to show that we write Vn bar to be equal to Vn plus Vn minus divided by root z0n, and this is actually an plus bn and in the same manner In bar becomes an minus bn. And we introduce 2 square matrixes 1 is a matrix of root z0n which is all the diagonal elements as root z0n and of the diagonal elements are zero. And another matrix where the diagonal elements are 1 by root z0n.

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Using these two matrices we can write

$$[V] = \left[\sqrt{z_{0n}}\right][\bar{V}]$$

$$[I] = \left[\frac{1}{\sqrt{z_{0n}}}\right][\bar{I}]$$

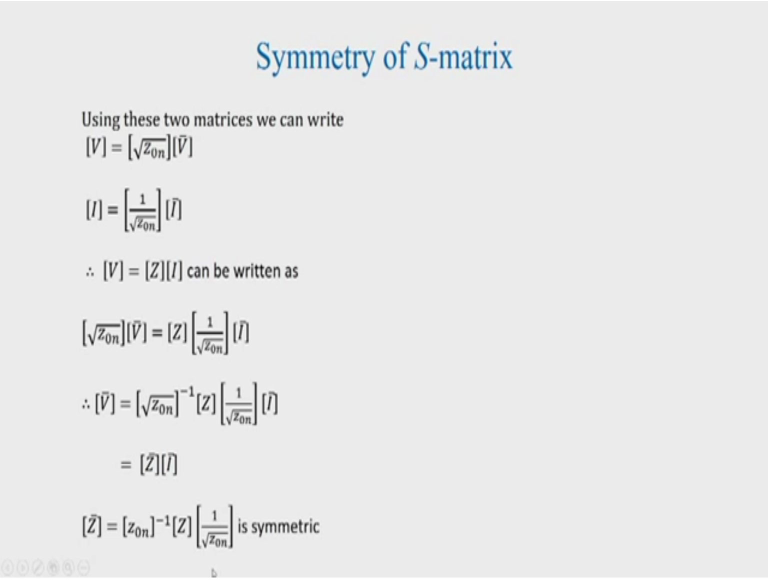
$\therefore$ $[V] = [Z][I]$ can be written as

$$\left[\sqrt{z_{0n}}\right][\bar{V}] = [Z]\left[\frac{1}{\sqrt{z_{0n}}}\right][\bar{I}]$$

$$\therefore [\bar{V}] = \left[\sqrt{z_{0n}}\right]^{-1}[Z]\left[\frac{1}{\sqrt{z_{0n}}}\right][\bar{I}]$$

$$= [\bar{Z}][\bar{I}]$$

$[\bar{Z}] = [z_{0n}]^{-1}[Z]\left[\frac{1}{\sqrt{z_{0n}}}\right]$ is symmetric



Now if we introduce these 2 matrices we can write the voltage matrix V can be related to normalized voltage V bar and similarly, current matrix I can be related to normalized current matrix I bar. And this equation V matrix is equal to z matrix into I matrix can be written as shown here, in terms of the normalized voltages and currents. And then we find that this normalized voltage matrix is related to the current matrix in this manner, where we have a matrix root z0n inverse then, the matrix z another matrix 1 by root z0n.

So, we denote this as Z bar and therefore, we have now V bar is equal to Z bar, I bar. Please note that this matrix Z bar is symmetric, because the component matrix z0n, 1 by z0n, z they are all symmetric matrices and Z bar is the product of these symmetric matrices.

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$$[a] + [b] = [\bar{V}]$$

$[a] - [b] = [\bar{I}]$

$[a] + [b] = [\bar{V}] = [\bar{Z}][\bar{I}]$

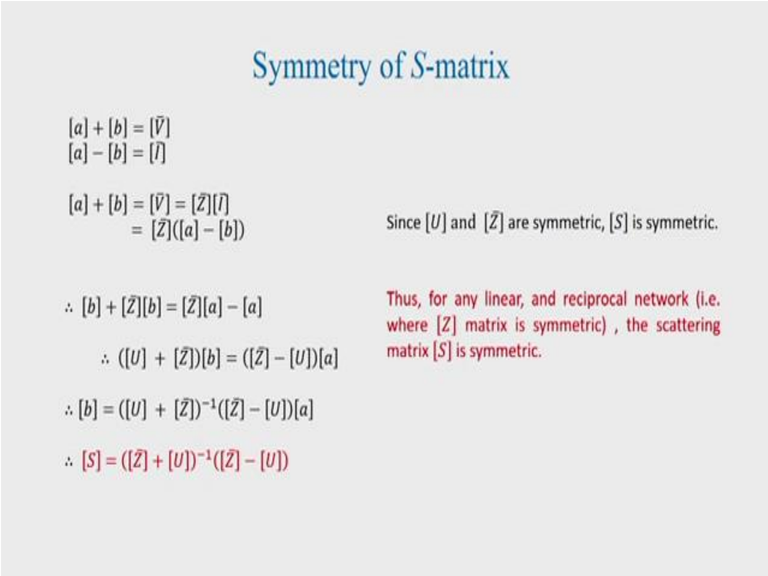
$= [\bar{Z}]([a] - [b])$

$\therefore\ [b] + [\bar{Z}][b] = [\bar{Z}][a] - [a]$

$\therefore\ ([U] + [\bar{Z}])[b] = ([\bar{Z}] - [U])[a]$

$\therefore\ [b] = ([U] + [\bar{Z}])^{-1}([\bar{Z}] - [U])[a]$

$\therefore\ [S] = ([\bar{Z}] + [U])^{-1}([\bar{Z}] - [U])$



So, we can relate the normalized voltage matrix to matrices a and b as shown. Similarly, the normalized current matrix can be related to matrices a and b as, a minus b equal to I bar. Now V bar can be replaced at Z bar I bar and then, we replaced I bar and then, rearranging the terms we can write, b to be U plus Z bar inverse into Z bar minus U a. now this essentially gives the s matrix the scattering matrix S in terms of the unity matrix U normalized Z matrix.

Now since, U and Z bar they are symmetric S is also symmetric and therefore, for any linear and reciprocal network, that is where Z matrix is symmetric, the scattering matrix S is also symmetric.

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We have seen that $P_n \;=\; \frac{1}{2}(a_n\, a_n^* - b_n\, b_n^*)$

For a lossless junction total power leaving the all N ports must be equal to sum of the incident powers.

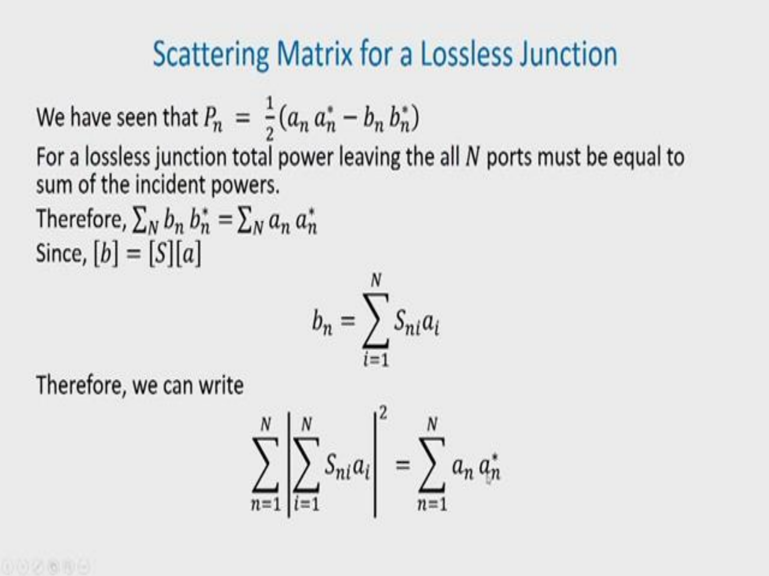
Therefore, $\sum_N b_n\, b_n^* = \sum_N a_n\, a_n^*$

Since, $[b] = [S][a]$

$$b_n = \sum_{i=1}^{N} S_{ni} a_i$$

Therefore, we can write

$$\sum_{n=1}^{N}\left|\sum_{i=1}^{N} S_{ni}a_i\right|^2 = \sum_{n=1}^{N} a_n\, a_n^*$$



Now let us consider the scattering matrix representation for a lossless junction. By a lossless junction, we mean that whatever power it enters the network, through different ports same power will leave the network and we have seen that the power in the Nth port Pn is given by half an an conjugate minus bn bn conjugate, this is the incident power and this is the reflected power. So, when the junction is lossless total power leaving all N ports must be equal to the sum of the incident powers, and we will have sigma Pn equal to 0, and from there, we can write, summation of bn bn conjugate, is equal to summation of an an conjugate.

Now since, we have b is equal to Sa we can write, bn to be equal to sum of Sni ai. So, bn is the Nth element of the b vector and that can be obtained by multiplying the corresponding Nth elements of Nth row with the elements of vector I and summing them, over I to N. and then, if we substitute here, in this expression, bn and bn conjugate then essentially we will get modules of summation of Sni ai square and this side we will get, summation of an an conjugate.

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Since a_n parameters are independent, if we set all the $a_n$except a_i to be zero.

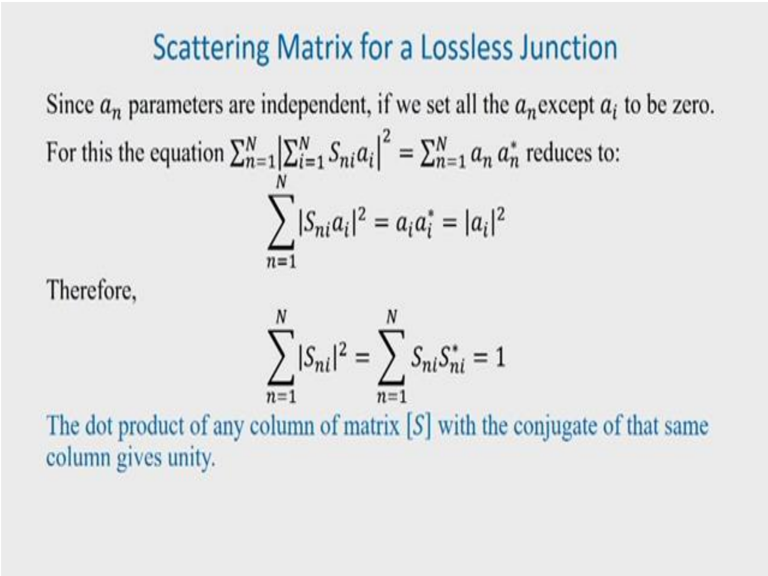
For this the equation $\sum_{n=1}^{N}\left|\sum_{i=1}^{N} S_{ni}a_i\right|^2 = \sum_{n=1}^{N} a_n\, a_n^*$ reduces to:

$$\sum_{n=1}^{N}|S_{ni}a_i|^2 = a_i a_i^* = |a_i|^2$$

Therefore,

$$\sum_{n=1}^{N}|S_{ni}|^2 = \sum_{n=1}^{N} S_{ni}S_{ni}^* = 1$$

The dot product of any column of matrix $[S]$ with the conjugate of that same column gives unity.



Now this parameters an these are the input parameters and therefore, they can be chosen independently and if we set all the an except ai to be 0 then, this equation what we have written, reduces to this. So it becomes this summation becomes, mod of ai square and this summation it becomes a single term Snai ai (magnitu) root square and therefore, what we see that, summation of Sni square which is actually summation of Sni Sn conjugate is equal to 1.

Now this Sni essentially represents the element of the Ith column of the scattering matrix. And this is the conjugate of the elements of the Ith column and therefore, we can write that, the dot product of any column of scattering matrix S with the conjugate of that of the same column gives unity. And please note that here, this I may take any value from 1 to N. so, this relationship will be held for any column of the S matrix.

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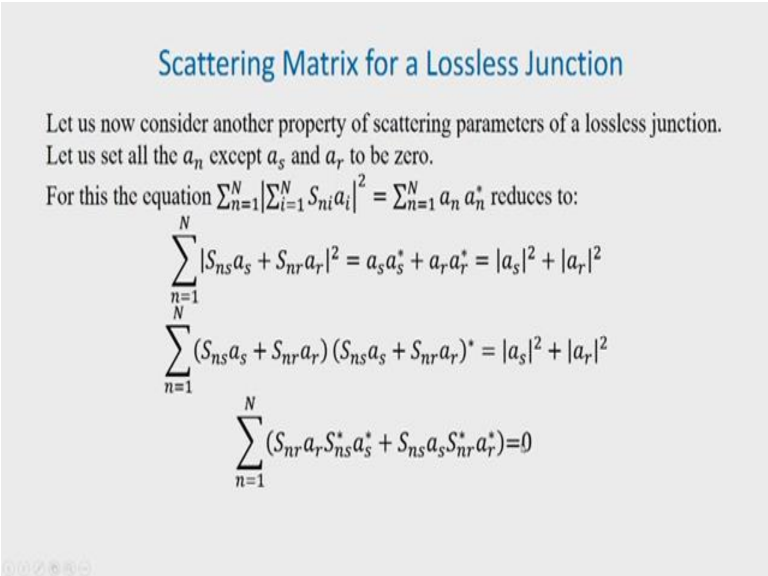
Let us now consider another property of scattering parameters of a lossless junction.

Let us set all the a_n except a_s and a_r to be zero.

For this the equation $\sum_{n=1}^{N}\left|\sum_{i=1}^{N} S_{ni}a_i\right|^2 = \sum_{n=1}^{N} a_n\, a_n^*$ reduces to:

$$\sum_{n=1}^{N}|S_{ns}a_s + S_{nr}a_r|^2 = a_s a_s^* + a_r a_r^* = |a_s|^2 + |a_r|^2$$

$$\sum_{n=1}^{N}(S_{ns}a_s + S_{nr}a_r)\,(S_{ns}a_s + S_{nr}a_r)^* = |a_s|^2 + |a_r|^2$$



Now let us consider another property of the scattering parameters of a lossless junction. Now in this case what we will do, we consider as and ar to be non-zero, and all other an s is zero. Then, from this equation if we expand, we can write summation of 1 to N Sns as plus Snr ar magnitude square, and on the right-hand side it becomes, as conjugate plus ar conjugate, and this becomes as square plus ar square. So, if we expand these product terms then, we find that we will have terms like summation of small n is equal to 1 to capital N. mod of Sns square mod of as square.

Now these summations of mod of Sns square as we have seen it will become 1. So, we will have mod of as square. Similarly, product of these 2 terms will give mod of ar square and that will get cancelled with these 2 terms and making this simplification from this expression, we will get a relation of these form, which is summation of 1 to N Snr ar Sns conjugate as conjugate is plus Sns as Snr conjugate ar conjugate, is equal to zero.

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Since a_s and a_r are independent, if we chose $a_s = a_r$ from

$$\sum_{n=1}^{N}(S_{nr}a_r S_{ns}^* a_s^* + S_{ns}a_s S_{nr}^* a_r^*)=0$$

we get

$$\sum_{n=1}^{N}(S_{ns}S_{nr}^* + S_{nr}S_{ns}^*)=0$$

If, instead, we chose $a_s = ja_r$ with a_r real, from

$$\sum_{n=1}^{N}(S_{nr}a_r S_{ns}^* a_s^* + S_{ns}a_s S_{nr}^* a_r^*)=0$$

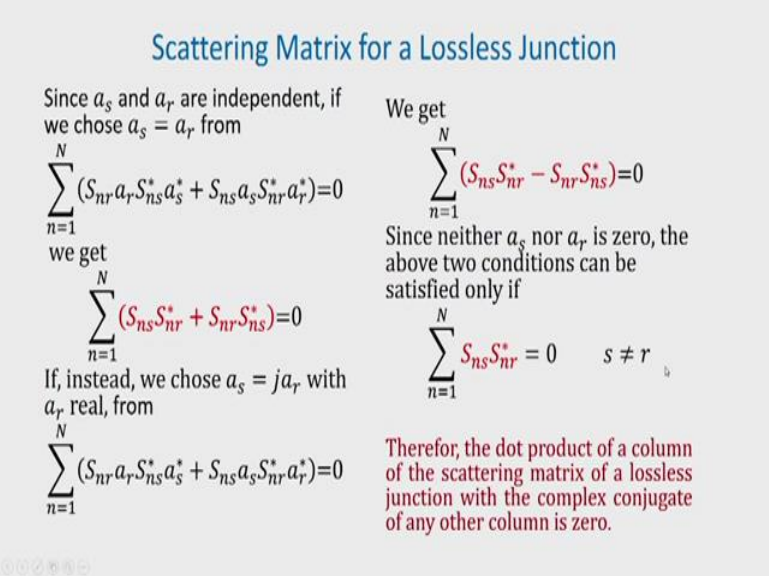
We get

$$\sum_{n=1}^{N}(S_{ns}S_{nr}^* - S_{nr}S_{ns}^*)=0$$

Since neither a_s nor a_r is zero, the above two conditions can be satisfied only if

$$\sum_{n=1}^{N} S_{ns}S_{nr}^* = 0 \qquad s \neq r$$

Therefor, the dot product of a column of the scattering matrix of a lossless junction with the complex conjugate of any other column is zero.



Now since, as and ar they are independent, if we choose as equal to ar and from this relation, so in this equation when we substitute as equal to ar we get, summation of Sns into Snr conjugate plus Snr into Sns conjugate is equal to zero. And if instead, we choose as equal jar that means with ar equal to real, that means this is as is purely imaginary and ar is real, if we chose as and ar in this form, then we get another equation and we get, summation of Sns Snr conjugate minus Snr Sns conjugate equal to 0.

Now in this equation, you see that, we are getting the sum of these 2 terms and, here we are getting the difference of these 2 terms and in the both cases right hand side is equal to 0. So, neither as nor ar is zero, the above 2 conditions can be satisfied only, if we have summation of Sns Snr conjugate is equal to 0. and s is not equal to r. so, this can be put in this form therefore, we find that the dot product of a column of a scattering matrix of a lossless junction. With the complex conjugate of any another column is zero. So only condition we require is that, s is not equal to r that means the 2 columns are to be different.

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We can also prove the properties of the S matrix for a lossless junction following a different approach as follows:

We have,

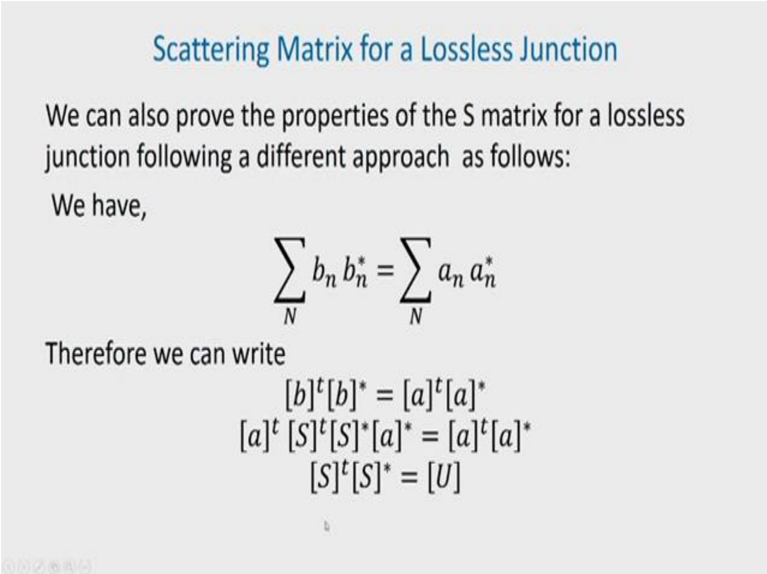
$$\sum_{N} b_n\, b_n^* = \sum_{N} a_n\, a_n^*$$

Therefore we can write

$$[b]^t[b]^* = [a]^t[a]^*$$

$$[a]^t\,[S]^t[S]^*[a]^* = [a]^t[a]^*$$

$$[S]^t[S]^* = [U]$$



Now whatever, we discuss so far regarding the properties of the lossless junction, this can be derived using a different approach which is as follows, we have for a lossless junction. summation of bn bn conjugate, is equal to summation of an an conjugate. And therefore, this term is essentially transpose of b into b conjugate and similarly, an conjugate. Can be written as transverse of a and a conjugate.

Now if we substitute b to be equal to sa and then we take the transpose we get, a transpose s transpose, s conjugate, a conjugate, equal to a transpose a conjugate. That means this term will be a unity matrix.

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$$\sum_{n=1}^{N} (S_{ns}a_s + S_{nr}a_r)\,(S_{ns}a_s + S_{nr}a_r)^* = |a_s|^2 + |a_r|^2$$

$$|a_s|^2 \sum_{n=1}^{N} S_{ns}\,S_{ns}^* + \sum_{n=1}^{N} (S_{nr}a_r S_{ns}^* a_s^* + S_{ns}a_s S_{nr}^* a_r^*) \ + |a_r|^2 \sum_{n=1}^{N} S_{nr}\,S_{nr}^* = |a_s|^2 + |a_r|^2$$

$$\sum_{n=1}^{N} (S_{nr}a_r S_{ns}^* a_s^* + S_{ns}a_s S_{nr}^* a_r^*) = 0$$

$[S]^t[S]^* = [U]$ can be written as

$$\sum_{k=1}^{N} S_{ki}S_{kj}^* = \delta_{ij}$$

$\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$

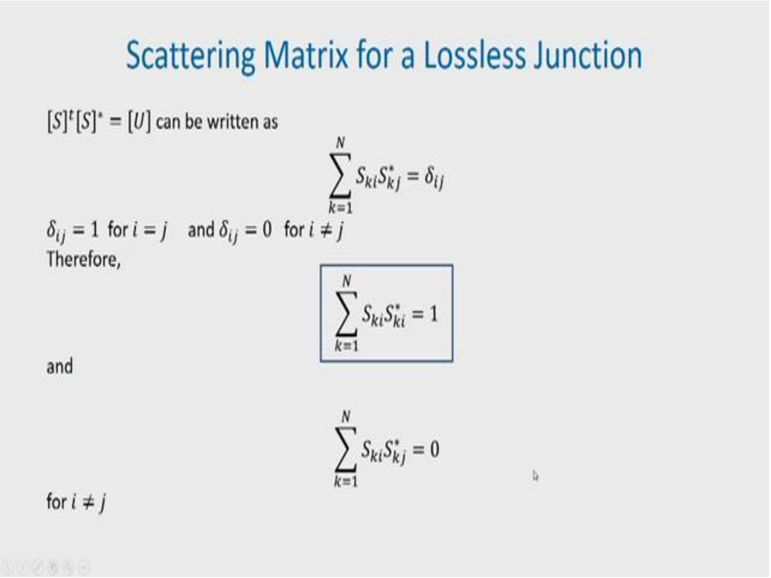
Therefore,

$$\sum_{k=1}^{N} S_{ki}S_{ki}^{*} = 1$$

and

$$\sum_{k=1}^{N} S_{ki}S_{kj}^{*} = 0$$

for $i \neq j$



So, s transverse s conjugate equal to u can be written as summation k equal to 1 to N, Ski Skj conjugate is equal to delta ij. Now this delta ij is equal to 1 for i equal to j and delta ij is equal to 0 for i not equal to j. and therefore, whenever i equal to j., we can write k is equal to 1 to N Ski Ski conjugate is equal to 1. So, this the condition we derived earlier, this essentially represents the dot product of the elements of 1 column, with the conjugate of the same column that means the Ith column. And when i is not equal to j, that means the columns i and j are different in that case, if we take the dot product of the Ith and Jth column, Ski and Skj conjugate then this summation becomes zero.

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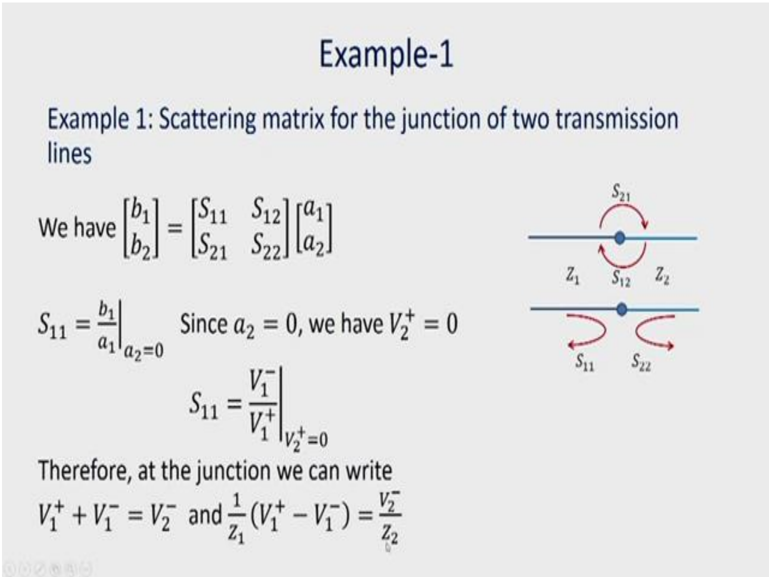
We have $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$S_{11} = \frac{b_1}{a_1}\Big|_{a_2=0}$ Since $a_2 = 0$, we have $V_2^+ = 0$

$$S_{11} = \frac{V_1^-}{V_1^+}\Bigg|_{V_2^+=0}$$

Therefore, at the junction we can write

$V_1^+ + V_1^- = V_2^-$ and $\frac{1}{Z_1}(V_1^+ - V_1^-) = \frac{V_2^-}{Z_2}$



Now let us consider one example to explain some of the things that we have discussed in this lecture, so we take this example, scattering matrix for the junction of 2 transmission lines, so it is shown here, here we can see a junction between 2 transmission line, this transmission line has the characteristic impedance of Z1 and this is characteristic impedance of Z2. And this is S11, and S22 these are essentially the reflection coefficient at the junction. And S21 and S12 these are the transmission coefficient for the junction.

And let us find out these S parameters S11, S12, S21, S22, so we have b1, b2 this is equal to S11, S12, S21, S22 in to a1, a2. And from this matrix relation we see that, S11 is essentially b1 by a1 when, a2 equal to 0. Now we have defined a2 to be V2 plus divided by root z0 to or here it is z2. So when a2 equal to zero. It means V2 plus is equal to 0. So we can write similarly, V1 can be written as V1 minus by root z1 and a1 can be written as V1 plus divided by root z1 and hence, S11 we can write as V1 minus divided by V1 plus when V2 plus equal to zero.

Now if we take, this junction we can write, the voltage on the left hand side of the junction. It is V1 plus, plus V1 minus. On the right hand side of the junction we have already argued that, V2 plus is zero, so we can write, V1 plus V1 minus is equal to V2 minus, and if we equate the currents, then we can write, 1 by z1 V1 plus minus V1 minus this is equal to V2 minus by z2.

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Therefore, $V_1^+ + V_1^- = \frac{Z_2}{Z_1}(V_1^+ - V_1^-)$

and $S_{11} = \left.\frac{V_1^-}{V_1^+}\right|_{V_2^+=0} = \frac{Z_2 - Z_1}{Z_2 + Z_1}$

In the same manner,

$S_{22} = \left.\frac{V_2^-}{V_2^+}\right|_{V_1^+=0} = \frac{Z_1 - Z_2}{Z_2 + Z_1} = -S_{11}$

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0}$$

$$S_{21} = \left.\frac{\sqrt{Z_1}}{\sqrt{Z_2}}\frac{V_2^-}{V_1^+}\right|_{V_2^+=0}$$

$$V_1^+ + V_1^- = V_2^-$$

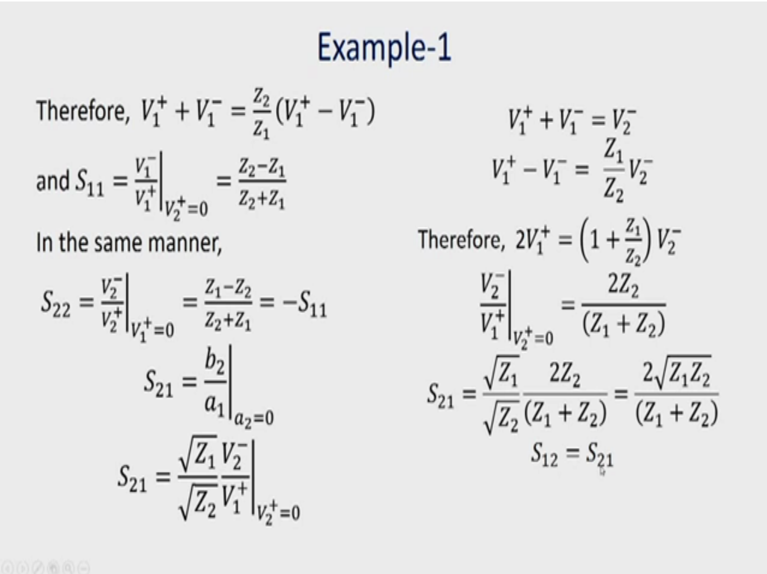
$$V_1^+ - V_1^- = \frac{Z_1}{Z_2}V_2^-$$

Therefore, $2V_1^+ = \left(1 + \frac{Z_1}{Z_2}\right)V_2^-$

$$\left.\frac{V_2^-}{V_1^+}\right|_{V_2^+=0} = \frac{2Z_2}{(Z_1 + Z_2)}$$

$$S_{21} = \frac{\sqrt{Z_1}}{\sqrt{Z_2}}\frac{2Z_2}{(Z_1 + Z_2)} = \frac{2\sqrt{Z_1 Z_2}}{(Z_1 + Z_2)}$$

$$S_{12} = S_{21}$$



And therefore, rearranging the terms we can write, V1 plus plus V1 minus is z2 by z1, V1 plus minus V1 minus and therefore, S11 which is V1 minus divided by V1 plus, when V2 plus equal to zero. This can be put in this form, z2 minus z1 divided by z2 plus z1. And this is same as the reflection coefficient at the junction. And if you proceed in the same manner, you can show that, S22 is equal z1 minus z2 divided by z1 plus z2 and this is equal to minus S11.

Now let us evaluate S21, S21 is b2 by a1 when, a2 equal to zero, and this can written once, we substitute b2 and a1, we can write V2 is V2 minus by root z2, and a1 is V1 plus by root z1. So, it can be written as S21 is equal to root z1 by root z2, V2 minus by V1 plus for V2 plus equal to zero, so we have seen that V1 plus plus V1 minus is equal V2 minus and from the current relationship we can write 1 by z1 V1 plus minus V1 minus is equal to V2 minus by z2.and this can be once, we take z1 in this side we can write, in this form.

And therefore, if we add this 2, we get 2 V1 plus is equal to 1 plus z1 by z2 V2 minus, and therefore, V2 minus by V1 plus when V2 plus is equal to 0, is 2 z2 divided by z1 plus z2. So now we have this part evaluated and S21 we get, by scaling these with root z1 by z2 and then, we get S21 equal to 2 root z1, z2 divided by z1 plus z2 and since, this junction is the reciprocal junction we will have, S12 equal to S21.

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In this example we find the reflection coefficient at the port 1 of a two-port for which the port 2 is terminated to a short circuit

Suppose the S parameters for the given two-port be

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

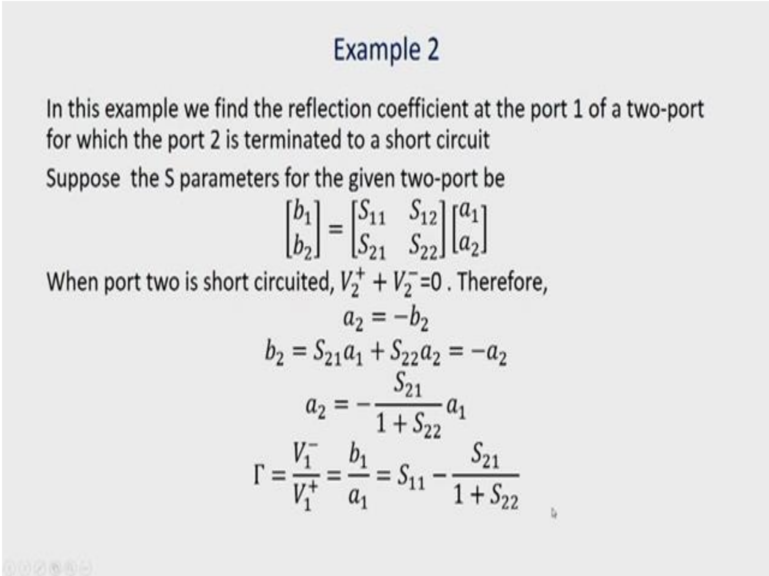
When port two is short circuited, $V_2^+ + V_2^-$=0 . Therefore,

$$a_2 = -b_2$$

$$b_2 = S_{21}a_1 + S_{22}a_2 = -a_2$$

$$a_2 = -\frac{S_{21}}{1 + S_{22}} a_1$$

$$\Gamma = \frac{V_1^-}{V_1^+} = \frac{b_1}{a_1} = S_{11} - \frac{S_{21}}{1 + S_{22}}$$



We now consider second example, where what we do we find the reflection coefficient at the port 1 of a 2 port, when the port 2 is terminated to a short circuit, so let the S parameters for the 2 port we given, as S11, S12, S21, S22. And next what we do, in this 2 port we connect a short circuit, at the port 2. So when we put a short circuit, at the port 2. V2 becomes 0 and V2 is V2 plus, plus V2 minus so, when this term becomes zero, essentially a2 becomes equal to minus b2, and now if we take this b2 is equal to S21 a1 plus S22 a2 this can be written as minus a2 and from here, we can solve a2 equal to minus S21 divided by 1 plus S22 a1 and if you consider b1 then b1 is equal to S11 a1 plus S12 a2.

So, the reflection coefficient is V1 minus divided by V1 plus which is b1 by a1 and can be found as S11 and we replace a2 by this term, so S11 minus S21 divided by 1 plus S22.