Lec 26: Index-guiding photonic crystal fibers

Hello students, welcome to lecture 26 of the online course on Photonic Crystals Fundamentals and Applications.

Lecture Outline

- **Index-Guiding Photonic-Crystal Fibers**
- **Index-Guiding Photonic-Crystal Fibers Dispersion Relation**
- **Endlessly single-mode fibers**
- **Scalar limit**
- **Linearly Polarized modes**

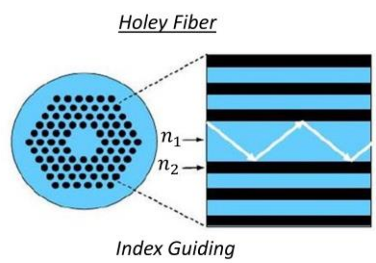
Source: F. Poli *et al*., "Photonic crystal fibers: properties and applications," Springer Science & Business, 2007.

Today's lecture will be on Index Guiding Photonic Crystal Fibers. So, here is the lecture outline. So, we will discuss about index guiding photonic crystal fibers, their dispersion relation, we will talk about endlessly single mode fibers how they work. the scalar limit and also we will discuss about the linearly polarized modes.

Index-Guiding Photonic-Crystal Fibers

Index-Guiding Photonic-Crystal Fibers

- The easiest photonic-crystal fibers to understand are those that employ index guiding.
- They guide light by virtue of the smaller average refractive index of the cladding relative to the core.
- A typical example is the *holey fiber*, in which the cladding has a cross-section that is a triangular lattice of air holes within an otherwise uniform dielectric medium.

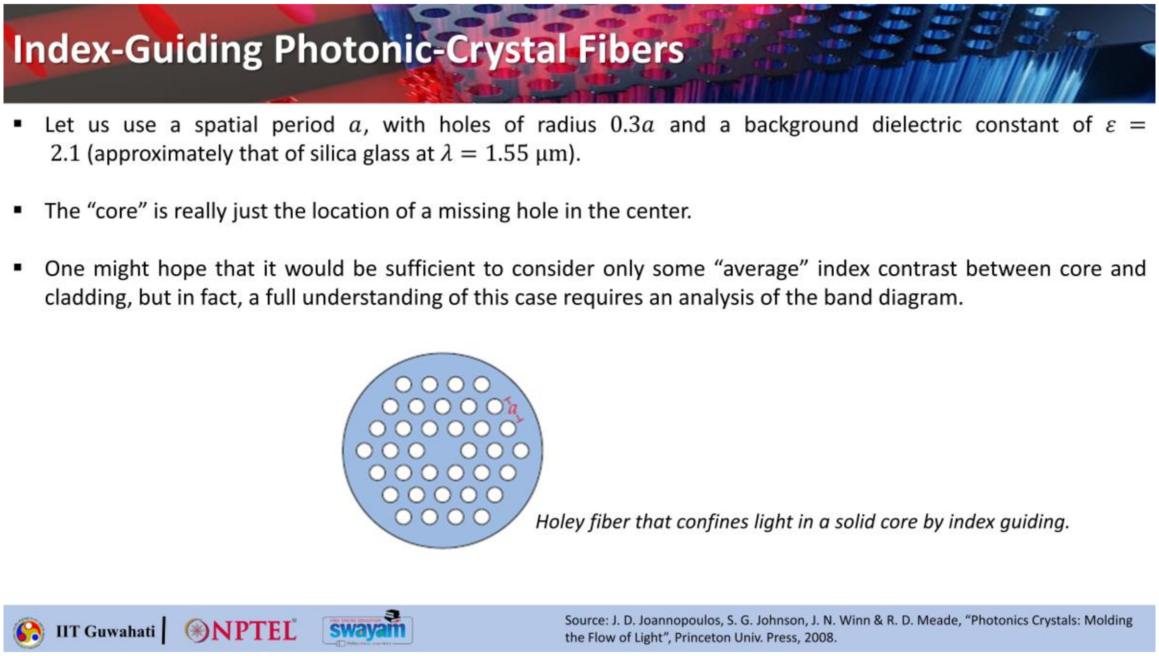


Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, let us look into this in more details we have briefly seen index guiding photonic crystal fibers in our previous lecture.

So, we will quickly go into the details of it in this lecture. So, if you remember so this is the easiest photonic crystal fiber to understand ok. So, this employs index guiding. So what happens here you can see okay you can actually create this is the core and you can create you know array of holes that reduce the refractive index of this cladding and that is how you know you can take the cross section and think of it that n_1 is high and n_2 is low and the light is basically guided through modified total internal reflection.

So, that is a very easy understanding of this holey fiber ok. So, here we need to ensure that the air holes are in a periodic pattern.



So, people choose triangular lattice usually ok and this is otherwise a uniform dielectric medium. So, if you consider a special period of a that is between you know there is a periodicity of the holes and if you consider the holes to have a radius of 0.3a, whereas the background is silica where epsilon equals 2.1. So, you can think of these parameters that will give you a holey fibre. So, what is happening at the core? The core is simply you know the location. with a missing hole in the center. So, you can ideally think of a hole air hole the same air hole can be here then it makes a completely uniform you know holey array or 2D photonic crystal. But you if you introduce a defect by removing this air hole where the laser pointer is currently placed then you create this solid core for this holey fiber.

Index-Guiding Photonic-Crystal Fibers

- Because a fiber has translational symmetry along the fiber axis (which we take to be the z axis), k_z is conserved, and we can write the field in the usual Bloch form:

$$\mathbf{H}(x, y, t) = \mathbf{H}(x, y)e^{ik_z z - i\omega t}$$

- Then plot ω versus k_z to obtain the band diagram (or **dispersion relation**), which is shown as an inset diagram in figure:

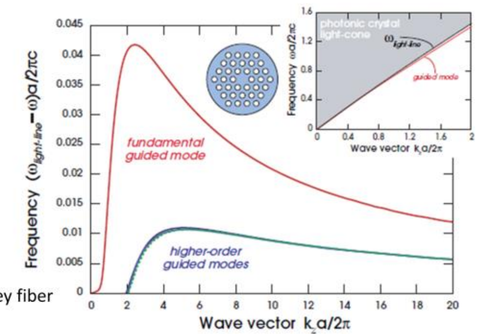


Figure: Band diagram of solid-core holey fiber as a function of axial wave vector k_z.

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, “Photonics Crystals: Molding the Flow of Light”, Princeton Univ. Press, 2008.

So, one might hope that it would be sufficient to consider some average index contrast between the core and the cladding, but in fact it is not the case you require a full understanding of the band diagram to analyze you know how holey fiber can actually help you guide light. Now because the fiber has translational symmetry along the fiber axis which is considered as z axis okay you can take k_z that is the z component of the wave vector to be conserved and then you can write the field in the usual block form you can write H(x,y,t) equals H(x,y) okay and it propagates along z direction and it is oscillating in time. So, this is how you can write the magnetic field as a function of space and time okay. So, then we plot you know the ω versus k_z that is basically this diagram the inset okay and it tells you about the dispersion relation. So, if you carefully look into this diagram what you see it is a usual ω-k plot.

So, this is where the normalized frequency is mentioned and this is where the normalized wave vector okay and here if you see carefully that there is a black line and there is another solid line. So, black line corresponds to the light line okay and the other line the straight line corresponds to the guided mode. Now what you can see that the difference between the frequency of you know in airline and light line light line and the guided mode is very, very small. So, you can actually plot the difference $\Delta\omega$ as function of the wave vector k_z and this is this particular plot the big one that you see here. So, this basically frequency the difference $\Delta\omega$ again the normalized one versus the wave vector which is also normalized one.

Index-Guiding Photonic-Crystal Fibers

- The usual ω plot is inset, but for clarity we also plot the $\Delta\omega$ between the guided bands and the light line.
- The higher-order guided modes are three bands that are nearly on top of one another.

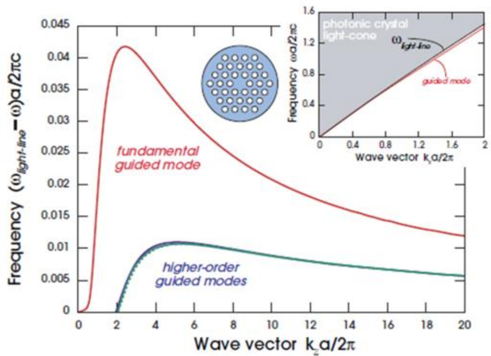


Figure: Band diagram of solid-core holey fiber as a function of axial wave vector k_z.

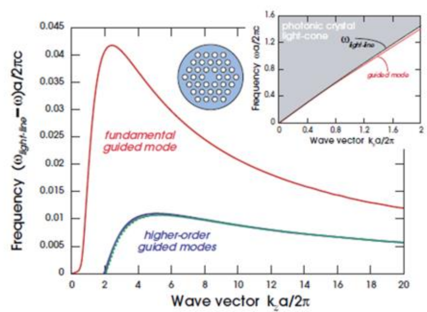
IIT Guwahati | NPTEL swayam

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, this is how you know the fundamental mode varies and this is how the higher order guided modes varies with know your different wave vectors. So, the projected band diagram here consists of two parts okay.

Index-Guiding Photonic-Crystal Fibers

- The projected band diagram consists of two parts:
  - A continuum of frequencies (the light cone) representing all of the possible extended states within the cladding
  - A discrete set of guided bands with frequencies lying below the light cone.
- If the cladding material were uniform with a dielectric constant ε (independent of ω)
  - the light line (the lower boundary of the light cone) would be a straight line, $\omega = ck_z/\sqrt{\varepsilon}$.

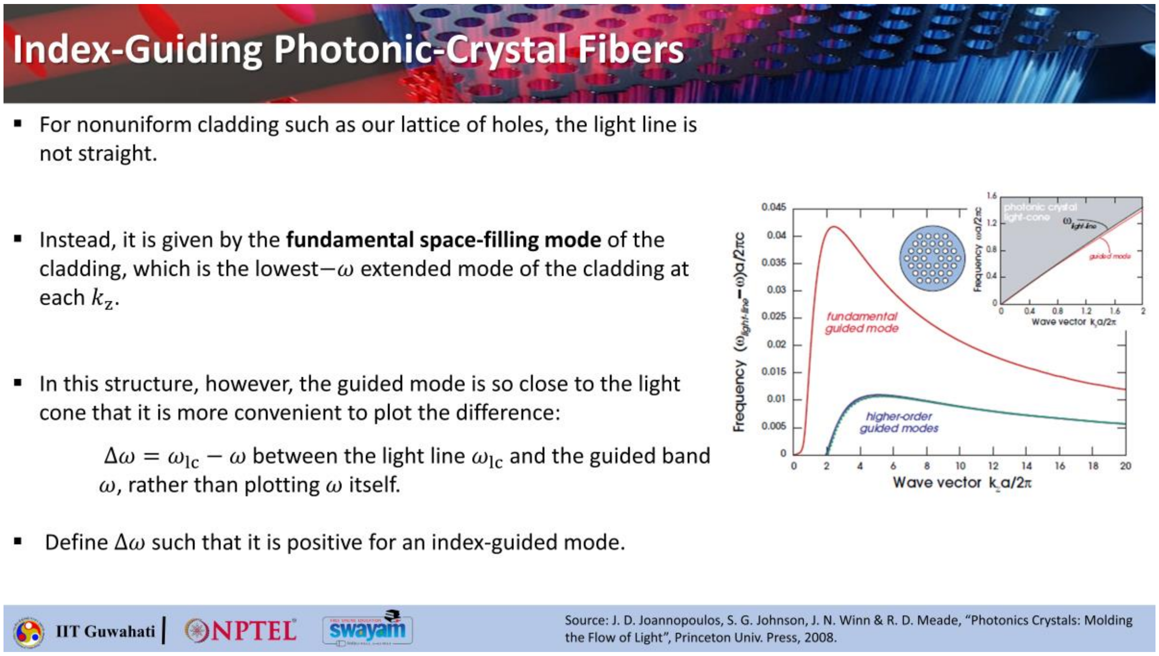


IIT Guwahati | NPTEL swayam

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, this is the band diagram that consists of basically two parts one is the continuum of frequencies that is basically the light cone okay. So, that represents all the possible extended modes or extended states within the cladding. And after that it tells you about a discrete set of guided bands with frequencies below this particular light cone.

Now if the cladding material were uniform say it had a dielectric constant of ε and that is also independent of ω . In that case the light line you could have drawn the you know lower boundary of this light cone as $\omega = ck_z/n$ or you can write ε. So, this entire thing is giving you the light cone and this line which is basically the lower boundary of the light cone is given by this straight line $\omega = ck_z/\sqrt{\varepsilon}$.



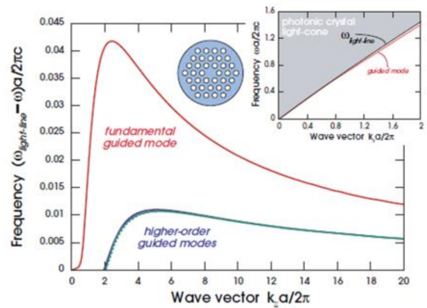
But now we do not have that uniform line. or homogeneous material in the cladding.

Instead of that what do you have? You have a non-uniform cladding such as you know a lattice of holes that in that case the light line is also not straight. So, instead it will be given by fundamental you know space filling mode of the cladding which is basically the lowest frequency extended mode of the cladding at each k_z. So, this is the one for this case. So, in this structure however the guided mode is basically so close as you can see here. So, it does not follow this light line which is for the uniform rather for this case you basically have the line which is very close to the light line.

And usually because of this closeness it is more convenient to plot the difference that is $\Delta\omega$ which is $\omega_{lc} - \omega$ that is the difference between the light line and the guided band you know rather than plotting the frequency itself. So, this is basically this that exists $\Delta\omega$ versus the wave vector. And we you define this $\Delta\omega$ in such a way that it is positive for the index-guided mode. So, you can see here for any k value the light line has got higher frequency. So, better you do it like you know $\omega_{lc} - \omega$ other than doing the you know reverse of it.

Index-Guiding Photonic-Crystal Fibers

- For each k_z, let us find all of the extended modes of the infinite periodic cladding (i.e., without the core) for all possible transverse wave vectors (k_x, k_y).
- Then plot the resulting frequencies as a function of k_z.
- The lowest frequency for each k_z defines ***the light line***.
- These extended modes are analyzed by considering the periodic cladding by itself.
- One need only consider (k_x, k_y) in the irreducible Brillouin zone of the triangular lattice.
- The extended modestake the Bloch form of a plane wave $e^{i\mathbf{k}.\mathbf{r}}$ multiplied by a periodic envelope function $\boldsymbol{H_k}(\mathrm{x, y})$.



Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

IIT Guwahati | NPTEL swayam

So, you make sure that that $\Delta\omega$ is basically positive. So for each k_z let us now find all the extended modes of the infinite periodic cladding okay that means without any core okay for all possible transverse wave vectors that is k_x and k_y. And then you will be plotting the resulting frequencies as function of k_z okay and again the lowest frequency for each k_z will basically define the light line that is basically the lower boundary of the light cone. Now this extended modes can be analyzed by considering the periodic cladding by itself okay and one need to consider only k_x, k_y in the irreducible Brillouin zone of a triangular lattice okay. And the extended modes take the block form of a plane wave that is $e^{(i\mathbf{k}.\mathbf{r})}$ okay and it has to be multiplied by the periodic envelope function that is $\boldsymbol{H_k}$ (x,y) okay.

Index-Guiding Photonic-Crystal Fibers

- The increased ε of the core introduces one or more guided modes, by pulling down modes beneath the light line.
- Because they are below the light line, these modes must decay exponentially into the cladding.
- The farther below the light line they are pulled, the faster the transverse decay.
- For the case of ***Holey fiber***, a doubly degenerate band is localized in the core, whose field patterns are shown in figure.
- We call this the **fundamental mode.**

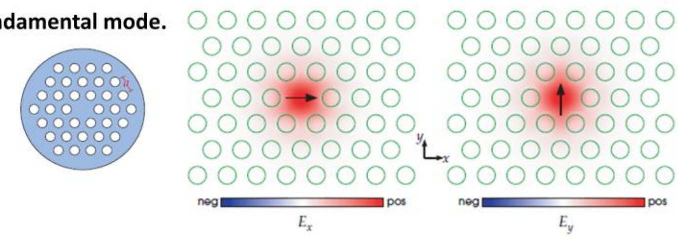


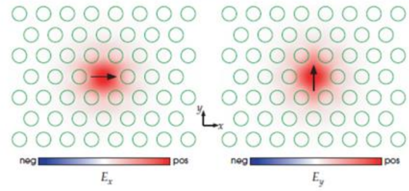
Figure : Electric-field pattern for the doubly degenerate fundamental mode.

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, H is the magnetic field and we have discussed why dealing with magnetic field is easier than dealing with the electric fields right. But if you have information about one you can always find out the other one. So the increased permittivity of the core introduces one or more guided modes by pulling down modes below or beneath the light line okay. And because they are below the light line that means you know this modes must decay exponentially into the cladding.

And further below the light line they are pulled the faster the decay will be right make sense. And for the case of a holey fiber okay you can see a doubly degenerate band is localized in the core. So, you can have either the you know polarization along x or y okay. So, you can see E_x or E_y like this. So, it can actually support a doubly degenerate band which is localized in the core and this is how the field pattern will look like.

Index-Guiding Photonic-Crystal Fibers



- Electric-field polarizations are nearly orthogonal everywhere.
- The mode pictured at left is mostly E_x, and the mode pictured at right is mostly E_y.
- The green circles show the locations of the air holes.
- In general, the fundamental mode is defined as the mode with the largest k_z for any given ω or, equivalently, the smallest ω for any given k_z.
- This is the analogue of the two degenerate, orthogonal, linearly polarized "LP01" modes that propagate within standard "single-mode" silica fibers.
- Here, due to the large index contrast and six-fold symmetry, the two orthogonal modes are neither purely linearly polarized nor are they exactly related by a 90° rotation.

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

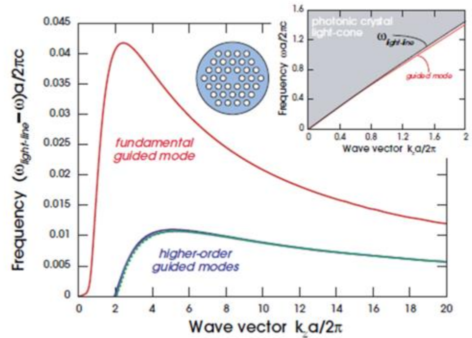
So, this one we call as fundamental mode because you know it is like a circular kind of pattern of the same you know electric positive electric charges. So, what we can see here is that electric field polarization are nearly orthogonal everywhere and here you can see that the left one is mostly this mode is purely E_x and on the right side here you can see this mostly E_y okay and the green circles over here they actually show the location of the air holes in this triangular lattice.

So, what you see here that in general the fundamental mode is defined as the mode with the largest you know k_z for a given ω or you can say they have the smallest you know ω for the given k_z. So, it means it will have the lowest or smallest largest k_z will mean you will have smallest wavelength associated. So, that will be the fundamental mode.

So this is analog to of the two degenerate orthogonal linearly polarized LP_{01} mode that typically propagates within a you know standard single mode silica fiber. In this case due to large index contrast and 6-fold symmetry the two orthogonal modes are neither purely linearly polarized nor they are exactly related by a 90 degree rotation.

Index-Guiding Photonic-Crystal Fibers

- For larger values of k_z, three additional guided bands are localized.
- These appear below the light line at $k_z a/2\pi \approx 2$.
- One of these bands is doubly degenerate, and is essentially a higher-order version of the fundamental mode, with an extra nodal plane perpendicular to the direction of polarization.
- The other two bands are nondegenerate.
- None of these higher-order modes can be excited by a plane wave source incident in the z-direction because the source and the modes would have different symmetries.

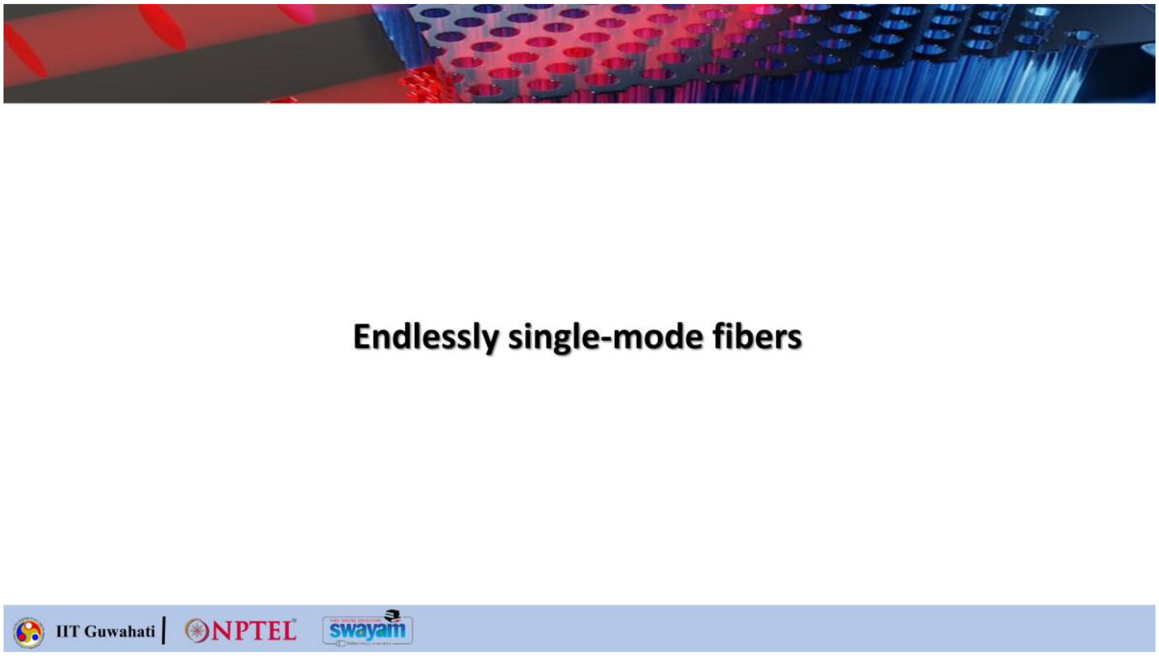


Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, here also you can see from the figure that for the larger values of k_z three additional guided bands are basically localized here you can see. okay. So, the here you can basically see that the 3 different values are overlapping and where do they come from they actually start appearing below the light line okay at $k_z a/2\pi$ equals 2.

So, from here they start appearing okay and you can see that one of this band okay is doubly degenerate. So this is the fundamental mode. So why it is called doubly degenerate because for the same frequency you can actually have two different orientation or two different modes possible okay. So this modes are doubly degenerate and is essentially a higher order version of the fundamental mode here you will see okay. Here also you can see doubly degenerate okay. for certain frequency band okay and with an extra nodal plane perpendicular to the direction of propagation right.

However, the other two bands are basically non degenerate. So, none of these higher order modes can be excited by a plane wave incident in the z direction because the source and the modes would have different symmetries.



So, with the understanding of the basic modes which are excited in the holey fiber we can now see how this fiber can act as endlessly single mode fiber ok. So, in an ordinary index guided waveguide as one goes to higher and higher ω that means smaller and smaller λ you will see that more and more number of guided modes are basically pulled down below the light line. So, this is your light line and these are the extended modes ok.

Endlessly single-mode fibers

- In an ordinary index-guided waveguide, as one goes to higher and higher ω (smaller wavelength λ), more and more guided modes are pulled below the light line.
- Eventually, one approaches the ray-optics limit, in which the guided modes are described by a continuum of angles greater than the critical angle for total internal reflection.
- However, as first pointed out by Birks et al. (1997), this need not be true of photonic-crystal fibers:

  They can remain endlessly single-mode, regardless of wavelength (limited only by the material properties).

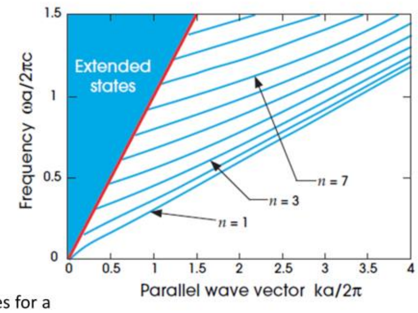


Figure: Harmonic mode frequencies for a plane of glass of thickness a and $\varepsilon = 11.4$.

IIT Guwahati | NPTEL swayam

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, you will see more number of you know with higher ω or smaller λ you will see more number of guided modes are pulled below the light line that is true. And eventually one approaches the ray optics limit where you know the guided modes are described by continuum of angles which are basically greater than the critical angle of the total internal reflection. So, you can actually consider those as you know the modes propagating. However, at as first pointed out by Birks et al. in 1997 okay this need not be true of the photonic crystal fibers right.

So, this particular plot shows the frequencies for a plane of glass of thickness a okay and that has been normalized okay and permittivity to be 11.4 okay. Now, photonic crystal fibers can remain endlessly single mode regardless of the wavelength. So, that means it is only limited by the material properties and we will see how it is possible. So, we also saw that you know the holey fiber that we considered could guide up to 4 bands ok.
So, fundamental guided mode and then there could be like 3 higher order guided modes. So, it is not definitely endlessly single mode there are possibility of other modes as well. However, it will still display the essential feature of this phenomena that is single endlessly single mode because as one go goes to higher and higher k_z or you can say larger ω you do not get more and more bands. It means the number of bands typically never exceed 4. so you can actually think of you know that you know one could reduce the number of modes to just one okay by having the you know whole radius to 0.15a however it will weaken the strength of confinement.

Endlessly single-mode fibers

- *Why are the higher-order modes absent?*
- The reason is that the effective index contrast between the core and the cladding in the holey fiber *decreases* at smaller wavelengths, rather than remaining fixed as it would for a homogeneous cladding.
- Thus, the strength of confinement is weaker at smaller wavelengths, and higher-order guided modes remain above the (lowered) light line.
- To be more concrete, we define the ***effective index*** of a mode as $\frac{ck_z}{\omega}$.
  - The factor by which the phase velocity $\frac{\omega}{k_z}$ is decreased from c.
  - The effective index equals the ordinary refractive index for a plane wave in a homogeneous medium.

IIT Guwahati | NPTEL swayam

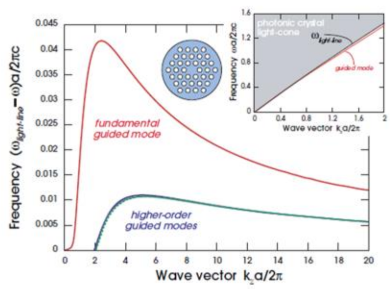
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

It means if you if you go below this one you will never excite this modes okay and you will be only dealing with the single mode case. So, only one compromise will be there it will weaken the strength of the confinement. Now the question comes why are the higher order modes absent? The reason is that the effective index contrast between the core and the cladding in the case of holey fiber decreases at small wavelengths rather than remaining fixed as it would have been the case for any homogeneous cladding. Because of that, the strength of confinement becomes weaker for smaller wavelengths.

And higher-order guided modes remain above the lowered light line. And to be more concrete, you can define the effective index of a mode as $(ck_z)/\omega$. So the factor by which the phase velocity ω/k_z decreases is this one ok. So, typically the phase velocity would have been c, but now it decreases by this particular factor to give you the new phase velocity ok. And this effective index equals the ordinary refractive index for plane wave in a homogeneous medium.

Endlessly single-mode fibers

- Therefore, at a given ω, an index-guided mode obviously has a larger effective index than that of the light line.
- To show the decrease in effective-index contrast with wavelength, figure *right* shows the effective index versus vacuum wavelength $\lambda/a = 2\pi c/\omega a$, based on the band diagram given in figure *left*.
- In the limit of small λ, the effective indices of both the modes and the light line approach the index 1.45 of the bulk silica.



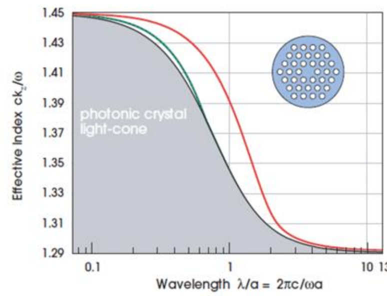


Figure: Bands of the solid-core holey fiber, from figure *left*, plotted as "effective index" ck_z/ω versus wavelength, along with the light cone (gray region).

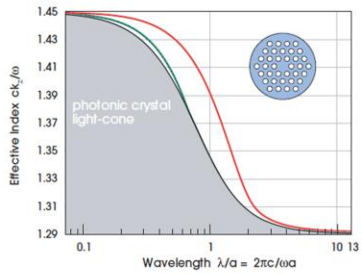
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, with this understanding you can say that at a given ω or frequency an index guided mode will obviously have you know larger effective index than that of the light line. So to decrease the you know to show the decrease in this effective index contrast with wavelength you can consider this particular figure okay which plots the effective index $(ck_z)/\omega$ versus the vacuum wavelength which is λ/a or you can say it is $2\pi c/\omega a$. right. So, what do you what do you see here? So, this is basically based on the band diagram which is given here right. So, here you can see the light line ok and these are some of the modes.

So, in the in the limit of small λ that is here effective indices of both modes okay and the light line you can see both are basically approaching that of the bulk silica that is 1.45 only when you are going for larger wavelength you will see there is a deviation okay. from the guided modes and the light line right.

Endlessly single-mode fibers

- *There is an intuitive explanation for why the effective-index contrast decreases with wavelength.*
- The fundamental (light-line) mode of the cladding "wants" to be concentrated as much as possible in the high-dielectric material.
- For long wavelengths $\lambda \gg a$, the light cannot lie entirely within the high dielectric, because the field cannot vary faster than the wavelength.



IIT Guwahati | NPTEL swayam

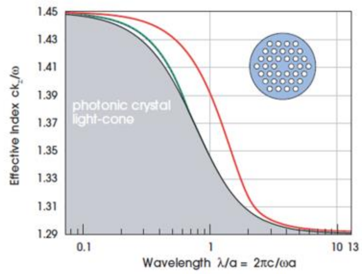
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, it means there is an intuitive explanation for why the effective index contrast would decrease with wavelength right.

So, here also it again merges. So, the fundamental that is the light line mode of the cladding wants to be concentrated as much as possible in the high dielectric region right. So, if you consider $\lambda \gg a$ to be much much larger than 1 that is you know your λ the periodicity is much much or λ the wavelength is much much larger than a. That means what it will look like the periodic media will look more or less like a homogeneous medium in that you know light cannot entirely lie within the high dielectric ok because the field cannot vary faster than the wavelength.

Endlessly single-mode fibers

- As we go to shorter and shorter λ, however, more of the light can "fit" in the dielectric between the holes.
- In the limit of small $\lambda \ll a$, the ray-optics limit applies, and the light is guided by total internal reflection and remains almost entirely within the dielectric material, with an effective index that approaches the index of the dielectric material.

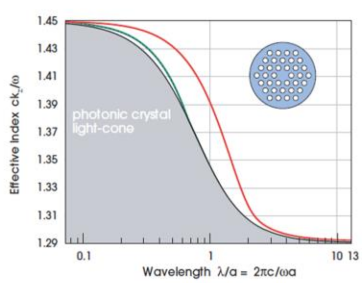


Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, in that case what will happen as we go to shorter and shorter λ more of the light will be able to fit in the dielectric between the holes. So, when you consider the limit of λ much much smaller than a that is the case when your ray optics limit will be applied and light can be simply guided by total internal refraction and it remains entirely within the dielectric material.

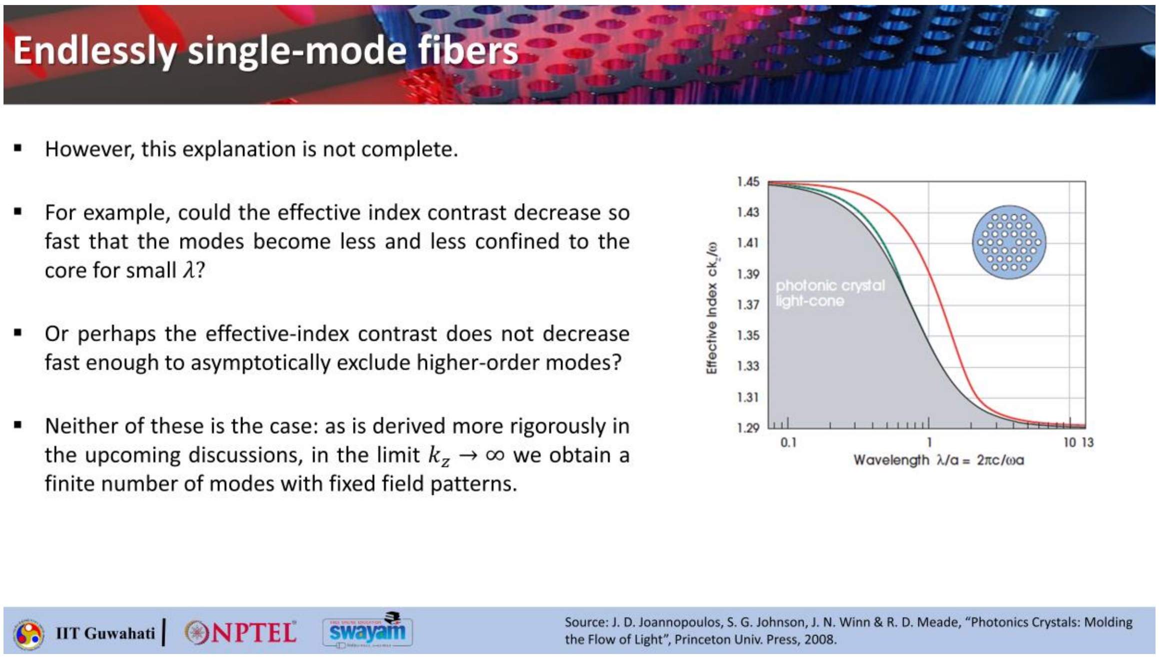
Endlessly single-mode fibers

- Since the core is made of the same dielectric material, the effective index of the guided mode must also approach the same value as the wavelength decreases.
- Precisely these limits are seen in **figure**, where both the light line and the guided mode approach the index $\sqrt{\varepsilon} = 1.45$ of the bulk silica.



Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

And in that case the effective index will be approaching the index of the dielectric material itself that is 1.45. Now, since the core is also made up of the same dielectric material. the effective index of the guided mode must always approach the same value as the wavelength decreases that is what we see here. So, precisely these are the limits seen in the figure where both the light line and guided mode approach the index which is 1.45 that with that of the bulk silica in this particular case. So, here the value of the effective mode is basically 1.29 and this is the case when your λ/a is like 10 which is much much less larger than 1.



However, as you can see that this explanation is not complete because if you take an example and think could the effective index contrast decrease so fast that the modes become less and less confined to the core for small λ or perhaps the effective index contrast does not decrease fast enough to asymptotically exclude the higher order modes. So, which is the case? But I actually you know neither of these is basically the case that is happening.

Because when you consider the limit of k_z going to infinity. we obtain a finite number of modes with fixed field patterns okay.

Endlessly single-mode fibers

- Of course, in a real material, we must eventually take into account the fact that ε is a function of frequency, and indeed the material may cease to be transparent at some ω.
- On the other hand, we can equivalently keep ω fixed and rescale the *structure*, in which case the above analysis is exact.
- The endlessly single mode property, in this case, means that we can make the waveguide arbitrarily large and still guide only a single (doubly degenerate) waveguide mode.
- This could be useful for reducing the effects of material nonlinearities, although one is eventually limited by the fact that bending losses tend to increase with mode size.

IIT Guwahati | NPTEL swayam

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

And of course in a real material we must eventually take into account the fact that you know ε is basically a function of frequency then it is never fixed okay and the material may cease to be transparent to some frequency means it may become lossy as well. And on the other hand we can equivalently keep ω fixed and try to rescale the structure in that case the above analysis is exact you can apply that particular analysis there. So, the endlessly single mode property in this case means that we can make the waveguide arbitrarily large and still guide a single or actually a doubly degenerate waveguide mode through it.

So, this could be useful for reducing the effects of material non-linearities although this is eventually limited by the fact that the bending loss tend to increase with the increase in the mode size right.

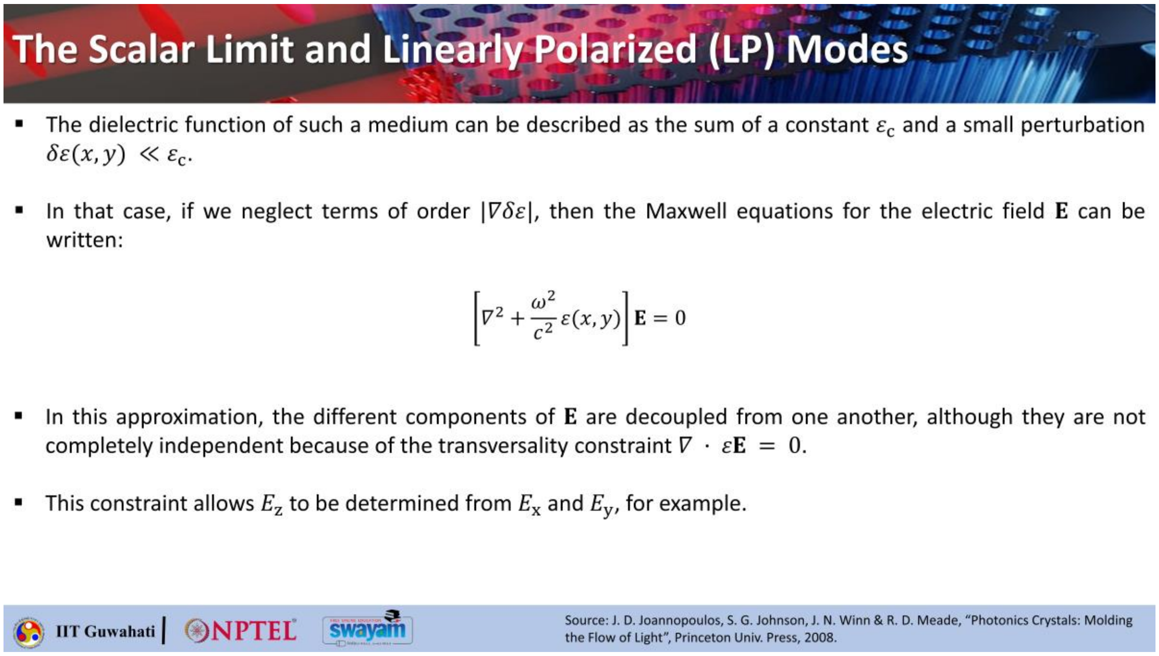
The Scalar Limit and Linearly Polarized (LP) Modes

The Scalar Limit and Linearly Polarized (LP) Modes

- The key to a quantitative understanding of the large-k_z limit is to realize that this regime is asymptotically described by a *scalar* wave equation that is independent of k_z.
- Consequently, for large k_z, the modes approach k_z-independent "linearly polarized" (**LP**) field patterns.
- Indeed, we shall see that this scalar limit is useful for understanding other fiber phenomena as well, such as the existence of photonic band gaps.
- Traditionally, the scalar approximation in electromagnetism is formulated only for structures with a *small* dielectric contrast.

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

Let us now discuss the scalar limit and linear polarized modes in the case of this photonic crystal fiber. The key to a quantitative understanding of the large k_z limit is to realize that this particular regime is asymptotically described by a scalar wave equation which is independent of k_z. Consequently for a large k_z the modes approach k_z independent ok, linearly polarized field patterns. Indeed we shall see that you know this scalar limit is useful for understanding other fiber phenomena as well such as the existence of photonic band gaps. Traditionally the scalar approximation in electromagnetism is formulated only for structures with the small dielectric contrast.



So the dielectric function for such a medium can be described as a sum of a constant epsilon c and a small perturbation which is $\delta\varepsilon(x, y)$ which is much smaller than the ε_c. Now in that case if we neglect terms which are of the order of delta you know $|\nabla\delta\varepsilon|$ then the Maxwell's equation for the electric field can be written as this okay. So, this is the typical form that we have already seen before okay. So, in this approximation the different components of electric field are decoupled from one another although they are not completely independent because of the transversality constant which is basically $\nabla \cdot \varepsilon\mathbf{E}$ equals 0 okay.

So, this constraint allows E_z to be determined from E_x and E_y for example, okay.

The Scalar Limit and Linearly Polarized (LP) Modes

- If we combine these results with Bloch's theorem for the waveguide modes, it follows that we can write the *transverse* (xy) components of $\mathbf{E}$ in terms of a single *scalar* function $\psi(x, y)$ (an LP mode):

$$E_{\mathrm{t}} = [p_x \hat{x} + p_y \hat{y}]\psi(x, y)^{e^{ik_z z}}$$

- Here p_x and p_y are constants that specify the amplitude and the direction of polarization, and the subscript t stands for *transverse*.

- The function ψ satisfies the eigenequation:

$$\left[-\nabla_{\mathrm{t}}^2 - \frac{\omega^2}{c^2}\delta\varepsilon(x, y)\right]\psi = k_{\mathrm{t}}^2\psi$$

reminiscent of the Schrödinger equation of quantum mechanics.

IIT Guwahati | NPTEL swayam

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

And if we consider this results with Bloch's theorem for the waveguide modes, it follows that we can write the transverse xy components of the electric field vector E in terms of scalar function $\psi(x, y)$ which describes an LP mode in this particular form that E_{t}=[$p_x \hat{x}$+$p_y \hat{y}$] $(\psi(x, y))^{(e\,\hat{}\,(ikz\,z))}$. So, here p_x and p_y are basically constants that specify the amplitude and the direction of polarization and the subscript t stands for transverse. So we can understand that this particular function ψ basically satisfies the eigen equation. So if you put it there you can see that some operator operating on ψ gives you back the ψ and some constant.

So this is an eigen equation or eigen mode equation and this is reminiscent of the Schrodinger's equation of the quantum mechanics.

The Scalar Limit and Linearly Polarized (LP) Modes

$$\left[-\nabla_t^2 - \frac{\omega^2}{c^2}\delta\varepsilon(x,y)\right]\psi = k_t^2\psi$$

- In this equation, ∇_t represents the transverse (x and y) components of ∇, and k_t is a *transverse wave number* defined as:

$$k_t \triangleq \sqrt{\frac{\omega^2}{c^2}\varepsilon_c - k_z^2}$$

- In contrast to this traditional approximation, a photonic-crystal fiber generally has a large index contrast.
- For this reason, it may be surprising that a photonic crystal fiber can also be accurately described by a scalar approximation.

IIT Guwahati | NPTEL swayam

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, in this equation if you see that this ∇_t okay represents the x and y components of the ∇ operator or you can say ∇_t okay. So, or nabla whatever you want to call it and here you can also see k_t that is basically the transverse wave number okay and it is defined as square root of $(\omega^2/c^2)\ \varepsilon_c - k_z^2$. So, these are this is the basically the transverse component of the wave factor right. So, in contrast to this traditional approach a photonic crystal fiber generally has a large index contrast and for this reason it may be surprising that a photonic crystal fiber can be accurately described by a scalar approximation.

The Scalar Limit and Linearly Polarized (LP) Modes

- Suppose that in addition to the small variations $\delta\varepsilon$, we also have some very low-index regions (e.g., the air holes) with a dielectric constant $\varepsilon = \varepsilon_c - \Delta\varepsilon$.
- In this case, $\Delta\varepsilon$ is large and positive. The key fact is that for large k_z, the fields within the very low-index regions become very small, because of index guiding.
- We may therefore use the scalar approximation in the regions where $\Delta\varepsilon = 0$, and simply set $\psi = 0$ where $\Delta\varepsilon \neq 0$ for the equation:

$$\left[-\nabla_t^2 - \frac{\omega^2}{c^2}\delta\varepsilon(x,y)\right]\psi = k_t^2\psi$$

- Seen this way, the effect of the air holes is to impose a boundary condition on ψ.

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, how does it work? Suppose that in addition to the small variation which you consider to be $\delta\varepsilon$, we also have some low some very low index region something like air holes with dielectric constant given by you know ε which is $\varepsilon_c - \Delta\varepsilon$. So, in this case you know because this has got a very low index it means this $\delta\varepsilon$ is basically large and positive okay. And the key fact is that for large k_z the fields within this low index regions will also become very small it means index guiding is happening right. So we may therefore use scalar approximation in the region where I know this $\Delta\varepsilon$ equals 0 okay and simply said ψ equals 0 where $\Delta\varepsilon$ is not equals 0.

for the equation this one right. So, these are the two regions we can understand.

The Scalar Limit and Linearly Polarized (LP) Modes

- To be more explicit, the fields fall off exponentially into a low-index region with a spatial decay constant of:

$$\kappa = \sqrt{k_z^2 - \frac{\omega^2}{c^2}\varepsilon}$$

- In terms of k_t, $\kappa \approx k_z\sqrt{\Delta\varepsilon/\varepsilon_c}[1 - O(k_t^2/k_z^2)]$
- The fields in this region can therefore be neglected when $\kappa \sim k_z \gg k_t$ (i.e. when the field decays much faster than the transverse ψ oscillations).
- However, the condition $k_z \gg k_t$ is equivalent to the condition that the effective index ck_z/ω approaches $\sqrt{\varepsilon_c}$, which we already saw must be true at high frequencies (large k_z).

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, where it is positive you actually have this particular wave being guided right and where the difference is not there you can simply put that you know the field is also 0 that means no wave guiding taking place. So seen in this particular way the effect of air holes is basically to impose the boundary condition on ψ. That makes sense if you go back to the first cross-sectional schematic we have shown those are the air holes giving you the boundary between the solid core and the you know the cladding okay and that is where you know you can think of high and low index medium and light is basically getting guided based on modified total internal reflections. So, to be more explicit the fields fall off exponentially into the low index region with a special decay constant of κ which is given as square root of $(k_z^2 - \omega^2/c^2)\ \varepsilon$.

And in terms of k_t you can write κ as this okay. So, you can actually find out what is the decay constant for the field to fall inside this low index region that is the air region. The field in this region can therefore be neglected when κ is of the order of k_z and which are much much larger than k_t. That means the field decays much faster than the transverse ψ oscillations okay. However, the condition k_z much greater than k_t is equivalent to the condition that the effective index of the mode that is ck_z/ω approaches the material index that is $\sqrt{(\varepsilon_c)}$ and that we have already seen that it is true for large frequencies right. which also correspond to large k_z.

The Scalar Limit and Linearly Polarized (LP) Modes

- **First:**
- if $\delta\varepsilon = 0$, as it is for our holey fiber, then ψ satisfies an eigenequation:

  $-\nabla_t^2 \psi = k_t^2 \psi$ (with $\psi = 0$ in the holes) in which neither k_z nor ω appear explicitly.

- Thus, the values of neither k_z nor ω will affect the solution ψ or the eigenvalue k_t^2.

- We can conclude that, for large k_z, the modes approach fixed field patterns obeying a dispersion relation:

$$\omega^2 = c^2(k_t^2 + k_z^2)/\varepsilon_c$$

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, why is it important? The scalar limit for large k_z has several interesting consequences. The first one is that if $\delta\varepsilon$ is 0 as it is for our holy fiber then ψ basically satisfies an eigen equation and it can be written as $-\nabla_t^2 \psi = k_t^2 \psi$ and with ψ equals 0 in the holes okay. So in this equation okay neither k_z nor ω appear explicitly right. So, thus the values of neither k_z nor ω is going to affect the solution of ψ that you can get from here or the eigenvalue k_t^2. that you are going to obtain from this equation. So, what we can conclude? We can conclude that for large k_z the modes basically approach fixed field pattern obeying a dispersion relation which is basically this one $\omega^2=c^2 (k_t^2+k_z^2)/\varepsilon_c$.

The Scalar Limit and Linearly Polarized (LP) Modes

- **Second:**
- Each mode ψ in the scalar limit, a so-called **LP mode** corresponds to *several* vectorial solutions of the Maxwell equations for the *same* $|\psi|^2$ intensity pattern and the same eigenvalue k_t.
- There are two possibilities. If ψ is a nondegenerate mode, then we get two vectorial modes $\psi\hat{\boldsymbol{x}}$ and $\psi\hat{\boldsymbol{y}}$, corresponding to a doubly degenerate "linearly polarized" state.
- If ψ itself is a doubly degenerate state with two solutions $\psi^{(1)}$ and $\psi^{(2)}$, then we get *four* vectorial modes $\psi^{(l)}\hat{x}$ and $\psi^{(l)}\hat{y}$ for = 1, 2.
- For finite k_z, the scalar approximation is not exact and such degeneracies break (leaving at most doubly degenerate pairs).

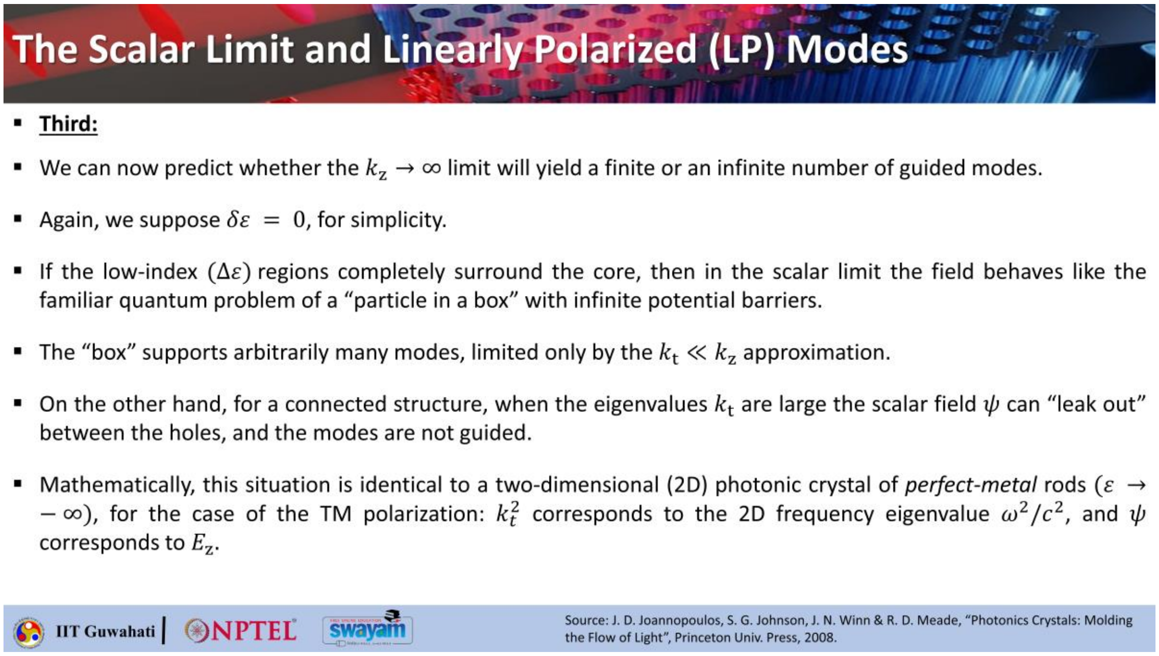
IIT Guwahati | NPTEL swayam

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

The second point is that you know each mode ψ in the scalar limit which is also the so called LP mode right it corresponds to a several vectorial solutions of the Maxwell's equation. for the same ψ^2 intensity pattern and the same value k_t, right. So, there are basically two possibilities. If ψ is a non-degenerate mode then we get two vectorial modes something like $\psi\ \hat{x}$ and $\psi\hat{y}$ which corresponds to doubly degenerate linearly polarized modes as we saw in that you know diagram electric field diagram before. And now, if ψ itself is a doubly degenerate state with two solutions, which is like ψ^1 and ψ^2, in that case, we get four vectorial modes, okay, like $\psi^1\ \hat{x}$ or you can say $\psi^l\hat{x}$ and $\psi^l\hat{y}$ for l equals 1 and 2.

So, l is missing here it should be l equals 1 and 2 right. So, for finite k_z the scalar approximation is not exact and such degeneracies will break that means you know leaving at most doubly degenerate pairs.

So, the states basically divide into linear combinations corresponding to different vectorial eigenmodes. It is precisely such LP modes which are nothing but a pair of doubly degenerate modes which corresponds to a non-degenerate or you can say monopole like ψ and for nearly you know degenerate modes which includes one doubly degenerate. pair corresponding to a doubly degenerate that is like you know dipole like ψ.



And the third case would have been that we can now predict whether the k_z tends to infinity limit is going to yield a finite or an infinite number of guided modes.

Again if we suppose that you know $\delta\varepsilon$ is 0 for simplicity. So we can write that in the low index region that is $\delta\varepsilon$ region which completely surrounds the core. Then in the scalar limit the field will behave like the familiar quantum problem of particle in a box with infinite potential barriers. So, here the box supports you know arbitrarily many modes which are limited only by the approximation that your k_t must be much much lesser than k_z right. So, on the other hand for a connected structure okay like the one we have when the eigenvalues k_t are large. the scalar field ψ will be able to leak out between the holes because there is still material and that can act as leakage between the holes and the modes are not guided.

So, mathematically this situation is identical to a two-dimensional photonic crystal of perfect metal rods okay where ψ can be or where epsilon can be considered as minus infinity. So, these are like perfectly metal rods and for the case of TM polarization right. So, there k_t square corresponds to the 2D frequency eigenvalues that is ω^2/c^2 and ψ will correspond to 2D. Ez, okay.

The Scalar Limit and Linearly Polarized (LP) Modes

- **Third:**
- The band diagram of this analogous 2D metallic structure is shown in the figure.
- It exhibits a well-known property of **metallo–dielectric photonic crystals**: there is a band gap starting at $k_t = 0$ and extending to the minimum of the first band.
- This finite gap, in turn, corresponds to a finite number of discrete-k_t localized modes supported by a defect.
- Another important feature that appears in figure is that (in this scalar/metallic limit) there are also "ordinary" photonic band gaps that appear between higher bands of the structure.

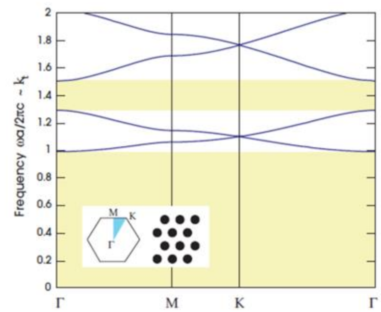


Figure: Band diagram for TM-polarized modes of a two-dimensional triangular lattice of perfect-metal cylinders.

IIT Guwahati | NPTEL swayam

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, the band diagram for this analogous 2D metallic structure which is like this.

So, these are like metal with perfect metal, but they are forming an array right now. So, it is a triangular array of metallic rods. So, you can see this will be the Brillouin zone, okay. and the band structure is shown here for this analogous 2D metallic structure. So, it exhibits a well-known property of metallo-dialectic photonic crystal where there is a band gap starting at k_t equals 0 and extending so this is one band gap that you can see okay.

And this is also another band gap which starts from k_t because this is k_t okay and you can yeah. So, here the frequency $\omega a/2\pi c$ is of the order of k_t okay. So, it starts or you can say it is equivalent to k_t here. So, it starts from k_t equals 0 and extending to the minimum of the first band. So, this much is the band gap for this metallo dielectric photonic crystal right.

So, this finite gap in turn corresponds to a finite number of discrete k_t localized modes which are supported by a defect okay. So, another important feature that you can see in this particular figure is that you know in this scalar metallic limit. There are also ordinary photonic band gaps that appear here between the higher bands of the structure. So, the two band gaps which are shown in this shaded yellow. So, this one the lower lowest band has a low cutoff frequency characteristic of the metallic structure and these bands are equivalent to the modes of the holey fiber in the scalar limit that is basically the limit where k_z is very large okay.

So with that we will stop here and we will start the discussion of in detail analysis of band structure or band gap guidance in holey fiber in the next lecture. So if you have got any queries regarding this lecture you can drop an email to this particular email address mentioning MOOC and photonic crystal on the subject line. Thank you.